Transformations and equation reductions in finite elasticity IV: Illustration of the general integral for a plane strain similarity deformation

by

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Abstract

In three previous parts of this work, for plane strain, plane stress and axially symmetric deformations a number of new first integrals are deduced for the so-called perfectly elastic Varga materials. These results constitute a considerable advance in the theory of finite elastic deformations, there being no similar results to these in existing theory. The new integrals, together with the constraint of incompressibility, mean that certain highly nonlinear fourth order partial differential equations admit second order systems, every solution of which is a solution of the corresponding problem. Moreover, many of the second order partial differential equations admit linearization to either the harmonic equation or the Helmholtz equation, thus giving rise to the possibility of generating quite general solutions. However, it is not immediately clear how such solutions relate to solutions of the full system and in this part of the work the complex question as to the extent to which the solutions of these new first integrals span the solutions of the full space is tackled. In this part, the general integral for plane strain deformations, given in Part III, is illustrated with reference to a specific similarity deformation, which maps one wedge shaped region into another such region and for which the general solution of the full system can be obtained in closed form.

CONTENTS

I. INTRODUCTION

In Parts I and II and III of this work (Hill and Arrigo [1,3] and Arrigo and Hill [2]) the authors have deduced a number of first integrals for plane strain, plane stress and axially symmetric deformations of the perfectly elastic Varga materials. Results of this type have not been given previously in the literature and constitute a major advance in the theory of large elastic deformations, in the sense that these results mean that solutions may be determined by examining a second order partial differential equation rather than a fourth order one. Moreover, many of the second order equations admit linearization and therefore the number of possible exact solutions is considerably extended. In this part, we examine the general integral, given in Part III for plane strain deformations, and attempt to provide some insight into the extent to which solutions of the general integral span the entire solution space. Although this is a complex question, the problem is tackled by an examination of a specific plane strain similarity deformation, for which the general solution is available and explicit expressions are obtained for the constants arising in the general integral in terms of the four arbitrary constants involved in the general solution.

In Part III, it is shown that certain plane strain deformations

$$x = x(X, Y), \quad y = y(X, Y), \quad z = Z, \tag{1.1}$$

of the perfectly elastic incompressible material with strain-energy function Σ given by

$$\Sigma = \alpha_1 \left(\lambda_1 + \lambda_2 + \lambda_3 - 3\right) + \alpha_2 \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} - 3\right), \tag{1.2}$$

satisfy the first integral

$$\frac{\partial x}{\partial Y} - \frac{\partial y}{\partial X} = \tan\psi \left(\frac{\partial x}{\partial X} + \frac{\partial y}{\partial Y}\right), \tag{1.3}$$

where $\psi(X, Y, x, y)$ is defined to be

$$\psi = 2\tan^{-1}\left(\frac{ax + by + cX + dY + e}{-bx + ay + dX - cY + f}\right) + k, \tag{1.4}$$

where $a, b, c, d, e, f,$ and k denotes seven arbitrary real constants. In the above (X, Y, Z) and (x, y, z) denote respectively material and spatial rectangular Cartesian coordinates, α_1 and α_2 are constants such that $\alpha_1 + \alpha_2 = 2\mu$ where μ is the usual infinitesimal shear modulus and $\lambda_i (i = 1, 2, 3)$ denote the principal stretches, which for an incompressible material satisfy $\lambda_1 \lambda_2 \lambda_3 = 1$. The "modified" Varga strain-energy function (1.2) was first proposed in [4] and the extra constant α_2 means that the range of the physical applicability of this strain-energy function is an improvement on that of the standard Varga material, which is fully referenced in Part I of this work.

It is important to emphasize that every solution of (1.3) and the incompressibility condition

$$\frac{\partial(x, y)}{\partial(X, Y)} = \frac{\partial x}{\partial X} \frac{\partial y}{\partial Y} - \frac{\partial x}{\partial Y} \frac{\partial y}{\partial X} = 1, \tag{1.5}$$

is a bonafide solution of the appropriate equilibrium conditions corresponding to the perfectly elastic material defined by (1.2). However, of course not every solution of the equilibrium conditions is necessarily a solution of the reduced system of equations defined by (1.3) - (1.5) and the purpose of this paper is to attempt to identify the relationship between this "solution sub-space" and the full solution space. In addition, by examining a particular plane deformation, we attempt to generate some insight into the meaning of the integral. We emphasize that the above integral is a curious result and although a formal linearization is given in Part III, questions concerning the extent to which the reduced system spans the solution space are clearly complex and non-trivial.

