Rat-hole stress profiles for shear-index granular materials

James M. Hill[1] and Grant M. Cox

School of Mathematics and Applied Statistics, University of Wollongong, 2522, N.S.W., Australia

Summary The formation of stable circular and almost vertical cylindrical holes in stockpiles and hoppers is a familiar phenomena in granular material industries and is often the cause of serious disruption to the industry. These holes are referred to as "rat-holes" and the precise conditions under which they may form or the stability of an existing rat-hole are two issues which have yet to be properly resolved in the literature. We do not address these issues here, but for an existing rat-hole, and assuming a shear-index granular material, we determine the limiting stress profiles. Existing theory applies only to the Coulomb-Mohr yield function. A shear-index granular material is one for which failure due to frictional slip between particles, occurs when the shear and normal components of stress τ and σ satisfy the so called Warren Spring equation $(|\tau|/c)^n = 1 - (\sigma/t)$, where c, t and n are positive constants which are referred to as the cohesion, tensile strength and shear-index respectively, and the known numerical values of the shear-index indicate that for certain materials n lies between the values 1 and 2. The value $n = 1$ corresponds to the standard Coulomb-Mohr yield function while the value $n = 2$ permits some further analytical investigation, and stress profiles for these two values constitute bounds for those shear-index materials for which $1 < n < 2$.

1 Introduction

The formation of stable circular and almost vertical cylindrical holes in stockpiles and hoppers, as indicated in Figures 1(a) and 1(b) respectively, are the cause of sig-

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[1] e-mail: jhill@uow.edu.au

nificant disruption in many industries such as mining, minerals, grains and chemical industries. This frequently occurring phenomena is referred to as "piping" and the holes themselves are known as "rat-holes". Often once a rat-hole forms it tends to remain there because the material dries and sets as a solid. The precise conditions under which a rat-hole may form or the conditions prescribing the stability of an existing rat-hole are as yet unresolved issues. We do not address these issues here, but rather we determine the limiting equilibrium stress profiles assuming an existing vertical cylindrical rat-hole in a shear-index granular material. This work extends existing theory for the Coulomb-Mohr yield function.

Shear-index yield functions and certain plane and axially symmetric problems are analysed in detail in Hill and Wu [2, 3]. Failure of powders or granular materials is due to frictional slip between particles, and at yield the magnitude of the shear component of stress $|\tau|$ varies according to the value of the normal component of stress σ, which here we take positive in tension. The locus of the values of (σ, τ) at which permanent deformation or yield occurs is called the yield locus. It can also be defined as the envelope of the Mohr stress circles at yield, because in plasticity the stress state must satisfy the stress equilibrium equations and cannot exceed the yield locus. For certain granular materials, the angle of internal friction is a constant and the yield locus is a straight line and is referred to as the Coulomb-Mohr yield function, thus

$$|\tau| = c - \sigma \tan \delta, \tag{1}$$

where c is a constant, which is called the cohesion of the material. This yield function has been interpreted by Shield [1] in terms of principal stress components to obtain the yield surface for a three dimensional stress field. Shield [1] showed that in principle stress space the yield surface is a right hexagonal pyramid equally inclined to the $\sigma_1, \sigma_2, \sigma_3$ axes, and with its vertex at the point $\sigma_1 = \sigma_2 = \sigma_3 = c \cot \delta$.

In general however, experimental evidence (see Hill and Wu [2] for detailed references) indicates that for most granular materials the angle of internal friction is not constant along the yield locus but decreases for decreasing σ from a maximum value $\pi/2$ at the vertex A and typically the corresponding yield locus is as indicated in Figure 2. In general therefore, the angle of internal friction is a stress dependent function $\delta(\sigma)$ which is defined incrementally from the equation

$$\mathrm{d}\tau = -\mathrm{d}\sigma \tan\delta. \tag{2}$$

Here, we consider the yield function sometimes referred to as the Warren Spring equation, namely

