

# On the problem of the determination of force distributions in granular heaps using continuum theory

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## Abstract

The determination of the horizontal and vertical force distributions at the base of a sand-pile is by now a famous problem in granular theory. In 1981 it was suggested from experimental work that the peak vertical force at the base does not occur directly beneath the vertex of the pile, but at some intermediate point so that there is a ring of maximum vertical pressure. Practising engineers have some reservation on this result and since that time numerous discrete theoretical and computational models of granular sand-piles have been proposed to explain this phenomenon, with varying degrees of success. For two-dimensional sand-piles, following other authors, we attempt to estimate the horizontal and vertical force distributions using the proper continuum mechanical theory of granular materials. For an infinite sand-pile we determine the force distributions at a certain height, and we argue that these forces should approximate those for a sand-pile of finite height resting on a horizontal surface. We exploit an adaptation of the classical Jenike radial flow solutions for granular flow in a converging wedge. This solution involves a stress function  $\psi$  which may be positive or negative. For both cases, numerical results indicate there is no solution satisfying all of the conditions of the stated problem, except in the special case when the angle of internal friction is equal to ninety degrees. Except in this special case, the difficulty in determining a numerical solution may well be due to the fact that the zero surface stress conditions are generally incompatible with the yield condition. For  $\psi$  negative a formal exact parametric solution is presented for the special case of the angle of internal friction equal to ninety degrees and which coincides with an independent numerical solution. This special exact solution which is a bona fide solution of the stated problem may well be the ‘only’ solution of the problem, but does not predict the dip in the vertical force which is suggested from experimental work.

## 1. Introduction

Granulated materials in the chemical industries are frequently stored in heaps. In order to evaluate storage conditions, a knowledge of the stress distribution throughout the material is important, and particularly at the base of the heap, since these stresses influence settlement, caking, comminution and overall deterioration of the stored material. Smid and Novosad [1] showed experimentally that the horizontal and vertical force distributions were as shown in Figure 1. Of particular importance is the counter-intuitive outcome, that the maximum vertical pressure does not occur directly beneath the sand-pile vertex, but rather at some intermediate point giving rise to a ring of maximum vertical pressure. This result has attracted much attention, both in the popular scientific literature (see for example [2, 3]) and has produced numerous discrete and computational models which attempt to explain this

