Stress profiles for tapered cylindrical cavities in granular media

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Abstract

The formation of almost vertical cylindrical tunnels known as piping or rat-holes in stockpiles and hoppers cause serious disruptions to the reclaiming of material. The authors have recently shown that the classical rat-hole theory proposed by Jenike and his coworkers involving the so-called "stable rat-hole equation" is not as accurate as it might be. Specifically it is shown that the function appearing in the stable rat-hole equation which is conventionally denoted by $G(\phi)$ and referred to as the rat-holing function, is not a good approximation of the exact numerical solution. Jenike's original theory assumes a symmetrical stress distribution which is independent of height. In practice however, rat-holes tend to exhibit some tapering with height and the purpose of this paper is to determine the stress profiles corresponding to a symmetrical but slightly tapered circular cavity. Stress distributions are found which are a perturbation of those arising from classical theory, and separable solutions involving exponential functions in the height are used to "mimic" a slightly tapered cavity. Four numerical examples are presented and departures from standard theory are shown graphically. For slightly tapered rat-holes occurring in stock-piles, the work presented here constitutes the first rigorous mathematical analysis of this important problem.

Key words: tapered cylindrical cavities; rat-holes; granular materials; Coulomb-Mohr yield condition; plastic regimes; principal stresses

1. Introduction

Stockpiles and hoppers are widely used throughout many mineral and mining industries to store and recover material. From a practical perspective, we would like to efficiently remove material from the stockpile or hopper at a uniform and uninterrupted rate of flow. Therefore, the occurrence of almost vertical tunnels inside stockpiles or hoppers, which prevents the flow of material, is an unwanted phenomenon, and we would like to understand the conditions under which such phenomenon occurs. These tunnels are commonly known as "rat-holes" and the process of their formation is referred to as "piping". Once a rat-hole has formed in a stockpile, the material around the surface of the hole often dries out and sets as a solid material. This makes the removal of rat-holes more difficult, and often they have to be destroyed manually and the stockpile completely reshaped. Practising engineers believe that the classical rat-hole theory enunciated by Jenike (1962) and Jenike and Yen (1962) does not accurately reflect actual material behaviour. In Hill and Cox (1999), the present authors showed that the stable rat-hole equation proposed by Jenike and his coworkers is in fact not a good approximation of the exact numerical solution. The purpose of this present paper is to determine the stress distributions for stockpile rat-holes which are slightly tapered, and we exploit the classical stress distributions as the basis for a perturbation scheme. We comment that for rat-holes occurring in bins, an approximate analysis, based on the method of "slices", which does incorporate some height variation is provided by Johanson (1969). We emphasize that for slightly tapered stockpile rat-holes the work presented here constitutes the first rigorous full mathematical analysis of the problem.

Typically, a stockpile rat-hole appears as indicated in Fig. 1, where θ denotes the angle of repose, and α and γ denote small angles. For the idealised situation of a symmetrical cylindrical rat-hole and with the axis as shown in Fig. 1, the limiting equilibrium equations become

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\phi\phi}}{r} = 0, \quad \frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} = \rho g, \tag{1}$$

where ρ is the bulk density of the material, g is the acceleration due to gravity, σ_{rr}, σ_{rz}, etc. denote the stresses in a cylindrical polar coordinate system (r, ϕ, z), and which are assumed to be independent of ϕ. In addition, the material is assumed to satisfy the Coulomb-Mohr yield condition

$$|\tau| = c - \sigma^* \tan \delta, \tag{2}$$

where c is the cohesion, δ is the angle of internal friction, and σ^* and τ denote the normal and tangential components of compressive traction, which here we assume positive in tension. Namely, we adopt the usual convention in continuum mechanics that positive forces are assumed to produce positive extensions.

In this paper we assume that slightly tapered cylindrical cavity profiles such as those depicted in Fig. 1 can be represented by an expression of the form

$$r = r_0 + \varepsilon R(z), \tag{3}$$

where r_0 is assumed to be independent of the height z, and ε is a small non-dimensional parameter. The distinction from the classical theory is shown schematically in Fig. 2. We emphasize that we have in mind slightly tapered cylindrical cavities for which the correction terms of order ε are much smaller than the corresponding terms for a perfectly circular cylindrical vertical cavity. Further, $R(z)$ is a function of z which we assume can be approximated by an expression of the form

$$R(z) = \sum_{n=1}^{N} R_{\mathrm{n}} e^{-\alpha_{\mathrm{n}} z}, \tag{4}$$

for certain constants α_{n} and R_{n} $(n = 1, 2, ..., N)$. For example, we show that the cavity profile shown in Fig. 1(a) can be adequately approximated by the two term expression

$$R(z) = R_1 \left(1 + e^{-\alpha_2 z}\right), \tag{5}$$

assuming $\tan \alpha = \varepsilon$. On the other hand we may show that the cavity profile shown in Fig. 1(b) can be approximated by the three term expression

$$R(z) = R_1 + R_2 e^{-\alpha_2 z} + R_3 e^{-\alpha_3 z}, \tag{6}$$

assuming $\tan\alpha = \varepsilon$ and $\tan\gamma = K\varepsilon$ for some $K > 1$.