Formally, the number of arbitrary constants appearing in (1.4) can be reduced by appropriate choice of rotational and translational constants. However,to do this at the outset overly restricts the solutions of (1.3) - (1.5), since our objective is to relate these constants to four arbitrary integration constants. In the first part of the paper we merely set the translational constants $e, f,$ and k to zero and in material and spatial cylindrical polar coordinates (R, Θ, Z) and (r, θ, z) respectively, we examine the plane strain similarity deformation

$$r = Rf(\Theta), \quad \theta = g(\Theta), \quad z = Z. \tag{1.6}$$

This is a standard deformation which has been investigated in the context of stress singularities and in the determination of critical wedge angles for finite elastic stress boundary value problems involving wedge shaped regions, and additional references are given in [5]. In this latter paper, closed expressions are given for the functions $f(\Theta)$ and $g(\Theta)$ and the objective here is to determine to what extent such solutions are embodied in the above integral.

On performing the constant rotations of the material and spatial coordinates

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos\Phi & \sin\Phi \\ -\sin\Phi & \cos\Phi \end{pmatrix} \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \tag{1.7}$$

it is not difficult to show that (1.5) remains unchanged, while (1.3) has the same structure except that ψ is replaced by $\psi_1 = \psi - (\phi - \Phi)$. Now on examination of (1.4) we can always translate our coordinates such that the constants e and f are zero. Moreover, by an appropriate choice of Φ and ϕ in the rotations (1.7) we may eliminate two of the constants a, b, c, and d. Thus, for example if we select

$$\tan\Phi = \frac{c}{d}, \quad \tan\phi = -\frac{b}{a}, \tag{1.8}$$

and choose the constant $k = \phi - \Phi$, then equation (1.4) becomes

$$\psi_1 = 2\tan^{-1}\left(\frac{(a^2+b^2)^{1/2}\, x_1 + (c^2+d^2)^{1/2}\, Y_1}{(a^2+b^2)^{1/2}\, y_1 + (c^2+d^2)^{1/2}\, X_1}\right), \tag{1.9}$$

and therefore we can if necessary assume that $\psi(X, Y, x, y)$ is given by

$$\psi = 2\tan^{-1}\left(\frac{ax + dY}{ay + dX}\right), \tag{1.10}$$

but this does immediately restrict the possible solutions of (1.3) and (1.5). Clearly there are other choices of Φ and ϕ which would allow other possibilities for constants to be set to zero. In addition, ψ given by (1.10) involves only one essential constant, namely either of the ratios d/a or a/d. However, it is convenient to leave the two constants a and d so that the special cases $a = 0, d \neq 0$ and $d = 0, a \neq 0$ recover the two integrals studied in Part I (Hill and Arrigo [1]).

In the following section we state the basic equations for the general integral in terms of cylindrical polar coordinates and we summarize the known results for (1.6). Assuming that $e = f = k = 0$, we show in the subsequent section that in order that the solutions given in [5] arise from the above integral, the constants a, b, c, and d must be such that the two expressions (3.7) and (3.9) coincide. The identity of these equations is not a trivial matter but from an examination of their Taylor series expansions, we are able to provide various relations (see equation (3.11)) for the three ratios $b/a, c/a$, and d/a in terms of the four arbitrary constants arising in the general solution for (1.6). In the section thereafter, we examine four special cases of a, b, c, and d which may be integrated and we identify these closed expressions with the general solution given in [5]. In the final section of the paper we suppose that $b = c = 0$ and assume that ψ is given by equation (1.10). Appendices A and B list the ratios $b/a, c/a$, and d/a for the various cases arising from equating equations (3.7) and (3.9).

II. CYLINDRICAL POLAR COORDINATES AND SIMILARITY WEDGE DEFORMATIONS

In terms of cylindrical polar coordinates the deformation (1.1) becomes

$$r = r(R, \Theta), \quad \theta = \theta(R, \Theta), \quad z = Z, \tag{2.1}$$

while equations (1.5),(1.3) and (1.4) with $e = f = k = 0$ become respectively

$$r_R\theta_\Theta - r_\Theta\theta_R = \frac{R}{r}, \tag{2.2}$$

$$\frac{r_\Theta}{R} - r\theta_R = \tan(\psi + \theta - \Theta)\left(r_R + \frac{r}{R}\theta_\Theta\right), \tag{2.3}$$

$$\tan\left(\frac{\psi + \theta - \Theta}{2}\right) = \left(\frac{(ar + cR)\cos\left(\frac{\theta + \Theta}{2}\right) + (br + dR)\sin\left(\frac{\theta + \Theta}{2}\right)}{-(br - dR)\cos\left(\frac{\theta + \Theta}{2}\right) + (ar - cR)\sin\left(\frac{\theta + \Theta}{2}\right)}\right), \tag{2.4}$$

where subscripts denote partial derivatives and equation (2.4) is most easily established from the left-hand side and expanding as $\tan[\psi/2 + (\theta - \Theta)/2]$ and then using (1.4) to determine an expression for $\tan(\psi/2)$.