$$\left(\frac{|\tau|}{c}\right)^n = 1 - \frac{\sigma}{t}, \tag{3}$$

where c, t and n are positive constants which are referred to as the cohesion, tensile strength and shear index respectively. A number of authors (see Hill and Wu [2]) have performed experiments which confirm the validity of (3). Using the Jenike shear tests Farley and Valentin [4] suggest that the cohesion c is usually of the order of twice the tensile strength t and that the shear index n for a particular powder is independent of the bulk density of the compact, and can therefore be used to classify powders according to their flow properties. In addition, Farley and Valentin [4] give simple expressions relating n to the ratio of volume to surface mean diameter and t to the bulk density. The known numerical values of shear index n such as those cited in Hill and Wu [2] all lie between 1 and 2 and Table 1 gives the typical values of n, t, c and ρ as determined by Farley and Valentin [4].

For a general yield function Hill and Wu [2] show that the yield function in terms of principal stress components is given parametrically as follows,

$$\begin{aligned}(\sigma_I - \sigma_{III})\cos\delta &= 2f\left[(\sigma_I + \sigma_{III})/2 + (\sigma_I - \sigma_{III})(\sin\delta)/2\right], \\ \tan\delta &= -\frac{\mathrm{d}}{\mathrm{d}\sigma}f\left[(\sigma_I + \sigma_{III})/2 + (\sigma_I - \sigma_{III})(\sin\delta)/2\right],\end{aligned} \tag{4}$$

where the stress dependent angle of internal friction $\delta = \delta(\sigma)$ is the parameter. Thus, for example, if the angle of internal friction is constant and $\tau = f(\sigma)$ is the linear yield condition (1) then $(4)_1$ gives the well known Coulomb-Mohr yield condition

$$\sigma_I - \sigma_{III} = 2c\cos\delta - (\sigma_I + \sigma_{III})\sin\delta. \tag{5}$$

Granular material (Particle size)	Shear-index n	Tensile strength t (g/cm^2)	Cohesion c (g/cm^2)	Density ρ (gm/cm^3)
Alumina				
(+37 μ)	1.19	0.226	0.385	1.018
(20-30 μ)	1.52	8.09	12.2	1.113
(9 - 15 μ)	1.40	10.55	15.2	1.176
Zinc dust				
(Standard)	1.39	2.54	5.33	3.706
(Ultrafine)	1.86	4.89	11.7	3.238
Precipitated CaCO$_3$				
($<$ 12 μ)	1.53	2.95	8.09	0.888
(12 - 14 μ)	1.46	0.76	1.83	1.112

Table 1: Typical values of n, t, c and ρ (Farley and Valentin [4], g denotes 981 dynes).

In the case of the Warren Spring equation (3) we have

$$f(\sigma) = c\left(1 - \frac{\sigma}{t}\right)^{1/n}, \tag{6}$$

and (4) gives the yield condition in parametric form

$$\frac{\sigma_I}{t} = 1 + \frac{\beta c}{t}(\sec\delta - \tan\delta) - \beta^n, \quad \frac{\sigma_{III}}{t} = 1 - \frac{\beta c}{t}(\sec\delta + \tan\delta) - \beta^n, \tag{7}$$

where β is a function of δ defined by

$$\beta = \left(\frac{nt}{c}\tan\delta\right)^{1/(1-n)}, \tag{8}$$

and we note that the special cases $n = 1$ and $n = 2$ become respectively

$$\begin{aligned} \left(1 - \frac{\sigma_I}{t}\right)^{1/2} &= \left\{\left(1 + \left(\frac{c}{t}\right)^2\right)^{1/2} - \frac{c}{t}\right\}\left(1 - \frac{\sigma_{III}}{t}\right)^{1/2} \quad (n = 1), \\ \left(1 - \frac{\sigma_I}{t}\right)^{1/2} &= \left(1 - \frac{\sigma_{III}}{t}\right)^{1/2} - \frac{c}{t} \quad (n = 2). \end{aligned} \tag{9}$$