Corresponding to a slightly tapered cylindrical cavity of the form (3) we assume that the non-zero stresses are a small perturbation of those for the classical theory, namely

$$\begin{aligned} &\sigma_{rr}(r,z) = \sigma_{rr0}(r) + \varepsilon\sigma_{rr1}(r,z), \quad \sigma_{rz}(r,z) = \sigma_{rz0}(r) + \varepsilon\sigma_{rz1}(r,z), \\ &\sigma_{zz}(r,z) = \sigma_{zz0}(r) + \varepsilon\sigma_{zz1}(r,z), \quad \sigma_{\phi\phi}(r,z) = \sigma_{\phi\phi0}(r) + \varepsilon\sigma_{\phi\phi1}(r,z), \end{aligned} \tag{7}$$

where ε is the small parameter defined by equation (3), and the quantities $\sigma_{rr1}, \sigma_{rz1}$, etc. are unknown functions of r and z. We assume that the stresses (7) obey a pressure boundary condition at the cavity wall, of the form

$$\sigma_j = -Pn_j, \tag{8}$$

where σ_j $(j = 1, 2, 3)$ denotes the stress vector, P is the assumed external pressure and n_j $(j = 1, 2, 3)$ denotes the components of the normal vector to the cavity surface. From Fig. 3, the normal vector to the surface of the sidewall of the unstable rat-hole can be seen to be given by

$$\mathbf{n} = \left(-\cos\theta(z),\ 0,\ \sin\theta(z)\right), \tag{9}$$

where $\theta(z)$ is the angle the normal vector $\mathbf{n}$ makes with the r axis. Therefore, from (8) and (9) on assuming that the external pressure P is zero, we find

$$\sigma_r = 0, \quad \sigma_z = 0, \tag{10}$$

and upon expanding (10), using the fact that $\sigma_j = \sigma_{ij}n^i$, we obtain

$$\begin{aligned} &-\sigma_{rr}(r_0 + \varepsilon R(z), z)\cos\theta(z) + \sigma_{rz}(r_0 + \varepsilon R(z), z)\sin\theta(z) = 0, \\ &-\sigma_{rz}(r_0 + \varepsilon R(z), z)\cos\theta(z) + \sigma_{zz}(r_0 + \varepsilon R(z), z)\sin\theta(z) = 0. \end{aligned} \tag{11}$$

Now at the cavity wall, we see from Fig. 3

$$\theta(z) \ = \tan^{-1}\left(\mathrm{d}r/\mathrm{d}z\right), \tag{12}$$

and upon expanding (11) and (12) we may deduce the following conditions,

$$\sigma_{rr0}(r_0) = 0, \quad \sigma_{rz0}(r_0) = 0, \tag{13}$$

$$\sigma_{rr1}(r_0, z) = -R(z)\left(\frac{\mathrm{d}\sigma_{rr0}}{\mathrm{d}r}\right)_{r=r_0}, \quad \sigma_{rz1}(r_0, z) = -R(z)\left(\frac{\mathrm{d}\sigma_{rz0}}{\mathrm{d}r}\right)_{r=r_0} + R'(z)\sigma_{zz0}(r_0). \tag{14}$$

and we note that the zeroth order conditions are simply those used in the classical theory.

In the following section we present the governing equations for the slightly tapered rat-hole assuming separable solutions for the stresses. In the subsequent section we derive a second order ordinary differential equation from which we can determine the stresses in the slightly tapered rat-hole. In section 4 we consider various two and three term approximations for $R(z)$ of the form (4), and apply these to the single and double slightly tapered rat-holes as shown in Figs. 1(a) and 1(b) respectively.

2. The Governing Ordinary Differential Equations

In this section we determine the governing ordinary differential equations for a slightly tapered rat-hole. To do this, we assume that the stockpile is at equilibrium, and that the rat-hole is on the point of collapse, so that the equilibrium equations (1) apply and the stresses are given by (7). We then assume that the unknown functions $\sigma_{rr1}, \sigma_{rz1}, \sigma_{zz1}$, and $\sigma_{\phi\phi1}$ can be expressed as a sum of separable variable functions, where the z dependence is uniform for each of the stresses. Thus, we assume

$$\sigma_{rr1}(r, z) = \sum_{i=1}^{N} A_i(r)E_i(z), \quad \sigma_{rz1}(r, z) = \sum_{i=1}^{N} B_i(r)E_i(z),$$
$$\sigma_{zz1}(r, z) = \sum_{i=1}^{N} C_i(r)E_i(z), \quad \sigma_{\phi\phi1}(r, z) = \sum_{i=1}^{N} D_i(r)E_i(z). \tag{15}$$