In order to investigate to what extent the solutions of (2.1) - (2.4) span the entire solution space, we need to examine a class of deformations for which the general solution is known. Accordingly, we examine the similarity wedge deformations (1.6) which for the Varga material have general solutions for $f(\Theta)$ and $g(\Theta)$. In [5] it is shown that the equilibrium equations can be integrated to yield

$$(I + 2)^{-1/2} = -(\delta_1/4\mu)(1 + f^{-2})^{-1} + \delta_2, \tag{2.5}$$

where δ_1 and δ_2 denote arbitrary constants and I is the strain invariant such that

$$I + 2 = f'^2 + \left(f + \frac{1}{f}\right)^2, \tag{2.6}$$

where primes denote differentiation with respect to Θ and $f^2g' = 1$ for incompressibility. ¿From (2.5) and (2.6) it is not difficult to show that

$$f' = \pm\frac{(f^2 + 1)[f^2 - (\delta_1^* f^2 + \delta_2)^2]^{1/2}}{f(\delta_1^* f^2 + \delta_2)}, \tag{2.7}$$

where δ_1^* is a different arbitrary constant defined by $\delta_1^* = \delta_2 - \delta_1/4\mu$. From (2.7) and incompressibility we have

$$\Theta = \int \frac{df}{f'} + \delta_3, \quad \theta = \int \frac{df}{f^2 f'} + \delta_4, \tag{2.8}$$

where δ_3 and δ_4 denote additional arbitrary constants. The integrals appearing in (2.8) can if necessary be evaluated in closed form and examples can be found in [5]. For our purposes it is sufficient to note that the above equations define the general solution for the deformation (1.6) in terms of four arbitrary constants $\delta_1^*, \delta_2, \delta_3$, and δ_4. We observe however, that a much simpler integral is obtained by addition of the integrals (2.8), thus

$$\theta + \Theta = \int \frac{(f^2+1)}{f^2 f'} df + \delta_3 + \delta_4, \tag{2.9}$$

so that from (2.7) and on making the substitution

$$\cos\omega = \delta_1^* f + \frac{\delta_2}{f}, \tag{2.10}$$

we may deduce

$$\theta + \Theta = \mp \int \frac{\cos\omega d\omega}{(1 - 4\delta_1^*\delta_2 - \sin^2\omega)^{1/2}} + \delta_3 + \delta_4, \tag{2.11}$$

so that in the case $4\delta_1^*\delta_2 < 1$ we obtain

$$\theta + \Theta = \mp \tan^{-1} \left[\frac{\left[1 - \left(\delta_1^* f + \frac{\delta_2}{f} \right)^2 \right]^{1/2}}{\left(\delta_1^* f - \frac{\delta_2}{f} \right)} \right] + \delta_3 + \delta_4, \tag{2.12}$$

In the following section we examine how solutions of (2.2) - (2.4), of the form (1.6), relate to the above general solution.

III. SIMILARITY WEDGE DEFORMATIONS ARISING FROM THE GENERAL INTEGRAL

¿From (1.6) and (2.2) - (2.4) we obtain

$$f^2 g' = 1, \qquad f' = \left(f + \frac{1}{f} \right) \tan\psi^*,$$

$$\tan\frac{\psi^*}{2} = \left(\frac{(af+c) + (bf+d)\tan\left(\frac{\theta+\Theta}{2}\right)}{-(bf-d) + (af-c)\tan\left(\frac{\theta+\Theta}{2}\right)} \right), \tag{3.1}$$

where ψ^* denotes $\psi + \theta - \Theta$. Now from $(3.1)_2$ and (2.5) and (2.6) it is a simple matter to show

$$\cos\psi^* = \delta_1^* f + \frac{\delta_2}{f}, \tag{3.2}$$

and therefore ψ^* coincides with the variable ω defined by (2.10), thus $\omega = \psi + \theta - \Theta$. With appropriate assumptions on the constants δ_1^* and δ_2 we have from (2.10) or (3.2) the following two possible expressions for $f(\Theta)$,

$$f = \frac{1}{2\delta_1^*}\left(\cos\omega + (\cos^2\omega - 4\delta_1^*\delta_2)^{1/2}\right), \; f = \frac{1}{2\delta_1^*}\left(\cos\omega - (\cos^2\omega - 4\delta_1^*\delta_2)^{1/2}\right), \tag{3.3}$$

and from equation (2.12) we may deduce

$$\theta + \Theta = \mp\tan^{-1}\left[\frac{\tan\omega}{[1 - 4\delta_1^*\delta_2(1+\tan^2\omega)]^{1/2}}\right] + \delta_3 + \delta_4. \tag{3.4}$$