These relations become more transparent by expressing the parametric solution (7) in the form

$$\begin{aligned} 1-\frac{\sigma_I}{t} &= \frac{1}{\beta^n}\left\{\left(\left[\beta^{2n}+\left(\frac{\beta\lambda}{n}\right)^2\right]^{1/2}-\frac{\beta\lambda}{2}\right)^2-(\beta\lambda)^2\left(\frac{1}{n}-\frac{1}{2}\right)^2\right\}, \\ 1-\frac{\sigma_{III}}{t} &= \frac{1}{\beta^n}\left\{\left(\left[\beta^{2n}+\left(\frac{\beta\lambda}{n}\right)^2\right]^{1/2}+\frac{\beta\lambda}{2}\right)^2-(\beta\lambda)^2\left(\frac{1}{n}-\frac{1}{2}\right)^2\right\}, \end{aligned} \tag{10}$$

where λ denotes c/t and from which it is clear that $n = 1$ and $n = 2$ play special roles and we may readily deduce (9). It appears that $n = 1$ and $n = 2$ are the only values of n giving rise to simple analytical yield functions such as (9). However, other special values of n such as $n = 4/3, 3/2$, and $8/5$ permit further analytical investigation but the final results are still complicated (see the Appendix of Hill and Wu [2]).

In the following section we provide the mathematical formulation to determine the limiting equilibrium stresses for a perfectly vertical and circular cylindrical rat-hole for a granular material subject to the yield condition given parametrically by (7). The corresponding problem for the Coulomb-Mohr yield function (1) is examined in Hill and Cox [5] while Hill and Cox [6] determine stress profiles for slightly tapered vertical circular cavities again assuming (1). In the subsequent section we present further analytical details for the cases $n = 1$ and $n = 2$, and in the final section we show how the numerically determined results for general n such that $1 < n < 2$ are bounded by those for $n = 1$ and $n = 2$ respectively.

2 Mathematical formulation

For the idealized situation of a vertical circular cylindrical rat-hole with the axis as shown in Figure 1, the mathematical problem is to solve the equilibrium equations

$$\frac{d\sigma_{rz}}{dr}+\frac{\sigma_{rz}}{r}=\rho g, \quad \frac{d\sigma_{rr}}{dr}+\frac{(\sigma_{rr}-\sigma_{\phi\phi})}{r}=0, \tag{11}$$

subject to the boundary conditions that the surface of the hole of radius r_0 is stress free

$$\sigma_{rz} = \sigma_{rr} = 0 \quad \text{for} \quad r = r_0, \tag{12}$$

where ρ is the bulk density of the material, g is the acceleration due to gravity, σ_{rr}, σ_{rz}, etc. denote the stresses in a cylindrical polar coordinate system (r, ϕ, z), and which following Jenike [7] are assumed to be independent of ϕ and z. In addition, the material is assumed to satisfy the Warren Spring yield condition (3) where σ and τ denote the normal and tangential components of compressive traction, which we assume to be positive in tension. Namely, we adopt the usual convention in continuum mechanics that positive forces are assumed to produce positive extensions. From $(11)_1$ and $(12)_1$ it is a simple matter to deduce

$$\sigma_{rz} = \frac{\rho g}{2}\left(r - \frac{r_0^2}{r}\right), \tag{13}$$

while the maximum and minimum principal stresses are given respectively by

$$\begin{aligned} \sigma_I &= \frac{1}{2}\left\{(\sigma_{rr} + \sigma_{zz}) + \left[(\sigma_{rr} - \sigma_{zz})^2 + 4\sigma_{rz}^2\right]^{1/2}\right\}, \\ \sigma_{III} &= \frac{1}{2}\left\{(\sigma_{rr} + \sigma_{zz}) - \left[(\sigma_{rr} - \sigma_{zz})^2 + 4\sigma_{rz}^2\right]^{1/2}\right\}. \end{aligned} \tag{14}$$