We then find that upon substituting (7) and (15) into $(1)_1$ that for each i $(i = 1, 2, ..., N)$ we require

$$\frac{\mathrm{d}A_i}{\mathrm{d}r}E_i(z) + B_i(r)\frac{\mathrm{d}E_i}{\mathrm{d}z} + \frac{A_i(r)E_i(z) - D_i(r)E_i(z)}{r} = 0, \tag{16}$$

and therefore each $E_i(z)$ must satisfy an equation of the form

$$\frac{\mathrm{d}E_i}{\mathrm{d}z} = -\alpha_i E_i(z), \tag{17}$$

for certain constants α_i. Therefore, from (17) we find $E_i(z) = e^{-\alpha_i z}$, on incorporating the constant of integration into the functions of r. From (16) we find that the equation becomes

$$\frac{\mathrm{d}A_i}{\mathrm{d}r} - \alpha_i B_i(r) + \frac{A_i(r) - D_i(r)}{r} = 0, \tag{18}$$

and similarly for $(1)_2$ we obtain

$$\frac{\mathrm{d}B_i}{\mathrm{d}r} - \alpha_i C_i(r) + \frac{B_i(r)}{r} = 0. \tag{19}$$

Now, on assuming that the granular material satisfies the Coulomb-Mohr yield condition defined by (2) we find from Hill and Cox (1999) or Hill and Wu (1992), that this yield condition becomes

$$\sigma_I = (f_\mathrm{c} + \sigma_{III}) \left(\frac{1-\beta}{1+\beta} \right),$$

where $\beta = \sin\delta$, f_c is the unconfined yield strength defined by $\sigma_I = 0$ when $\sigma_{III} = -f_\mathrm{c}$, and f_c can be written as

$$f_\mathrm{c} = 2c \left(\frac{1+\beta}{1-\beta} \right)^{1/2},$$

and σ_I, σ_{II}, and σ_{III} denote the maximum, intermediate, and minimum principal stresses respectively. Further, we have also assumed that of the seven possible plastic regimes available for axially symmetric stress states, the material is in plastic regime A, which means that the stresses satisfy the inequality $\sigma_I > \sigma_{II} = \sigma_{\phi\phi}$. The seven plastic regimes are well known and can be found in tabular form in either Hill and Wu (1992) or Cox, Eason and Hopkins (1961). It is clear that a relation between the principal stresses and the stresses in the rat-hole is needed. Hill and Cox (1999) show that the maximum and minimum principal stresses for the classical rat-hole are given by

$$\sigma_{I_0} = \frac{1}{2}\left\{(\sigma_{rr0}+\sigma_{zz0}) + \left[(\sigma_{rr0}-\sigma_{zz0})^2 + 4\sigma_{rz0}^2\right]^{1/2}\right\},$$

$$\sigma_{II_0} = \sigma_{\phi\phi0}, \tag{20}$$

$$\sigma_{III_0} = \frac{1}{2}\left\{(\sigma_{rr0}+\sigma_{zz0}) - \left[(\sigma_{rr0}-\sigma_{zz0})^2 + 4\sigma_{rz0}^2\right]^{1/2}\right\}.$$

In order to determine the principal stresses for the slightly tapered rat-hole we note that the principal stresses are defined by the eigenvalue equation

$$\begin{vmatrix} \sigma_{rr}-\mu & \sigma_{rz} & 0 \\ \sigma_{rz} & \sigma_{zz}-\mu & 0 \\ 0 & 0 & \sigma_{\phi\phi}-\mu \end{vmatrix} = 0,$$

where μ denotes a principal stress for the slightly tapered rat-hole. We then assume that we can write $\mu = \mu_0 + \varepsilon\mu_1$ where μ_0 denotes a principal stress for the uniform rat-hole, and μ_1 is an unknown function of r and z. Therefore, upon substituting (7) and (15) into the eigenvalue equation and noting that μ_0 satisfies the equation

$$(\sigma_{\phi\phi0}-\mu_0)\left[(\sigma_{rr0}-\mu_0)(\sigma_{zz0}-\mu_0) - \sigma_{rz0}^2\right] = 0,$$

we can solve for μ_1, obtaining the expression

$$\begin{aligned}\mu_1 \;=\; & \left[(\sigma_{zz0}-\mu_0)(\sigma_{\phi\phi0}-\mu_0)\sum_{i=1}^{N} A_i(r)e^{-\alpha_i z} + (\sigma_{rr0}-\mu_0)(\sigma_{\phi\phi0}-\mu_0)\sum_{i=1}^{N} C_i(r)e^{-\alpha_i z}\right. \\ & +(\sigma_{rr0}-\mu_0)(\sigma_{zz0}-\mu_0)\sum_{i=1}^{N} D_i(r)e^{-\alpha_i z} \\ & \left. -2\sigma_{rz0}(\sigma_{\phi\phi0}-\mu_0)\sum_{i=1}^{N} B_i(r)e^{-\alpha_i z} - \sigma_{rz0}^2\sum_{i=1}^{N} D_i(r)e^{-\alpha_i z}\right] \Big/ \\ & \left[(\sigma_{rr0}-\mu_0)(\sigma_{zz0}-\mu_0) + (\sigma_{rr0}-\mu_0)(\sigma_{\phi\phi0}-\mu_0) + (\sigma_{zz0}-\mu_0)(\sigma_{\phi\phi0}-\mu_0) - \sigma_{rz0}^2\right].\end{aligned} \tag{21}$$