Now on introducing

$$\beta = \tan(\delta_3 + \delta_4), \quad \xi = \tan\omega, \tag{3.5}$$

we have from (3.4)

$$\tan(\theta + \Theta) = \frac{\beta[1 - 4\delta_1^*\delta_2(1+\xi^2)]^{1/2} \mp \xi}{[1 - 4\delta_1^*\delta_2(1+\xi^2)]^{1/2} \pm \beta\xi}, \tag{3.6}$$

which on introducing $t = \tan[(\theta + \Theta)/2]$, equation (3.6) gives a quadratic equation in t, which may be solved to yield

$$\tan\left(\frac{\theta + \Theta}{2}\right) = \frac{[-(\Delta \pm \beta\xi) \pm \gamma\sec\omega]}{(\beta\Delta \mp \xi)}, \tag{3.7}$$

where γ and $\Delta(\xi)$ are defined by

$$\gamma = (1 - 4\delta_1^*\delta_2)^{1/2}\sec(\delta_3 + \delta_4), \quad \Delta(\xi) = [1 - 4\delta_1^*\delta_2(1+\xi^2)]^{1/2}, \tag{3.8}$$

and where the term $\pm\gamma\sec\omega$ arises from solving the quadratic in t and is separate from the other $\pm$ originating from (2.7). But from equation $(3.1)_3$ we may deduce a second expression for $\tan[(\theta + \Theta)/2]$, namely

$$\tan\left(\frac{\theta + \Theta}{2}\right) = \frac{[(af + c) + (bf - d)\tan(\omega/2)]}{[-(bf + d) + (af - c)\tan(\omega/2)]}, \tag{3.9}$$

noting that ψ^* in equation $(3.1)_3$ coincides with ω, and that for definiteness, we assume that f as a function of ω is defined by $(3.3)_1$. Corresponding details for $(3.3)_2$ can be found in Appendix B.

It is not immediately clear from (3.7) and (3.9) that the two expressions coincide for certain choices of the four arbitrary constants. We deduce the three ratios b/a, c/a, and d/a by expanding both expressions as power series in ω, using MAPLE, and then determine these three quantities by equating coefficients of ω^0, ω, and ω^2. For definiteness we take (3.7) to be given by

$$\tan\left(\frac{\theta+\Theta}{2}\right) = \frac{[-(\Delta+\beta\xi)+\gamma\sec\omega]}{\beta\Delta-\xi}, \tag{3.10}$$

and we find, after extensive use of MAPLE, that the three ratios are given by

$$\begin{aligned}
\frac{b}{a} &= -\frac{[2+(1-4\delta_1^*\delta_2)^{1/2}]\tan(\delta_3+\delta_4)}{(1-4\delta_1^*\delta_2)^{1/2}[\sec(\delta_3+\delta_4)-1]-2},\\
\frac{c}{a} &= \frac{1}{2}\frac{[1+(1-4\delta_1^*\delta_2)^{1/2}]((1-4\delta_1^*\delta_2)^{1/2}[\sec(\delta_3+\delta_4)-1]+2)}{((1-4\delta_1^*\delta_2)^{1/2}[\sec(\delta_3+\delta_4)-1]-2)\delta_1^*},\\
\frac{d}{a} &= \frac{1}{2}\frac{(2+(1-4\delta_1^*\delta_2)^{1/2}[1-(1-4\delta_1^*\delta_2)^{1/2}])\tan(\delta_3+\delta_4)}{((1-4\delta_1^*\delta_2)^{1/2}[\sec(\delta_3+\delta_4)-1]-2)\delta_1^*}.
\end{aligned} \tag{3.11}$$

We note that the corresponding values of these ratios for the three other cases of (3.7) are given in Appendix A. It is still not a trivial matter to establish the equality of (3.7) and (3.9) and only by expanding the two expressions as power series in ω, making extensive use of MAPLE, are we able to confirm the equality of (3.9) and (3.10) with f given by $(3.3)_1$. However, there are other cases examined in Appendices A and B which do not lead to equality of (3.7) and (3.9). We find that in order that (3.7) and (3.9) are identically equal, the expression $(3.3)_1$ must be used in conjunction with the positive case of (2.7), while $(3.3)_2$ must be coupled with the negative sign in (2.7). For all other combinations, (3.7) and (3.9) are not identically equal.

Alternatively as a check we may confirm that the two expressions (3.7) and (3.9) coincide for particular choices of ω. By the method of determining the three ratios, the two expressions trivially coincide for the value $\omega = 0$, namely

$$\frac{[1 + (1 - 4\delta_1^*\delta_2)^{1/2}] + 2\delta_1^*\dfrac{c}{a}}{-\dfrac{b}{a}[1 + (1 - 4\delta_1^*\delta_2)^{1/2}] - 2\delta_1^*\dfrac{d}{a}} = \tan\left(\frac{\delta_3 + \delta_4}{2}\right), \tag{3.12}$$

which is simply the condition that the coefficients of ω^0 coincide, on noting the identity

$$\tan\left(\frac{\delta_3 + \delta_4}{2}\right) = \frac{\sin(\delta_3 + \delta_4)}{1 + \cos(\delta_3 + \delta_4)}. \tag{3.13}$$