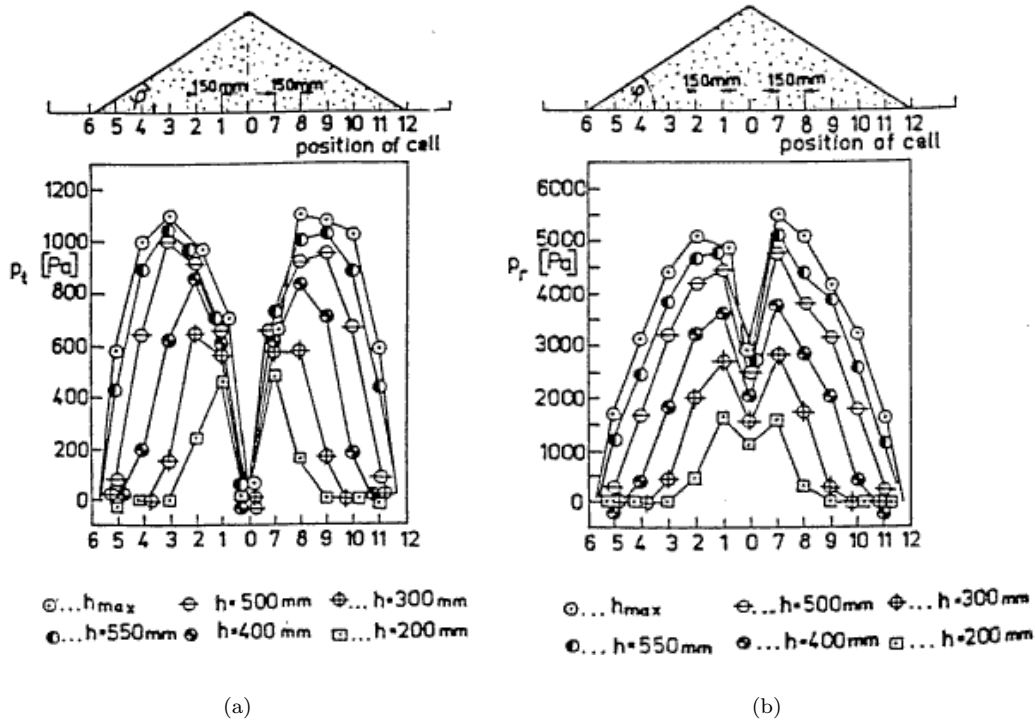


Figure 1: Horizontal (a) and vertical (b) force distributions as determined experimentally by Smid and Novosad [1].

curious phenomenon (see for example [4] to [11]). More recently, Vanel *et al.* [12] have undertaken systematic experiments on two and three-dimensional systems which demonstrate the influence of the mode of granular deposition on the stress distribution.

In this paper, following previous authors (van R. Marias [13], Sokolovsky [14] and Booker [15]) we examine a continuum mechanical approach for two-dimensional sand-piles and we attempt to utilize the solutions first used by Jenike [16, 17, 18] and Johanson [19] which describe gravity flows of granular materials in converging wedges and cones. More recently these solutions have been re-examined by Bradley [20] and Spencer and Bradley [21] and for convenience we follow the notation of these latter authors. We note however that gravity in our problem is acting in the opposite direction to their problem and that these authors adopt the unusual convention of the  $x$ -axis being vertical. We comment here that Savage [22] provides an extensive and critical review of the literature relating to the approach adopted here. The models proposed in [23], [24] and [25] all incorporate an inner elastic region and such models do indeed exhibit a dip in the pressure curves.

The major contribution of this paper is the determination of an explicit exact solution for the special case of the angle of internal friction equal to  $90^\circ$ , which numerical results indicate may well be the only solution satisfying all the conditions of the stated problem. While this special case corresponds to a certain extent to a non-physical material, there are however many materials which do indeed exhibit large angles of internal friction as shown in Table 1. Mathematically this solution might provide a limiting upper bound for physically meaningful materials and also might provide a benchmark for other numerical schemes.

In this paper we consider an ‘infinite’ two-dimensional sand-pile and we set up coordinate axes at the vertex of the sand-pile as indicated in Figure 2, with gravity shown acting in

Granular material	Measured values of $\beta = \sin \phi$			
Coal	0.939	0.958	0.973	0.985
Alumina cake	0.941			
Waste rock	0.974			
Silica	0.979			

Table 1: Measured values of  $\beta = \sin \phi$  for certain granular materials, where  $\phi$  is the angle of internal friction.

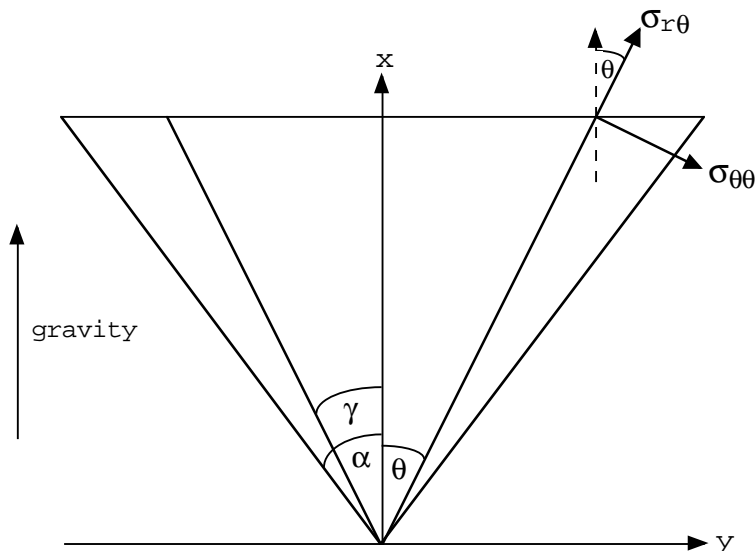


Figure 2: Coordinates for two-dimensional sand-piles.