From the boundary conditions (12), (14), and noting that for compression $\sigma_{zz} < 0$ and therefore $\sqrt{\sigma_{zz}^2}$ is $-\sigma_{zz}$, we observe that at $r = r_0$ we have

$$\sigma_I(r_0) = 0, \quad \sigma_{III}(r_0) = \sigma_{zz}(r_0), \tag{15}$$

and therefore $\sigma_{zz}(r_0) = -f_c$, where f_c is usually referred to as the unconfined yield strength which is defined by $\sigma_{III} = -f_c$ when $\sigma_I = 0$. Moreover, throughout the paper we assume the plastic regime conventionally referred to as A (namely one of the Haar-von Karmen regimes), and therefore

$$\sigma_{\phi\phi} = \sigma_{III}. \tag{16}$$

The above equations along with the shear-index yield function defined parametrically by (7) provide the complete mathematical prescription of the problem. In the following section we extend the mathematical analysis for the special cases $n = 1$ and $n = 2$ for which the yield condition in terms of σ_I and σ_{III} can be given explicitly.

3 Mathematical analysis for $n = 1$ and $n = 2$

In this section we present some limited mathematical analysis for the two special cases $n = 1$ and $n = 2$. For $n = 1$ this approach differs from that given by the authors (Hill and Cox [5]). Now we require to solve $(11)_2$ subject to σ_{rr} zero at $r = r_0$ and $\sigma_{\phi\phi}$ determined from $(14)_2$ and (16), thus

$$\sigma_{\phi\phi} = \frac{1}{2}\left\{(\sigma_{rr} + \sigma_{zz}) - \left[(\sigma_{rr} - \sigma_{zz})^2 + 4\sigma_{rz}^2\right]^{1/2}\right\}. \tag{17}$$

For $n = 1$ the standard Coulomb-Mohr yield condition becomes (see Hill and Cox [5])

$$\beta(\sigma_{rr} + \sigma_{zz}) - 2c(1-\beta^2)^{1/2} = -\left[(\sigma_{rr} - \sigma_{zz})^2 + 4\sigma_{rz}^2\right]^{1/2}, \tag{18}$$

where $\beta = \sin\delta$ and $\tan\delta = c/t = \lambda$. Now on squaring this equation and solving the resulting quadratic in σ_{zz}, we find

$$\sigma_{zz} = \sigma_{rr} - 2c\left\{\frac{c}{t}\left(1 - \frac{\sigma_{rr}}{t}\right) + \left(1 + \frac{c^2}{t^2}\right)^{1/2}\left[\left(1 - \frac{\sigma_{rr}}{t}\right)^2 - \left(\frac{\sigma_{rz}}{c}\right)^2\right]^{1/2}\right\}, \tag{19}$$

where we have taken only the negative root to ensure $\sigma_{zz} < 0$. Clearly for a real solution we require

$$1 - \frac{\sigma_{rr}}{t} > \frac{\sigma_{rz}}{c}, \tag{20}$$

noting that $\sigma_{rz}/c > 0$.

Moreover for the case $n = 2$, on writing $(9)_2$ as

$$\left(1 - \frac{\sigma_{III}}{t}\right)^{1/2} - \left(1 - \frac{\sigma_I}{t}\right)^{1/2} = \frac{c}{t},$$

and squaring this equation we may deduce from solving a quadratic equation

$$\sigma_{zz} = \sigma_{rr} - \frac{c^2}{t} - 2c\left[1 - \frac{\sigma_{rr}}{t} - \left(\frac{\sigma_{rz}}{c}\right)^2\right]^{1/2}, \tag{21}$$

where again we have taken only the negative root to ensure $\sigma_{zz} < 0$. Evidently, for a real solution we require that

$$1 - \frac{\sigma_{rr}}{t} > \left(\frac{\sigma_{rz}}{c}\right)^2. \tag{22}$$

Now on introducing $u(r)$ defined by

$$u(r) = \lambda + 2\left[1 - \frac{\sigma_{rr}}{t} - \left(\frac{\sigma_{rz}}{c}\right)^2\right]^{1/2}, \tag{23}$$