For a non-trivial check we consider the value of ω, say ω_0 for which $\Delta \equiv 0$, Thus

$$\cos\omega_0 = 2(\delta_1^*\delta_2)^{1/2}, \tag{3.14}$$

in which case we have from $(3.3)_1$ and the identity

$$\tan\frac{\omega}{2} = \left(\frac{1 - \cos\omega}{1 + \cos\omega}\right)^{1/2}, \tag{3.15}$$

we have $f = (\delta_2/\delta_1^*)^{1/2}$ and from (3.7) and (3.9) we obtain

$$\frac{\left(\dfrac{\delta_2}{\delta_1^*}\right)^{1/2} + \dfrac{c}{a} + \left(\dfrac{b}{a}\left(\dfrac{\delta_2}{\delta_1^*}\right)^{1/2} - \dfrac{d}{a}\right)\left(\dfrac{1 - 2(\delta_1^*\delta_2)^{1/2}}{1 + 2(\delta_1^*\delta_2)^{1/2}}\right)^{1/2}}{-\dfrac{b}{a}\left(\dfrac{\delta_2}{\delta_1^*}\right)^{1/2} - \dfrac{d}{a} + \left(\left(\dfrac{\delta_2}{\delta_1^*}\right)^{1/2} - \dfrac{c}{a}\right)\left(\dfrac{1 - 2(\delta_1^*\delta_2)^{1/2}}{1 + 2(\delta_1^*\delta_2)^{1/2}}\right)^{1/2}} = \tan\left(\frac{\delta_3 + \delta_4}{2} - \frac{\pi}{4}\right). \tag{3.16}$$

On using

$$\tan\left(\frac{\delta_3 + \delta_4}{2} - \frac{\pi}{4}\right) = \frac{\sin(\delta_3 + \delta_4) - 1}{\cos(\delta_3 + \delta_4)}, \tag{3.17}$$

we may confirm, again after extensive use of MAPLE, that the given ratios for $b/a, c/a$, and d/a satisfy (3.16). Such results become more transparent by an examination of special cases and this is done in the final two sections of the paper.

IV. INTEGRABLE SPECIAL CASES

In this section we obtain explicit expressions for the following four special cases:

(i) $a = b = c = 0$ and $d \neq 0$,

(ii) $b = c = d = 0$ and $a \neq 0$,

(iii) $a = b = 0$ and $c = d \neq 0$,

(iv) $c = d = 0$ and $a = b \neq 0$,

and we show how the solutions obtained relate to some of the results given in the previous section. These cases happen to give rise to readily integrable equations. Because of the observation made in Section 1 regarding the elimination of certain of the constants a, b, c, and d by constant rotations, and the inverse deformation results due to Adkins [6], we remark that the results presented here are not necessarily independent, nor do they represent an exhaustive list of integrable cases.

(i) $\mathbf{a = b = c = 0}$ and $\mathbf{d \neq 0}$

In this case we have immediately from $(3.1)_3$ that $\omega = \theta + \Theta$ and therefore from $(3.1)_2$ we obtain

$$\frac{f'}{f} = \left(1 + \frac{1}{f^2}\right)\tan(\Theta + g) = (1 + g')\tan(\Theta + g), \tag{4.1}$$

which can be integrated immediately to give

$$f = \frac{\sec(\Theta + g)}{(C_1^2 - 1)^{1/2}}, \tag{4.2}$$

where C_1 denotes an arbitrary constant. From this equation and (3.2) we see that $\delta_1^* = 0$ and $\delta_2 = (C_1^2 - 1)^{-1/2}$ and since δ_1^* is zero a number of the equations of the preceding section do not hold since they were derived on the basis that δ_1^* is non-zero. Now from $(3.1)_1$ and (4.2) we have

$$g' = (C_1^2 - 1)\cos^2(\Theta + g), \tag{4.3}$$

which on making the substitution $h = g + \Theta$, integrates to give

$$g(\Theta) = -\Theta + \tan^{-1}(C_1 \tan(C_1\Theta + C_2)), \tag{4.4}$$

where C_2 denotes an additional arbitrary constant. An explicit expression for f arises from $(3.1)_1$ and by differentiating (4.4), thus

$$f(\Theta) = \left[\frac{1 + C_1 \tan^2(C_1\Theta + C_2)}{C_1^2 - 1}\right]^{1/2}, \tag{4.5}$$

which is valid provided $C_1 \neq \pm 1$. We note that precisely the same deformation arises for the case $a = b = d = 0$ and $c \neq 0$.