the vertically upwards direction. The basic idea is that we attempt to determine the stress distribution in a sand-pile of infinite height and then we evaluate the horizontal and vertical forces acting along a horizontal plane at a finite height. Clearly the problem for a finite sand-pile and resting on a rigid horizontal plane is different to that examined here, but nevertheless we might expect the two force distributions to be similar.

In the following section for two-dimensional sand-piles we state briefly the basic equations of the continuum theory of granular materials and we give expressions for horizontal and vertical force resultants. As previously mentioned we follow the notation of Spencer and Bradley [21], noting again that in our problem gravity acts in the opposite direction and the  $x$ -axis denotes the vertical direction. In the subsequent section we derive a number of additional relations involving boundary values. These additional boundary relations are derived either directly from the basic governing equations (7) or from the assumption that the derivative of the stress angle  $\psi$  defined by (4) at the surface of the pile remains finite. The stress angle  $\psi$  may be positive or negative and for both cases numerical schemes making use of these additional relations indicate that no solution exists which satisfies either all of the stated boundary conditions or the additional boundary relations. However, in a recent paper [26] the authors derive a formal exact solution of the governing equations for two-dimensional gravity flow through a converging wedge shaped hopper for the special case of an angle of internal friction  $\phi = 90^\circ$ . Here, we exploit this general solution to determine an exact parametric solution for the determination of the stress distribution in a sand-pile for the special case of  $\phi = 90^\circ$ , which is done in section 4 for  $\psi$  negative. For  $\psi$  positive we are

able to determine a similar numerical solution to those obtained by previous authors such as van R. Marias [13], Sokolovsky [14] and Booker [15] but as with all previous solutions, one of the stated boundary conditions fails to be satisfied. It would appear then that only for the special case when  $\phi = 90^\circ$  does a solution of the full stated problem exist.

## 2. Basic equations of continuum theory

For quasi-static plane flow, the stress components in a cylindrical polar coordinate system  $(r, \theta, z)$  defined as shown in Figure 2, satisfy the equilibrium equations

$$\begin{aligned} \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} &= -\rho g \cos \theta, \\ \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{2\sigma_{r\theta}}{r} &= \rho g \sin \theta, \end{aligned} \quad (1)$$

where  $\rho$  is the density,  $g$  is the acceleration due to gravity and  $\sigma_{rr}, \sigma_{\theta\theta}$  and  $\sigma_{r\theta}$  denote the in-plane physical stress components. Following Spencer and Bradley [21] these components can be expressed in the standard form

$$\sigma_{rr} = -p + q \cos 2\psi, \quad \sigma_{\theta\theta} = -p - q \cos 2\psi, \quad \sigma_{r\theta} = q \sin 2\psi, \quad (2)$$

where  $p$  and  $q$  are defined by

$$p = -\frac{1}{2}(\sigma_{rr} + \sigma_{\theta\theta}), \quad q = \left\{ \frac{1}{4}(\sigma_{rr} - \sigma_{\theta\theta})^2 + \sigma_{r\theta}^2 \right\}^{1/2}, \quad (3)$$

while  $\psi$  is defined by

$$\tan 2\psi = 2\sigma_{r\theta}/(\sigma_{rr} - \sigma_{\theta\theta}), \quad (4)$$

and physically  $\psi$  is the angle between the maximum principal stress axis and the radial direction, in the direction of increasing  $\theta$ . For a cohesionless material, the Coulomb–Mohr yield condition takes the form

$$q = \beta p, \quad (5)$$

where  $\beta = \sin \phi$  and  $\phi$  is a material constant referred to as the angle of internal friction.