noting again that $\lambda = c/t$, it is not difficult to show that $(11)_2$, (17) and (21) yield the first order ordinary differential equation

$$(u-\lambda)\frac{\mathrm{d}u}{\mathrm{d}r} + \varepsilon\frac{\mathrm{d}\varepsilon}{\mathrm{d}r} = \frac{\lambda}{r}\left[u + \left(u^2+\varepsilon^2\right)^{1/2}\right], \tag{24}$$

which for $n = 2$ must be solved numerically subject to the boundary condition

$$u(r_0) = 2 + \lambda, \tag{25}$$

where $\varepsilon(r)$ denotes $\rho g(r - r_0^2/r)/c$. In the following section we present the general numerical scheme for n such that $1 < n < 2$ and we confirm that in the special case $n = 2$ the same result arise from the numerical integration of (24) and (25).

Although we make no essential use of the following analysis, it may be worthwhile noting that with the transformations

$$s = \log r, \quad s_0 = \log r_0, \tag{26}$$

the function $\varepsilon(r)$ becomes

$$\varepsilon(s) = \gamma \sinh(s - s_0), \tag{27}$$

where the constant γ is given by $\gamma = 2\rho g e^{s_0}/c$. Further, the differential equation (24) transforms to

$$(u-\lambda)\frac{\mathrm{d}u}{\mathrm{d}\varepsilon} + \varepsilon = \lambda\left\{\frac{u + (u^2+\varepsilon^2)^{1/2}}{(\varepsilon^2+\gamma^2)^{1/2}}\right\}, \tag{28}$$

which still must be solved numerically subject to $u = 2 + \lambda$ when $\varepsilon = 0$. However, it is at least worth noting that the r dependence in (24) may be transformed away in this manner.

In the numerical section below, it is clear that for all n there exists a point r at which the stress profiles no longer exist and for $n = 1$ and $n = 2$ this corresponds to where the square roots in (19) and (21) become imaginary. Thus, directly from (19) and (21), at this point r we have

$$1 - \frac{\sigma_{rr}}{t} = \frac{\sigma_{rz}}{c} \quad (n = 1), \quad 1 - \frac{\sigma_{rr}}{t} = \left(\frac{\sigma_{rz}}{c}\right)^2 \quad (n = 2), \tag{29}$$

and also from (19) and (21) we may deduce

$$\sigma_{zz} = \sigma_{rr} - 2\lambda\sigma_{rz} \quad (n = 1), \quad \sigma_{zz} = \sigma_{rr} - \lambda c \quad (n = 2). \tag{30}$$

As described in the Appendix we may utilize these expressions to determine the stress values at the point r at which the stress profiles fail to exist. The advantage of the limited mathematical analysis of this section is that the above formulae (29) and (30) are immediately apparent, but the question arises as to the appropriate extension for general n such that $1 < n < 2$. Since in the general case further mathematical analysis appears difficult, the question is nontrivial. However, in the Appendix we establish

$$1 - \frac{\sigma_{rr}}{t} = \left(\frac{\sigma_{rz}}{c}\right)^n, \quad \sigma_{zz} = \sigma_{rr} - \frac{2\lambda c}{n}\left(\frac{\sigma_{rz}}{c}\right)^{2-n}, \tag{31}$$

as the appropriate generalization of (29) and (30).

4 Numerical formulation for general n

For general n such that $1 < n < 2$, we find from $(11)_2$ and (17) we are required to solve

$$\frac{\mathrm{d}\sigma_{rr}}{\mathrm{d}r} = -\frac{1}{2r}\left\{(\sigma_{rr} - \sigma_{zz}) + \left[(\sigma_{rr} - \sigma_{zz})^2 + 4\sigma_{rz}^2\right]^{1/2}\right\}, \tag{32}$$

subject to $(12)_2$, where σ_{rz} is defined by (13) and σ_{zz} is determined from the Warren Spring yield condition (7). From Hill and Wu [2], we find that (7) gives rise to the yield condition