(ii) $\mathbf{b = c = d = 0}$ and $\mathbf{a \neq 0}$

In this case we have from $(3.1)_3$

$$\tan\frac{\omega}{2} = \cot\left(\frac{\theta + \Theta}{2}\right), \tag{4.6}$$

and therefore $\omega = \pi - (\theta + \Theta)$ and $(3.1)_2$ becomes

$$\frac{f'}{f} = -\left(1 + \frac{1}{f^2}\right)\tan(\Theta + g), \tag{4.7}$$

which integrates to give

$$f = \frac{(1 - C_1^2)^{1/2}}{C_1}\cos(\Theta + g), \tag{4.8}$$

and from this equation and (3.2) we see that $\delta_1^* = -C_1/(1 - C_1^2)^{1/2}$ and δ_2 is zero. From $(3.1)_1$ and (4.8) we obtain, after some re-arrangement and an integration,

$$g(\Theta) = C_1 \tan^{-1}(C_1 \tan(\Theta + g(\Theta)) + C_2. \tag{4.9}$$

This implicit equation for $g(\Theta)$, together with (4.8) defines $f(\Theta)$. We remark that the same deformation is obtained for the case $a = c = d = 0$ and $b \neq 0$.

(iii) $\mathbf{a = b = 0}$ and $\mathbf{c = d \neq 0}$

In this case equation $(3.1)_3$ yields

$$\tan\frac{\omega}{2} = \frac{1 + \tan\left(\dfrac{\theta + \Theta}{2}\right)}{1 - \tan\left(\dfrac{\theta + \Theta}{2}\right)} = \tan\left(\frac{\pi}{4} + \frac{\theta + \Theta}{2}\right), \tag{4.10}$$

and therefore $\omega = \pi/2 + \theta + \Theta$ and on using

$$\tan\omega = \frac{2\tan(\omega/2)}{1-\tan^2(\omega/2)} = \frac{2(1+t)(1-t)}{(1-t)^2-(1+t)^2} = -\cot(\theta+\Theta), \tag{4.11}$$

where t denotes $\tan[(\theta+\Theta)/2]$, equation $(3.1)_2$ gives

$$\frac{f'}{f} = -\left(1+\frac{1}{f^2}\right)\cot(\Theta+g). \tag{4.12}$$

This equation integrates to give

$$f = \frac{\operatorname{cosec}(\Theta+g)}{(C_1^2-1)^{1/2}}, \tag{4.13}$$

which on comparison with (3.2) shows that δ_1^* is zero and $\delta_2 = -(C_1^2-1)^{-1/2}$. On using $(3.1)_1$ we may deduce

$$g(\Theta) = -\Theta + \tan^{-1}\left(\frac{\tan(C_1\Theta+C_2)}{C_1}\right), \tag{4.14}$$

and on differentiating this equation and again using $(3.1)_1$ we may obtain

$$f(\Theta) = \left[\frac{1+C_1^2\cot^2(C_1\Theta+C_2)}{C_1^2-1}\right]^{1/2}, \tag{4.15}$$

and again this is well-defined provided $C_1 \neq \pm 1$.

(iv) $\mathbf{c} = \mathbf{d} = \mathbf{0}$ and $\mathbf{a} = \mathbf{b} \neq \mathbf{0}$

In this case we have from equation $(3.1)_3$

$$\tan\frac{\omega}{2} = -\left(\frac{1+\tan\left(\dfrac{\theta+\Theta}{2}\right)}{1-\tan\left(\dfrac{\theta+\Theta}{2}\right)}\right) = -\tan\left(\frac{\pi}{4}+\frac{\theta+\Theta}{2}\right), \tag{4.16}$$

and therefore $\omega = -(\pi/2+\theta+\Theta)$. As in the previous case we can show $(3.1)_2$ becomes

$$\frac{f'}{f} = \left(1+\frac{1}{f^2}\right)\cos(\Theta+g), \tag{4.17}$$

which integrates to give

$$f = \frac{(1-C_1^2)^{1/2}}{C_1}\sin(\Theta+g), \tag{4.18}$$

and from (3.2) we see that in this case we have $\delta_1^* = -C_1/(1-C_1^2)^{1/2}$ and δ_2 is zero. From (4.18) and $(3.1)_1$ we may deduce

$$g(\Theta) = \frac{1}{C_1}\tan^{-1}(C_1\tan(\Theta + g(\Theta))) + C_2, \tag{4.19}$$

and this implicit equation for $g(\Theta)$ together with (4.18) defines $f(\Theta)$.

In the following section we present some partial results for the case $b = c = 0$ and a and d both non-zero.

V. CASE OF b = c = 0 AND a AND d BOTH NON-ZERO

As noted in Section 1, we may eliminate two of the constants $a, b, c,$ and d by appropriately rotating the material and spatial rectangular Cartesian coordinates. If we assume that both b and c are zero and that both a and d are non-zero then the system of equations (3.1) becomes

$$\begin{gathered} f^2 g' = 1, \qquad f' = \left(f + \frac{1}{f}\right)\tan\omega, \\ \tan\frac{\omega}{2} = \frac{af + d\tan\left(\dfrac{\Theta+g}{2}\right)}{d + af\tan\left(\dfrac{\Theta+g}{2}\right)}, \end{gathered} \tag{5.1}$$

and a straightforward application of the half-tangent formula gives

$$f' = \left(f + \frac{1}{f}\right)\left\{\frac{2adf}{(d^2 - a^2f^2)}\sec(\Theta+g) + \frac{(d^2 + a^2f^2)}{(d^2 - a^2f^2)}\tan(\Theta+g)\right\}, \tag{5.2}$$

which appears not to admit an obvious first integral.