Following Jenike [17] and Spencer and Bradley [21] we examine solutions of the form

$$\psi = \psi(\theta), \quad q = -\rho g r F(\theta), \quad (6)$$

and from the above equations we deduce

$$\begin{aligned} \frac{dF}{d\theta} &= \frac{F \sin 2\psi + \beta \sin(2\psi + \theta)}{\beta + \cos 2\psi}, \\ \frac{d\psi}{d\theta} + 1 &= \frac{F(\beta^{-1} - \beta) + \cos \theta + \beta \cos(2\psi + \theta)}{2F(\beta + \cos 2\psi)}. \end{aligned} \quad (7)$$

Now for a symmetrical stress distribution and for zero stress along the sand-pile slope, we require the following conditions

$$\psi(0) = 0, \quad F(\alpha) = 0, \quad (8)$$

where  $\alpha$  denotes the semi-vertex angle. We observe from the equilibrium equations (1) that if  $\sigma_{rr}$  and  $\sigma_{\theta\theta}$  are assumed to be even functions of  $\theta$ , then  $\sigma_{r\theta}$  is necessarily an odd function or skew-symmetric and therefore  $\sigma_{r\theta}$  vanishes at the origin, and hence the condition

(8)<sub>1</sub>. We comment that given the nonlinear character of the problem and the likely non-uniqueness of the solution, the symmetry condition  $\psi(0) = 0$  may well represent a severe restriction.

We note here that for the special case of  $\beta = 1$  ( $\phi = \pi/2$ ) a possible alternative boundary condition to (8)<sub>2</sub> is  $\psi(\alpha) = \pm\pi/2$ . Noting that in this special case, this is sufficient to guarantee zero stress on the surface. We comment that the stress angle  $\psi$  may be positive or negative and that in principle, a numerical solution of (7) and (8) can be determined provided that  $\psi$  lies either below or above the first critical singularity of (7), namely the first value of  $\psi$  for which  $\beta + \cos 2\psi = 0$ . We further comment that while (7) and (8) appear to be well-posed in the sense that they constitute a second order system with two boundary conditions, in fact from (3) and (5), we see that the yield condition is not compatible with the stress-free conditions  $\sigma_{r\theta} = \sigma_{\theta\theta} = 0$  unless either  $\beta = 1$  or  $\sigma_{rr}$  vanishes along  $\theta = \alpha$ . Generally, the latter condition does not occur and accordingly this incompatibility may well account for the difficulty in determining a full numerical solution satisfying all the stated boundary conditions and the additional boundary relations derived in the following section.

We note that the following expressions for the horizontal and vertical force distributions acting along a plane  $x = \text{constant}$  may be deduced, namely

$$\sigma_x = \rho gh \frac{F(\theta)}{\cos \theta} \left\{ \frac{1}{\beta} - \cos 2[\psi(\theta) + \theta] \right\}, \quad \sigma_y = -\rho gh \frac{F(\theta)}{\cos \theta} \sin 2[\psi(\theta) + \theta]. \quad (9)$$

Finally, we note that (7) admits the special exact solution given by Sokolovsky [14]

$$\psi(\theta) = -\theta + \psi_0, \quad F(\theta) = -\beta(1 - \beta^2)^{-1} \{ \cos \theta + \beta \cos(2\psi_0 - \theta) \}, \quad (10)$$

where  $\psi_0$  is a constant, but in general, we observe that (10) can only satisfy one of the boundary conditions of (8). In the following section we determine some additional boundary relations from (7) and (8) which are necessary either to initiate the numerical procedure or to validate or otherwise a numerical solution.