$$\begin{aligned}&\left[\frac{n^2}{2\lambda^2(n-1)}\right]^{n/(2-n)}\left\{B - \left[B^2 - \frac{(n-1)}{n^2t^2}A\right]^{1/2}\right\}^{n/(2-n)} \\ &+\frac{n}{2(n-1)}\left\{B - \left[B^2 - \frac{(n-1)}{n^2t^2}A\right]^{1/2}\right\} - B = 0,\end{aligned} \tag{33}$$

where A and B are defined by

$$A = (\sigma_{rr} - \sigma_{zz})^2 + 4\sigma_{rz}^2, \quad B = 1 - \frac{\sigma_{rr} + \sigma_{zz}}{2t}. \tag{34}$$

In order to numerically solve (32), at each iteration in the numerical scheme we are required to numerically determine σ_{zz} from (33). We note that upon letting $n = 1$ in (33), the $n = 1$ yield equation $(9)_1$ arises after applying ℓ'Hopitals rule. Also, upon letting $n = 2$ in (33) gives rise to the $n = 2$ yield equation $(9)_2$, noting that we first need to raise both sides of (33) to $2 - n$.

Here we have used a fourth order Runge-Kutta scheme to determine the numerical solution of (32) and fsolve a root finding procedure in MAPLE to determine σ_{zz} numerically from (33) at each iteration. We comment that this scheme gives rise to numerical values which for the special cases $n = 1$ and $n = 2$ coincide with those obtained directly. The values of the constants t, c, ρ and the shear-index n are those obtained by Farley and Valentin [4] and we consider three specific granular shear-index materials, namely alumina, standard zinc dust, and precipitated calcium carbonate which have shear-indexes of $n = 1.19, 1.39$, and 1.53, respectively. For the sake of comparison and in order to emphasize the dependence on the shear-index n, the remaining constants are fixed as those for the standard zinc dust sample, namely $t = 2.54$ g/cm^2, $c = 5.33$ g/cm^2, and $\rho = 3.7062$ gm/cm^3. Farley and Valentin [4] adopt the unusual convention that g in reference to the units of t and c designates 981 dynes.

Figure 3 shows the yield function in terms of the variation of the maximum principal stress σ_I with the minimum principal stress σ_{III} for various values of n. It can be see that in the range shown these curves are almost straight lines. The unconfined yield strength f_c which is defined by $\sigma_{III} = -f_c$ when $\sigma_I = 0$ is an important physical property of the material, being the "compressive" stress that an initially unstressed material is able to sustain immediately prior to failure. The variation of this quantity with the shear-index n is shown in Figure 4. For $n = 1$

and $n = 2$ respectively, we may deduce the explicit values

$$f_c = 2c \left[\frac{c}{t} + \left(1 + \frac{c^2}{t^2} \right)^{1/2} \right] \ (n = 1), \quad f_c = c \left(\frac{c}{t} + 2 \right), \ (n = 2). \tag{35}$$

In the Appendix we propose the following expression as an appropriate generalization of (35) for general n, namely

$$f_c = \frac{2}{n}\frac{c^2}{t} + 2c \left(1 + \frac{c^2}{t^2} \right)^{(2-n)/2n}, \tag{36}$$

and this value is shown as a dotted line in Figure 4. We emphasize that although the expression (36) appears to provide a good numerical estimate of f_c it is only an educated guess based on the exact expressions (35).

Figures 5(a), 5(b), 5(c) and 5(d) show respectively the variations of $\sigma_{rr}, \sigma_{\phi\phi}, \sigma_{zz}$ and σ_I for five values of the shear-index n, noting that σ_{III} coincides with $\sigma_{\phi\phi}$. Finally, Figures 6(a), 6(b) and 6(c) shows the relative variation of all stresses for the three values $n = 1, 1.53$ and 2 respectively.