In the case $b = c = 0$, equation (3.12) and(3.16) constitute two equations for the determination of the ratio $\lambda = a/d$. ¿From (3.12) and (3.16) we obtain

$$\frac{\lambda}{2\delta_1^*}[1+(1-4\delta_1^*\delta_2)^{1/2}] = \tan\left(\frac{\delta_3+\delta_4}{2}\right),$$

$$\frac{\lambda\left(\frac{\delta_2}{\delta_1^*}\right)^{1/2} - \left(\frac{1-2(\delta_1^*\delta_2)^{1/2}}{1+2(\delta_1^*\delta_2)^{1/2}}\right)^{1/2}}{-1+\lambda\left(\frac{\delta_2}{\delta_1^*}\right)^{1/2}\left(\frac{1-2(\delta_1^*\delta_2)^{1/2}}{1+2(\delta_1^*\delta_2)^{1/2}}\right)^{1/2}} = \tan\left(\frac{\delta_3+\delta_4}{2}-\frac{\pi}{4}\right), \tag{5.3}$$

and the form of these equations suggests that we assume that the constants δ_1^* and δ_2 are such that

$$\delta_1^* = \cos^2\left(\frac{\rho}{2}\right), \quad \delta_2 = \sin^2\left(\frac{\rho}{2}\right), \tag{5.4}$$

then equations (5.3) simplify somewhat to give

$$\lambda = \tan[(\delta_3+\delta_4)/2],$$

$$\frac{\lambda\tau(1+\tau)-(1-\tau)}{\lambda\tau(1-\tau)-(1+\tau)} = \tan\left(\frac{\delta_3+\delta_4}{2}-\frac{\pi}{4}\right), \tag{5.5}$$

where τ denotes $\tan(\rho/2)$. Thus for the case of f increasing we have

$$\frac{\lambda\tau(1+\tau)-(1-\tau)}{\lambda\tau(1-\tau)-(1+\tau)} = \frac{\lambda-1}{\lambda+1}, \tag{5.6}$$

and this equation simplifies to give

$$(\lambda\tau)^2 + 2\lambda\tau = 1. \tag{5.7}$$

Thus, with the assumption (5.4) the two equations are only consistent for the two values $\lambda\tau = \pm\sqrt{2}-1$, namely

$$\frac{a}{d}\left(\frac{\delta_2}{\delta_1^*}\right)^{1/2} = \pm\sqrt{2}-1, \tag{5.8}$$

which means that in this instance the four arbitrary constants $\delta_1^*, \delta_2, \delta_3$, and δ_4 must satisfy the condition

$$\left(\frac{\delta_2}{\delta_1^*}\right)^{1/2}\tan\left(\frac{\delta_3+\delta_4}{2}\right) = \pm\sqrt{2}-1. \tag{5.9}$$

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Appendix A: The three ratios $\mathbf{b/a, c/a}$, and $\mathbf{d/a}$ for other cases of equation (3.7)

In this Appendix we list the three ratios $b/a, c/a$, and d/a for the three other cases arising from (3.7) and (3.9), assuming that the function of f is defined by $(3.3)_1$. For the case when equation (3.7) takes the form

$$\tan\left(\frac{\theta+\Theta}{2}\right)=\frac{[-(\Delta+\beta\xi)\pm\gamma\sec\omega]}{\beta\Delta-\xi}, \tag{A1}$$

then the three ratios of $b/a, c/a$, and d/a are given by

$$\frac{b}{a}=\mp\frac{[2+(1-4\delta_1^*\delta_2)^{1/2}]\tan(\delta_3+\delta_4)}{(1-4\delta_1^*\delta_2)^{1/2}[\sec(\delta_3+\delta_4)\mp1]\mp2},$$

$$\frac{c}{a}=\frac{1}{2}\frac{[1+(1-4\delta_1^*\delta_2)^{1/2}]((1-4\delta_1^*\delta_2)^{1/2}[\sec(\delta_3+\delta_4)\mp1]\pm2)}{((1-4\delta_1^*\delta_2)^{1/2}[\sec(\delta_3+\delta_4)\mp1]\mp2)\delta_1^*}, \tag{A2}$$

$$\frac{d}{a}=\pm\frac{1}{2}\frac{(2+(1-4\delta_1^*\delta_2)^{1/2}[1-(1-4\delta_1^*\delta_2)^{1/2}])\tan(\delta_3+\delta_4)}{((1-4\delta_1^*\delta_2)^{1/2}[\sec(\underset{\cdot}{e}lta_3+\delta_4)\mp1]\mp2)\delta_1^*}.$$