### 3. Additional boundary relations

From (7) and (8) it is possible to derive additional relations involving boundary values. First at  $\theta = 0$  we have from (7) and (8)<sub>1</sub>,

$$F'(0) = 0, \quad \psi'(0) = \frac{1}{2} \left\{ \frac{1}{\beta} - 3 + \frac{1}{F(0)} \right\}, \quad (11)$$

where the prime throughout the section denotes differentiation with respect to  $\theta$ . Now at  $\theta = \alpha$  we need to make an assumption as to whether  $\psi'(\alpha)$  remains finite or not. From the numerous numerical solutions given by Bradley [20] for a different set of assumed boundary conditions, it is clear that this may or may not be the case. The essential point is that  $\psi'(\alpha)$  is either finite or infinite. If it is finite we are able to deduce certain additional boundary relations, but for both  $\psi(\theta)$  positive and negative, it appears not possible to construct a numerical solution which is entirely consistent with all of the derived boundary relations. Accordingly, in this case we are led to examine the case when  $\psi'(\alpha)$  is infinite. However, for completeness we first note the additional boundary relations which apply when  $\psi'(\alpha)$  is finite.

Assuming  $\psi'(\alpha)$  is finite and also that  $\beta \geq \cos \alpha$  we see from (7)<sub>2</sub> and (8)<sub>2</sub> that this can only occur provided

$$\cos \alpha + \beta \cos[2\psi(\alpha) + \alpha] = 0, \quad (12)$$

which gives rise to the four distinct roots between  $-\pi \leq \psi \leq \pi$ ,

$$\psi(\alpha) = \pm \frac{1}{2}\pi - \frac{1}{2}\alpha \pm \frac{1}{2} \cos^{-1}(\beta^{-1} \cos \alpha), \quad (13)$$

where the  $\pm$  signs are independent. For example, the first negative root is

$$\psi(\alpha) = -\frac{1}{2} \{ \pi + \alpha - \cos^{-1}(\beta^{-1} \cos \alpha) \}, \quad (14)$$

and from (7)<sub>1</sub> and (14) we may deduce

$$F'(\alpha) = \beta / [\sin \alpha - (\beta^2 - \cos^2 \alpha)^{1/2}], \quad (15)$$

provided that  $\beta > \cos \alpha$  and we note that  $F'(\alpha)$  becomes infinite for  $\beta = 1$ . Further, since we have assumed  $\psi'(\alpha)$  is finite, we observe that (12) is obtained on the basis that both the numerator and denominator of (7)<sub>2</sub> must vanish at  $\theta = \alpha$ . This means that we may apply l'Hopital's rule to (7)<sub>2</sub> from which we may deduce

$$\psi'(\alpha) + 1 = \frac{F'(\alpha)(\beta^{-1} - \beta) - \sin \alpha - \beta \sin[2\psi(\alpha) + \alpha][1 + 2\psi'(\alpha)]}{2F'(\alpha)[\beta + \cos 2\psi(\alpha)]}, \quad (16)$$

and on simplifying this equation using (14) and (15) and again assuming that  $\beta > \cos \alpha$  we obtain the remarkably simple result

$$\psi'(\alpha) = -1. \quad (17)$$

Similarly, for

$$\psi(\alpha) = \frac{1}{2} \{ \pi - \alpha + \cos^{-1}(\beta^{-1} \cos \alpha) \}, \quad (18)$$

we find that  $F'(\alpha)$  and  $\psi'(\alpha)$  are given by (15) and (17) respectively. However, for

$$\psi(\alpha) = \frac{1}{2} \{ \pm \pi - \alpha - \cos^{-1}(\beta^{-1} \cos \alpha) \}, \quad (19)$$

we find that  $F'(\alpha)$  is given by

$$F'(\alpha) = \beta / [\sin \alpha + (\beta^2 - \cos^2 \alpha)^{1/2}], \quad (20)$$

provided that  $\beta > \cos \alpha$  and  $\psi'(\alpha)$  is still given by (17).