Thus, if we define

$$\Delta_0 = \sqrt{(\sigma_{rr0}-\sigma_{zz0})^2 + 4\sigma_{rz0}^2}, \quad \Sigma_0 = \sigma_{rr0} - \sigma_{zz0}, \tag{22}$$

then it can then be shown that if we substitute $(20)_1$ into (21) for μ_0, that μ_1 becomes

$$\mu_1 = \sum_{i=1}^{N} \frac{e^{-\alpha_i z}}{2\Delta_0} \left[(\Sigma_0 + \Delta_0) A_i(r) + 4\sigma_{rz0} B_i(r) - (\Sigma_0 - \Delta_0) C_i(r) \right],$$

and similarly, if we substitute $(20)_3$ into (21) then we get

$$\mu_1 = \sum_{i=1}^{N} \frac{e^{-\alpha_i z}}{2\Delta_0} \left[(-\Sigma_0 + \Delta_0) A_i(r) - 4\sigma_{rz0} B_i(r) + (\Sigma_0 + \Delta_0) C_i(r) \right],$$

and finally, if we let $\mu_0 = \sigma_{\phi\phi 0}$ then we find that $\mu_1 = \sum_{i=1}^{N} D_i(r) e^{-\alpha_i z}$.

Therefore, the maximum and minimum principal stresses for the slightly tapered rat-hole are

$$\begin{aligned} \sigma_I &= \sigma_{I_0} + \varepsilon \sum_{i=1}^{N} \frac{e^{-\alpha_i z}}{2\Delta_0} \left[(\Sigma_0 + \Delta_0) A_i(r) + 4\sigma_{rz0} B_i(r) - (\Sigma_0 - \Delta_0) C_i(r) \right], \\ \sigma_{II} &= \sigma_{II_0} + \varepsilon \sum_{i=1}^{N} D_i(r) e^{-\alpha_i z}, \\ \sigma_{III} &= \sigma_{III_0} + \varepsilon \sum_{i=1}^{N} \frac{e^{-\alpha_i z}}{2\Delta_0} \left[(-\Sigma_0 + \Delta_0) A_i(r) - 4\sigma_{rz0} B_i(r) + (\Sigma_0 + \Delta_0) C_i(r) \right]. \end{aligned} \tag{23}$$

Therefore, upon substituting (23) into the Coulomb-Mohr yield equation we find that the stresses for the slightly tapered rat-hole are related by the equation

$$(\Sigma_0 + \beta\Delta_0) A_i(r) + 4\sigma_{rz0} B_i(r) - (\Sigma_0 - \beta\Delta_0) C_i(r) = 0,$$

for $i = 1, ..., N$. Now from (22), we find

$$A_i(r) \left[1 + \beta \left(1 + \frac{4\sigma_{rz0}^2}{(\sigma_{rr0} - \sigma_{zz0})^2} \right)^{1/2} \right] + \frac{4\sigma_{rz0} B_i(r)}{(\sigma_{rr0} - \sigma_{zz0})}$$

$$-C_i(r) \left[1 - \beta \left(1 + \frac{4\sigma_{rz0}^2}{(\sigma_{rr0} - \sigma_{zz0})^2} \right)^{1/2} \right] = 0.$$

Following Hill and Cox (1999) we introduce

$$\tan 2\psi_0 = \frac{2\sigma_{rz0}}{\sigma_{rr0} - \sigma_{zz0}}, \tag{24}$$

so that the Coulomb-Mohr yield condition becomes

$$A_i(r)(\cos 2\psi_0 + \beta) + 2B_i(r)\sin 2\psi_0 - C_i(r)(\cos 2\psi_0 - \beta) = 0, \tag{25}$$

for $i = 1, ..., N$ and where ψ_0 is the known function of r defined by equation (24).

To determine the fourth equation, we recall that we are in plastic regime A, which has the stress relation $\sigma_I > \sigma_{\phi\phi 1} = \sigma_{III}$. Therefore, from (7) and $(23)_2$ we find for each $i = 1, 2, ..., N$ that

$$A_i(r)(-\Sigma_0 + \Delta_0) - 4\sigma_{rz0}B_i(r) + C_i(r)(\Sigma_0 + \Delta_0) = 2\Delta_0 D_i(r),$$

and hence, from (25) we obtain the relation

$$D_i(r) = \frac{1}{2}(1+\beta)[A_i(r) + C_i(r)]. \tag{26}$$

Therefore, the four equations (18), (19), (25), and (26) constitute the four determining equations for the unknown functions $A_i(r), B_i(r), C_i(r), D_i(r)$ for each $i = 1, 2, ..., N$.

3. The Differential Equation For $\mathbf{B_i(r)}$

In this section we consider the four governing equations developed in the previous section, namely (18), (19), (25), and (26), and determine a second order ordinary differential equation for $B_i(r)$ by eliminating the other unknowns.