For the case when equation (3.7) takes the form

$$\tan\left(\frac{\theta+\Theta}{2}\right)=\frac{[-(\Delta-\beta\xi)\pm\gamma\sec\omega]}{\beta\Delta+\xi}, \tag{A3}$$

then the three ratios of $b/a, c/a$, and d/a are given by

$$\frac{b}{a}=\frac{\sin(\delta_3+\delta_4)}{\cos(\delta_3+\delta_4)\mp1},$$

$$\frac{c}{a}=\frac{1}{2}\frac{(1+(1-4\delta_1^*\delta_2)^{1/2})^2}{((1-4\delta_1^*\delta_2)^{1/2}-1)\delta_1^*}, \tag{A4}$$

$$\frac{d}{a}=\frac{1}{2}\left[\frac{\sin(\delta_3+\delta_4)}{\cos(\delta_3+\delta_4)\mp1}\right]\left[\frac{(1+(1-4\delta_1^*\delta_2)^{1/2})^2}{((1-4\delta_1^*\delta_2)^{1/2}-1)\delta_1^*}\right].$$

For the two latter cases we observe that the ratios c/a and d/b coincide and have the same value for both cases but it is only in the former case, when (3.7) takes the form (A1), that the two expressions (3.7) and (3.9) coincide.

Appendix B: The three ratios $\mathbf{b/a, c/a}$, and $\mathbf{d/a}$ for f defined by equation $(\mathbf{3.3})_2$

In this Appendix we list the three ratios $b/a, c/a$, and d/a for the case when then function f is defined by equation $(3.3)_2$. When (3.7) takes the form (A1) we obtain by equating coefficients of ω^0, ω and ω^2 in (3.7) and (3.9), the following expressions

$$\frac{b}{a} = \frac{\sin(\delta_3 + \delta_4)}{\cos(\delta_3 + \delta_4) \mp 1},$$

$$\frac{c}{a} = -\frac{1}{2}\frac{(1 - (1 - 4\delta_1^*\delta_2)^{1/2})^2}{[1 + (1 - 4\delta_1^*\delta_2)^{1/2}]\delta_1^*}, \tag{B1}$$

$$\frac{d}{a} = -\frac{1}{2}\left[\frac{\sin(\delta_3 + \delta_4)}{\cos(\delta_3 + \delta_4)\ mp1}\right]\left[\frac{(1 - (1 - 4\delta_1^*\delta_2)^{1/2})^2}{(1 + (1 - 4\delta_1^*\delta_2)^{1/2})\delta_1^*}\right].$$

When (3.7) takes the form (A2) we find in the same way that

$$\frac{b}{a} = \mp\frac{[(1 - 4\delta_1^*\delta_2)^{1/2} - 2]\tan(\delta_3 + \delta_4)}{(1 - 4\delta_1^*\delta_2)^{1/2}(\sec(\delta_3 + \delta_4) \mp 1] \pm 2},$$

$$\frac{c}{a} = \frac{1}{2}\frac{[1 + (1 - 4\delta_1^*\delta_2)^{1/2}]((1 - 4\delta_1^*\delta_2)^{1/2}[\sec(\delta_3 + \delta_4) \mp 1] \mp 2)}{((1 - 4\delta_1^*\delta_2)^{1/2}[\sec(\delta_3 + \delta_4) \mp 1] \pm 2)\delta_1^*}, \tag{B2}$$

$$\frac{d}{a} = \pm\frac{1}{2}\frac{((1 - 4\delta_1^*\delta_2)^{1/2}[1 + (1 - 4\delta_1^*\delta_2)^{1/2}] - 2)\tan(\delta_3 + \delta_4)}{((1 - 4\delta_1^*\delta_2)^{1/2}[\sec(\delta_3 + \delta_4) \mp 1] \pm 2)\delta_1^*},$$

but only in the latter case do the expressions (3.7) and (3.9) coincide.

REFERENCES

[1] Hill, J.M. and Arrigo, D.J. : Transformations and equation reductions in finite elasticity I: Plane strain deformations. *Mathematics and Mechanics of Solids*, 1, 155-175 (1996).

[2] Arrigo, D.J. and Hill, J.M. : Transformations and equation reductions in finite elasticity II: Plane stress and axially symmetric deformations. *Mathematics and Mechanics of Solids*, 1, 177-192 (1996).

[3] Hill, J.M. and Arrigo, D.J. : Transformations and equation reductions in finite elasticity III: A general integral for plane strain deformations. *Mathematics and Mechanics of Solids*, submitted for publication.

[4] Hill, J.M. and Arrigo, D.J. : New families of exact solutions for finitely deformed incompressible elastic materials. *IMA Journal of Applied Mathematics*, 54, 109 - 123 (1995).

[5] Hill, J.M. : Some Integrable and exact similarity solutions for plane finite deformations. *IMA Journal of Applied Mathematics*, 14, 111 -123 (1990).

[6] Adkins, J.E. : A reciprocal property of the finite plane strain equations. *Journal of Mechanics and Physics of Solids*, 6, 267-275 (1958).