It is a matter of common experience that the larger the radius of the rat-hole, the more prone the material is to collapse and the essential problem is to determine the smallest rat-hole radius for which the material is unstable. Accordingly, in Figures 7, 8 and 9 we show the variation of the stresses $\sigma_{rr}, \sigma_{\phi\phi}, \sigma_{zz}$ and σ_I for the three values of the shear-index $n = 1, 1.53$, and 2 respectively. The figures show conclusively that for each of the three values of n there is a definite rat-hole radius at which there is an abrupt change in the stress patterns and for each r_0 there is a value of r for which the stresses no longer exist. As noted in the previous section for $n = 1$ and $n = 2$ we may show that at this value of r the relations (29) and (30) apply, while for $1 < n < 2$ the relation (31) apply. Figures 10 and 11 show the variation of σ_{rr} and σ_{zz} respectively with n for both the exact numerically determined values and those estimates based on (A1) and the approximate equations (A10) and (A11) when $\sigma_I = 0$ and $\sigma_{III} = -f_c$. We note that these estimates provide reasonable overall agreement with the exact numerically determined values.

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Appendix: Determination of the stress values at the point at which they fail to exist

In this Appendix we are concerned with the determination of the stresses at the point at which the stresses fail to exist. We establish the relations (31) which for general n connect the stress values at the point of failure. As noted in Section 3 the relations (29) and (30) are immediately apparent for $n = 1$ and $n = 2$, because in these special cases further mathematical analysis is possible.

If we compare (29) with the Warren Spring equation (3), then for general n it is reasonable to speculate

$$1 - \frac{\sigma_{rr}}{t} = \left(\frac{\sigma_{rz}}{c}\right)^n . \tag{A1}$$

Moreover, on examination of (30) it would not be unreasonable to assume that for general n

$$\sigma_{zz} = \sigma_{rr} + \alpha \left(\frac{\sigma_{rz}}{c}\right)^m , \tag{A2}$$

where α and m are certain unknown constants, yet to be determined. Now on introducing $\omega = \sigma_{rz}/c$, we have from (A1) and (A2) that the quantities A and B defined by (34) become

$$A = \alpha^2\omega^{2m} + 4c^2\omega^2, \quad B = \omega^n - \frac{\alpha}{2t}\omega^m, \tag{A3}$$

and from these relations we may deduce

$$B^2 - \frac{(n-1)}{n^2t^2}A = \left[\omega^n + \frac{(n-2)\alpha}{2nt}\omega^m\right]^2 - 2\frac{(n-1)}{n}\left(\frac{\alpha}{t}\omega^{m+n} + \frac{2\lambda^2}{n}\omega^2\right). \tag{A4}$$

We now require that the quantity on the left hand side be a perfect square, then we conclude that we need to choose α and m such that $m + n = 2$ and $\alpha/t = -2\lambda^2/n$, from which we find that (A2) becomes

$$\sigma_{zz} = \sigma_{rr} - \frac{2\lambda c}{n}\left(\frac{\sigma_{rz}}{c}\right)^{2-n} . \tag{A5}$$

We observe that (A5) is entirely consistent with the special cases $n = 1$ and $n = 2$

given by (30) and that

$$B - \left[B^2 - \frac{(n-1)}{n^2t^2}A\right]^{1/2} = \omega^n + \frac{\lambda^2}{n}\omega^{2-n} - \left[\omega^n + \frac{\lambda^2}{n^2}(2-n)\omega^{2-n}\right] = 2\frac{\lambda^2}{n^2}(n-1)\omega^{2-n}, \tag{A6}$$

from which it is a trivial matter to show that the yield condition (33) is satisfied identically, and thus confirms (A5) as the correct expression.