Upon substituting (26) into (18), we find that we have eliminated $D_i(r)$ to get the equation

$$\frac{\mathrm{d}A_i}{\mathrm{d}r} - \alpha_i B_i(r) + \frac{(1-\beta)A_i(r) - (1+\beta)C_i(r)}{2r} = 0, \tag{27}$$

and we note from (19) that

$$C_i(r) = \frac{1}{\alpha_i}\left[\frac{\mathrm{d}B_i}{\mathrm{d}r} + \frac{B_i(r)}{r}\right], \tag{28}$$

and therefore, (27) becomes

$$\frac{\mathrm{d}A_i}{\mathrm{d}r} + \frac{(1-\beta)}{2r}A_i(r) = \frac{(1+\beta)}{2\alpha_i r}\left[\frac{\mathrm{d}B_i}{\mathrm{d}r} + \frac{B_i(r)}{r}\right] + \alpha_i B_i(r). \tag{29}$$

If we also substitute (28) into (25) then we find

$$A_i(r) = \frac{s_1(r)}{\alpha_i}\left[\frac{\mathrm{d}B_i}{\mathrm{d}r} + \frac{B_i(r)}{r}\right] - s_2(r)B_i(r), \tag{30}$$

where

$$s_1(r) = \frac{\cos 2\psi_0 - \beta}{\cos 2\psi_0 + \beta}, \quad s_2(r) = \frac{2\sin 2\psi_0}{\cos 2\psi_0 + \beta}. \tag{31}$$

Hence, upon substituting (30) into (29) we get the second order ordinary differential equation for $B_i(r)$

$$\begin{aligned} 0 &= \frac{\mathrm{d}^2 B_i}{\mathrm{d}r^2} + \left[\frac{s_1'}{s_1} + \frac{1}{r} - \alpha_i\frac{s_2}{s_1} + \frac{(1-\beta)}{2r} - \frac{(1+\beta)}{2rs_1}\right]\frac{\mathrm{d}B_i}{\mathrm{d}r} \\ &+ \left[\frac{s_1'}{rs_1} - \frac{1}{r^2} - \alpha_i\frac{s_2'}{s_1} + \frac{(1-\beta)}{2r^2} - \alpha_i s_2\frac{(1-\beta)}{2rs_1} - \frac{(1+\beta)}{2r^2 s_1} - \frac{\alpha_i^2}{s_1}\right]B_i(r), \end{aligned} \tag{32}$$

for $i = 1, ..., N$, and from (14), (15), and (30) we can see that the boundary conditions on B_i are

$$\begin{aligned} \sum_{n=1}^{N} e^{-\alpha_n z} B_n(r_0) &= -R(z)\left(\frac{\mathrm{d}\sigma_{rz0}}{\mathrm{d}r}\right)_{r=r_0} + R'(z)\sigma_{zz0}(r_0), \\ \left(\frac{\mathrm{d}B_i}{\mathrm{d}r}\right)_{r=r_0} &= \alpha_i\left(\frac{1+\beta}{1-\beta}\right)A_i(r_0) - \frac{1}{r_0}B_i(r_0), \end{aligned} \tag{33}$$

where $A_i(r_0)$ is determined from

$$\sum_{n=1}^{N} e^{-\alpha_n z} A_n(r_0) = -R(z)\left(\frac{\mathrm{d}\sigma_{rr0}}{\mathrm{d}r}\right)_{r=r_0}. \tag{34}$$

In order to simplify matters we make the transformations

$$r = \eta/\alpha_i, \quad B_i = \alpha_i \mathcal{B}_i, \quad A_i = \alpha_i \mathcal{A}_i, \tag{35}$$

which transforms (32) into

$$\begin{aligned} 0 &= \frac{\mathrm{d}^2 \mathcal{B}_i}{\mathrm{d}\eta^2} + \left[\frac{s_1'}{s_1} + \frac{1}{\eta} - \frac{s_2}{s_1} + \frac{(1-\beta)}{2\eta} - \frac{(1+\beta)}{2\eta s_1}\right]\frac{\mathrm{d}\mathcal{B}_i}{\mathrm{d}\eta} \\ &+ \left[\frac{s_1'}{\eta s_1} - \frac{1}{\eta^2} - \frac{s_2'}{s_1} + \frac{(1-\beta)}{2\eta^2} - s_2\frac{(1-\beta)}{2\eta s_1} - \frac{(1+\beta)}{2\eta^2 s_1} - \frac{1}{s_1}\right]\mathcal{B}_i(\eta), \end{aligned} \tag{36}$$

where s_1 and s_2 are now function of η, and upon noting (4) and expanding we find that the boundary conditions for $\mathcal{B}_i$ at $\eta = \alpha_i r_0$ become

$$\begin{aligned} \mathcal{B}_i(\alpha_i r_0) &= -R_i \left(\frac{\mathrm{d}\sigma_{rz0}}{\mathrm{d}\eta}\right)_{\eta=\alpha_i r_0} - R_i \sigma_{zz0}(\alpha_i r_0), \\ \left(\frac{\mathrm{d}\mathcal{B}_i}{\mathrm{d}\eta}\right)_{\eta=\alpha_i r_0} &= R_i \left(\frac{1+\beta}{1-\beta}\right) \left(\frac{\mathrm{d}\sigma_{rr0}}{\mathrm{d}\eta}\right)_{\eta=\alpha_i r_0} - \frac{1}{\alpha_i r_0}\mathcal{B}_i(\alpha_i r_0), \end{aligned} \tag{37}$$

and we note that we consider each $\mathcal{B}_i$ at the different initial values, namely $\eta = \alpha_i r_0$. Thus, we now have a differential equation for $\mathcal{B}_i$ with two explicit boundary conditions at $\eta = \alpha_i r_0$. Further, due to the complexity of the coefficient functions of $\mathcal{B}_i, \mathrm{d}\mathcal{B}_i/\mathrm{d}\eta$, and $\mathrm{d}^2\mathcal{B}_i/\mathrm{d}\eta^2$, we will solve (36) subject to (37) numerically. This is essential since in fact, we can only determine s_1 and s_2 numerically in general.

4. Special Cases For $\mathbf{R}(\mathbf{z})$

In this section we consider two possible shapes for the rat-hole and approximate the required shape using a sum of exponentials (4). Once we have determined the unknown constants in (4), these are then used to solve the system of differential equations defined by (36) with the boundary conditions (37). We note that there is no unique procedure for this approximation and indeed the procedure adopted here for three terms for the double slightly tapered rat-hole give rise to two possible solutions.

4.1 *Single slightly tapered rat-hole.*

For a single slightly tapered rat-hole as shown in Fig. 1(a), the equation describing the sidewall of the rat-hole is

$$r = r_0 + z \ \tan\alpha. \tag{38}$$

Assuming a finite height H_1 and that the sidewall of the rat-hole can be approximated by a sum of exponentials as defined by (4), then for a two term sum we find that we wish to approximate (38) by an expression of the form

$$r = r_0 + \varepsilon \left(R_1 e^{-\alpha_1 z} + R_2 e^{-\alpha_2 z} \right), \tag{39}$$

for certain unknown constants α_1, α_2, R_1, and R_2. For simplicity we assume $\alpha_1 = 0$, and we determine the unknown constants by firstly assuming that (38) and (39) coincide at the bottom of the rat-hole, namely at $z = 0$, from which we find that $R_2 = -R_1$. Secondly, we assume that (38) and (39) coincide at the top of the rat-hole, namely at $z = H_1$, from which we find

$$H_1 = R_1 \left(1 - e^{-\alpha_2 H_1} \right), \tag{40}$$

where we have assumed $\tan \alpha = \varepsilon$.

Thirdly, we assume that for each z that the maximum horizontal difference between (38) and (39) is ε, then we find

$$\begin{gathered} R_1 \left(1 - e^{-\alpha_2 z}\right) - z - 1 \leq 0, \\ \text{or} \\ z - 1 - R_1 \left(1 - e^{-\alpha_2 z}\right) \leq 0, \end{gathered} \tag{41}$$

where $(41)_1$ is used when the right hand side of (39) is larger than the right hand side of (38), or in other words, the approximate solution is on the "outside" of the rat-hole, and similarly, $(41)_2$ is used when the right hand side of (39) is smaller then the right hand side of (38), or in other words, the approximate solution is on the "inside" of the rat-hole.

Here we have assumed that the approximate solution is on the outside of the rat-hole, and therefore from $(41)_1$ we find that the maximum value occurs when

$$z = \frac{1}{\alpha_2} \ln \alpha_2 R_1, \tag{42}$$

which combined with $(41)_1$ gives the relation

$$R_1 \left(1 - \frac{1}{\alpha_2 R_1} \right) - \frac{1}{\alpha_2} \ln \alpha_2 R_1 = 1. \tag{43}$$

Therefore, from (40) and (43) we have two equations for the two unknowns R_1 and α_2, and hence $R(z)$ can be determined.

We note that the transformations (35) are only well defined for $\alpha_i \neq 0$. For $\alpha_1 = 0$, we solve the governing equations (32) subject to (33), from which we find the boundary conditions

$$B_1(r_0) = -R_1 \left(\frac{\mathrm{d}\sigma_{rz0}}{\mathrm{d}r}\right)_{r=r_0}, \qquad \left(\frac{\mathrm{d}B_1}{\mathrm{d}r}\right)_{r=r_0} = -\frac{1}{r_0} B_1(r_0). \tag{44}$$

4.2 *Double slightly tapered rat-hole.*

For a double slightly tapered rat-hole as shown in Fig. 1(b), where γ is a small angle such that $\gamma > \alpha$, we denote H_1 to be the height where the the sidewall of the rat-hole changes slope from $\tan\alpha$ to $\tan\gamma$, and H_2 to be the height of the double tapered rat-hole. We find that the equations of the sidewall of the rat-hole are given by

$$\begin{aligned} r &= r_0 + \varepsilon z, && \text{for } 0 \leq z \leq H_1, \\ r &= r_0 + \varepsilon\left((1-K)H_1 + Kz\right), && \text{for } H_1 \leq z \leq H_2, \end{aligned} \tag{45}$$

where we have assumed that $\tan\alpha = \varepsilon$ and $\tan\gamma = K\varepsilon$ for some $K > 1$.