From Figures 7, 8 and 9 it would seem that the stresses fail to exist where $\sigma_I = 0$ and $\sigma_{III} = -f_c$. From equation (14) these conditions become

$$\begin{aligned} \sigma_{rr} + \sigma_{zz} + \left[(\sigma_{rr} - \sigma_{zz})^2 + 4\sigma_{rz}^2\right]^{1/2} &= 0, \\ \sigma_{rr} + \sigma_{zz} - \left[(\sigma_{rr} - \sigma_{zz})^2 + 4\sigma_{rz}^2\right]^{1/2} &= -2f_c, \end{aligned} \tag{A7}$$

from which we may deduce

$$\begin{aligned} \sigma_{rr}\sigma_{zz} &= \sigma_{rz}^2, \\ \sigma_{rr}\sigma_{zz} + f_c(\sigma_{rr} + \sigma_{zz}) + f_c^2 &= \sigma_{rz}^2. \end{aligned} \tag{A8}$$

On using (A1) in conjunction with (A8) we find

$$\sigma_{rr}^2 + c^2\left(1 - \frac{\sigma_{rr}}{t}\right)^{2/n} + f_c\sigma_{rr} = 0, \tag{A9}$$

as an exact equation for the determination of σ_{rr} at the point at which the stresses no longer exist and both $\sigma_I = 0$ and $\sigma_{III} = -f_c$ are satisfied.

Now based on an exact analysis of (A9) for the two special cases of $n = 1$ and $n = 2$, we are led to propose the alternate equation for (A9), namely

$$\left(1 + \frac{c^2}{t^2}\right)^{(2-n)/n}\sigma_{rr}^2 + \left(f_c - \frac{2}{n}\frac{c^2}{t}\right)\sigma_{rr} + c^2 = 0. \tag{A10}$$

Further, on using (A10) and again making a comparison with the exact analysis for $n = 1$ and $n = 2$, we may propose the following alternative approximate expression for f_c for general n

$$f_c = \frac{2}{n}\frac{c^2}{t} + 2c\left(1 + \frac{c^2}{t^2}\right)^{(2-n)/2n}, \tag{A11}$$

which is chosen so as to be consistent with the known exact values (35) for $n = 1$ and $n = 2$ and so that the quadratic (A10) becomes a perfect square. Curiously, Figure 4 vindicates this judicious choice for f_c while Figures 10 and 11 demonstrate the utility of approximating (A9) by (A10). We emphasize that equations (A10) and (A11) are speculative but give reasonably accurate results.

Figure and Table Captions

Figure 1. (a) Rat-hole occurring in a typical stockpile.
(b) Rat-hole causing funnel flow in a hopper.

Figure 2. General yield locus.

Figure 3. Shear-index yield condition expressed in terms of the maximum and minimum principal stresses and for various values of n.

Figure 4. Variation of the unconfined yield strength f_c with shear-index n and the approximate analytical expression (36) shown as a dashed line.

Figure 5. Variation of stresses with position for various values of the shear-index n and r_0 taken to be unity. ((a) σ_{rr}, (b) $\sigma_{\phi\phi}$, (c) σ_{zz} and (d) σ_I).

Figure 6. Variation of stresses with position for the three shear-index materials. ((a) $n = 1$, (b) $n = 1.53$ and (c) $n = 2$).

Figure 7. Variation of stresses for various values of the rat-hole radius r_0 for the shear-index material $n = 1$. ((a) σ_{rr}, (b) $\sigma_{\phi\phi}$, (c) σ_{zz} and (d) σ_I).

Figure 8. Variation of stresses for various values of the rat-hole radius r_0 for the shear-index material $n = 1.53$. ((a) σ_{rr}, (b) $\sigma_{\phi\phi}$, (c) σ_{zz} and (d) σ_I).

Figure 9. Variation of stresses for various values of the rat-hole radius r_0 for the shear-index material $n = 2$. ((a) σ_{rr}, (b) $\sigma_{\phi\phi}$, (c) σ_{zz} and (d) σ_I).

Figure 10. Numerical variation of σ_{rr} with shear-index n when $\sigma_I = 0$ and $\sigma_{III} = -f_c$ compared with the approximate value determined from (A10).

Figure 11. Numerical variation of σ_{zz} with shear-index n when $\sigma_I = 0$ and $\sigma_{III} = -f_c$ compared with the approximate value determined from (A10) and using (A1) with $(A8)_1$.

Table 1. Typical values of n, t, c and ρ (Farley and Valentin [4], g denotes 981 dynes).