Now, assuming that (45) can be approximated by a sum of exponentials as defined by (4), then for $\alpha_1 = 0$ and assuming that (39) and (45) coincide at the bottom of the rat-hole and at the change of slope of the sidewall at $z = H_1$, we find that $R_2 = -R_1$ and that (40) holds. Further, we assume that (39) and (45) coincide at the top of the double slightly tapered rat-hole, namely at $z = H_2$, which gives us the relation

$$(1-K)H_1 + KH_2 = R_1\left(1 - e^{\alpha_2 H_2}\right), \tag{46}$$

and we see from (39) that (40) and (46) determines the unknown constants α_2 and R_1. We also note that the appropriate boundary conditions for the double slightly tapered rat-hole with the approximation (39) for $\alpha_1 = 0$, where α_2 and R_1 are determined from (40) and (46), are given by (37) and (44).

We now assume that the sidewall of the double slightly tapered rat-hole as shown in Fig. 1(b) can be approximated by a three term sum of exponentials as defined by (4). Hence, we approximate the sidewall of the rat-hole described by (45) by

$$r = r_0 + \varepsilon \left(R_1 e^{-\alpha_1 z} + R_2 e^{-\alpha_2 z} + R_3 e^{-\alpha_3 z} \right), \tag{47}$$

for some unknown constants α_i and R_i, for $i = 1, 2, 3$. Following the previous two term approximations, we again assume for simplicity that $\alpha_1 = 0$ and that (45) and (47) coincide at the bottom of the rat-hole, at the change of slope of the sidewall at $z = H_1$, and at the top of the rat-hole, which yield respectively

$$\begin{aligned} R_1 + R_2 + R_3 &= 0, \\ R_1 + R_2 e^{-\alpha_2 H_1} + R_3 e^{-\alpha_3 H_1} &= H_1, \\ R_1 + R_2 e^{-\alpha_2 H_2} + R_3 e^{-\alpha_3 H_2} &= (1-K)H_1 + KH_2. \end{aligned} \tag{48}$$

We have three equations for five unknowns, and therefore we require two additional constraints between the unknowns. Following section 4.1 we assume for $0 \leq z \leq H_1$, that the maximum horizontal difference between (45) and (47) is ε, so that

$$\begin{gathered} R_1 + R_2 e^{-\alpha_2 z} + R_3 e^{-\alpha_3 z} - z - 1 \leq 0, \\ \text{or} \\ z - 1 - R_1 - R_2 e^{-\alpha_2 z} - R_3 e^{-\alpha_3 z} \leq 0, \end{gathered} \tag{49}$$

where $(49)_1$ is used when the solution is on the outside of the rat-hole and $(49)_2$ is used when the solution is on the inside of the rat-hole. If we denote z_1 as the value of z that gives the maximum value of (49), then for both equations, z_1 is determined from

$$\alpha_2 R_2 e^{-\alpha_2 z_1} + \alpha_3 R_3 e^{-\alpha_3 z_1} + 1 = 0, \tag{50}$$

which is a transcendental equation for z_1. Once z_1 is determined we can then substitute z_1 into (49) to obtain a new single condition on the unknowns, which depends on whether the solution is on the outside or on the inside of the rat-hole. However, as the values of α_2, α_3, R_2, and R_3 are unknown, then we must treat z_1 as an unknown and include (50) as an additional condition.

Similarly, we assume for $H_1 \leq z \leq H_2$ we require that the maximum horizontal difference between (45) and (47) is ε, which yields

$$\begin{gathered} R_1 + R_2 e^{-\alpha_2 z} + R_3 e^{-\alpha_3 z} - (1-K)H_1 - Kz - 1 \leq 0, \\ \text{or} \\ Kz + (1-K)H_1 - 1 - R_1 - R_2 e^{-\alpha_2 z} - R_3 e^{-\alpha_3 z} \leq 0, \end{gathered} \tag{51}$$

where $(51)_1$ is used when the solution is on the outside of the rat-hole and $(51)_2$ is used when the solution is on the inside of the rat-hole. If we denote z_2 as the value of z that gives the maximum value of (51), then for both equations, z_2 is determined from

$$\alpha_2 R_2 e^{-\alpha_2 z_2} + \alpha_3 R_3 e^{-\alpha_3 z_2} + K = 0, \tag{52}$$

which is also a transcendental equation for z_2. Once z_2 is determined we can then substitute z_2 into (51) to deduce a new single condition on the unknowns, which depends on whether the solution is on the outside or on the inside of the rat-hole. However, as the values of α_2, α_3, R_2, and R_3 are unknown then we must treat z_2 as an unknown and include (52) as an additional condition.

We now have seven equations for seven unknowns which may be solved numerically. The appropriate boundary conditions for the double slightly tapered rat-hole with the approximation (47) for $\alpha_1 = 0$ are given by (37) and (44).

5. Conclusions

For slightly tapered cylindrical vertical cavities we have provided the first rigorous mathematical analysis of the limiting equilibrium equations (1) for the plastic regime A to determine an axially symmetric stress distribution which is a perturbation of the classical Jenike solution for a perfectly right circular cylindrical cavity. The perturbations are assumed to be separable functions of r and z, and it is shown that the only allowable dependence on z must be exponential. For a slightly tapered rat-hole with profile $r = r_0 + \varepsilon R(z)$ we have solved numerically a second order ordinary differential equation using boundary conditions arising from the fact that the cavity is stress free. We have numerically determined four solutions for four different shapes of the sidewall of the slightly tapered rat-hole using a possible, but not a unique set of constraints to determine $R(z)$ and we have evaluated the stress approximations on the plane $z = 0$. For all numerical solutions we assume the constant values of $\rho = 0.7, g = 9.8, \beta = 0.5$ and $f_c = 5.2$.

Fig. 4 shows the single slightly tapered rat-hole as the shaded area with the mesh boundary showing the approximate solution on the outside of the rat-hole for $R(z)$ defined by (39) with $\alpha_2 = 0.53$, $R_1 = 4.55$, and $\varepsilon = 1/12$. Fig. 5 shows the approximate stresses relative to the classical stresses applying to a right circular cylindrical rat-hole. In particular, σ_{rr} is initially higher but then is below the classical estimate, while σ_{rz} is always below the classical estimate. σ_{zz} starts at the classical estimate and then goes below while $\sigma_{\phi\phi}$ is initially higher and then goes below $\sigma_{\phi\phi 0}$.

Fig. 6 shows the double slightly tapered rat-hole with the boundary of the shaded area showing the approximate solution on the inside of the rat-hole for $R(z)$ defined by (39) with $\alpha_2 = -0.59$, $R_1 = -0.41$, $\varepsilon = 1/12$, and $\gamma = 25$ degrees. From Fig. 7 we see that the approximate stresses follow the classical stresses. In particular, σ_{rr} is very close to the classical estimate, while σ_{rz} starts higher, but then asymptotes to the classical estimate. σ_{zz} and $\sigma_{\phi\phi}$ are in excess of the classical estimates.

Fig. 8 shows a double slightly tapered rat-hole with the approximated cavity profile being the shaded area in the lower region and the mesh boundary in the upper region. Here $R(z)$ is defined by (47) with $\alpha_2 = -1.15$, $\alpha_3 = 0.70$, $R_1 = 3.802$, $R_2 = 0.004$, $R_3 = -3.806$, $\varepsilon = 1/12$, and $\gamma = 9$ degrees. In this case the approximate solution starts on the outside of the rat-hole, similar to Fig.4, and then goes on the inside of the rat-hole at the change of slope, similar to Fig. 6. Therefore, for the approximate stresses on the plane $z = 0$, it is not surprising to see that they behave in a similar fashion to those for Fig. 4.

Fig. 10 shows the double slightly tapered rat-hole with the approximate cavity profile being the mesh boundary in the lower region and the shaded area in the upper region. For the approximate solution shown in Figs. 8 and 9, there are two profiles namely that given in Fig. 8 and that given in Fig. 10 which is the shaded area in the lower region and the mesh boundary in the upper region. For this shape,

$R(z)$ is defined by (47) with $\alpha_2 = -0.55$, $\alpha_3 = -1.95$, $R_1 = -0.5144$, $R_2 = 0.5145$, $R_3 = 0.00004$, $\varepsilon = 1/12$, and $\gamma = 9$ degrees. Here the approximate solution is on the inside in the lower region, similar to Fig. 6, and on the outside in the upper region, similar to Fig. 4. Therefore, the approximate stresses on the plane $z = 0$ behave in a similar fashion to those shown in Fig. 6.

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Figure Captions

FIG. 1.(a). Schematic drawing showing a single tapered cylindrical cavity in a stockpile.

FIG. 1.(b). Schematic drawing showing a double tapered cylindrical cavity in a stockpile.

FIG. 2.(a). Right circular uniform cylindrical cavity.

FIG. 2.(b). Cylindrical cavity with height variation.

FIG. 3. Angle $\theta(z)$ for a cylindrical cavity with height variation.

FIG. 4. Single tapered rat-hole with mesh showing approximation on outside of rat-hole which is shown by the shaded area and with $R(z)$ defined by (39).

FIG. 5. Classical and approximate stresses corresponding to Fig. 4.

FIG. 6. Double tapered rat-hole with shaded region showing approximation on inside of rat-hole which is shown by the mesh boundary and with $R(z)$ defined by (39).

FIG. 7. Classical and approximate stresses corresponding to Fig. 6.

FIG. 8. Double tapered rat-hole with approximation as the mesh on the outside in the lower region and as the shaded area on the inside in the upper region and with $R(z)$ defined by (47).

FIG. 9. Classical and approximate stresses corresponding to Fig. 8.

FIG. 10. Double tapered rat-hole with approximation as the shaded area on the inside in the lower region and as the mesh on the outside in the upper region and with $R(z)$ defined by (47).

FIG. 11. Classical and approximate stresses corresponding to Fig. 10.