Alternatively, if we eliminate  $F$  from (7), we may deduce the following second order differential equation for  $\psi(\theta)$ ,

$$\begin{aligned} (\beta + \cos 2\psi)[\cos \theta + \beta \cos(2\psi + \theta)]\psi'' &= 2(\psi' + 1)\{\sin 2\psi[\cos \theta + \beta \cos(2\psi + \theta)]\psi' \\ &\quad - \beta \sin(\theta + 2\psi) [2(\beta + \cos 2\psi)(\psi' + 1) + \beta - \beta^{-1}]\}. \end{aligned} \quad (21)$$

This formulation forms the basis for the exact analysis for  $\beta = 1$  given by Hill and Cox [26] for flow through a hopper and it is also the basis for an independent numerical scheme for the special case of  $\beta = 1$ . Now, from (21) it is clear that one possibility when (12) holds is certainly  $\psi'(\alpha) = -1$ , which agrees with (17). However, another possibility arises from

$$2(\beta + \cos 2\psi)(\psi' + 1) + \beta - \beta^{-1} = 0, \quad (22)$$

and this equation gives rise to

$$\psi'(\alpha) = -\frac{3}{2} \pm \frac{\sin \alpha}{2(\beta^2 - \cos^2 \alpha)^{1/2}}, \quad (23)$$

where the minus sign in (23) refers to  $\psi(\alpha)$  given by (14) and (18), and the plus sign to (19) respectively. We note that in the derivation of (23) we are still assuming  $\beta > \cos \alpha$ . The essential point is that on the assumption  $\psi'(\alpha)$  remains finite we obtain additional constraints on  $\psi(\theta)$  which for both  $\psi$  positive and negative the numerical solution of (7) indicates that it is not possible to simultaneously satisfy. Accordingly, we infer that  $\psi'(\alpha)$  is infinite and that it occurs at the first zero of  $\beta + \cos 2\psi(\alpha) = 0$ . We also note that (23) can be derived from (16) when  $F'(\alpha)$  is infinite. In addition, we observe that the positive case of (23) again yields  $\psi'(\alpha) = -1$  in the special case of  $\beta = 1$ .

We remark that the condition  $\cos \alpha \leq \beta$  is equivalent to the statement that the angle of repose of the sand-pile is less than or equal to  $\phi$ . This in turn is the same condition as the requirement that for a particle at rest on the surface of the sand-pile the frictional force  $F$  and the normal reaction  $N$  are such that  $F \leq \mu N$  where  $\mu = \tan \phi$ .

#### 4. Exact parametric solution for the special case of $\beta = 1$

The special case  $\beta = 1$  corresponds to  $\phi = 90^\circ$ . If  $\beta = 1$ , (21) simplifies to give

$$\cos \psi \cos(\psi + \theta) \psi'' = -2(\psi' + 1) \{ \sin(\psi + \theta) \cos \psi (\psi' + 1) + \cos(\psi + \theta) \sin \psi \}, \quad (24)$$

for which Hill and Cox [26] derive the exact parametric solution given by

$$\begin{aligned} \tan \theta &= C_2 \left\{ 2e^{\omega/2} \omega^{-1/2} - I(\omega) \right\}^{-1}, \\ \tan \psi(\theta) &= C_2^{-1} I(\omega) \left\{ 1 - \frac{1}{2} \omega^{1/2} e^{-\omega/2} I(\omega) \right\} - \frac{1}{2} C_2 \omega^{1/2} e^{-\omega/2}, \\ F(\theta) &= \frac{1}{4} \omega^{-1/2} e^{-\omega/2} [C_2^2 + I(\omega)^2] \left\{ C_2^2 + \left[ 2e^{\omega/2} \omega^{-1/2} - I(\omega) \right]^2 \right\}^{-1/2}, \end{aligned} \quad (25)$$

where  $\omega$  is the parameter, the integral  $I(\omega)$  is defined by

$$I(\omega) = \int_0^\omega t^{-1/2} e^{t/2} dt + C_1, \quad (26)$$

and  $C_1$  and  $C_2$  denote two arbitrary constants of integration. In this special case of  $\beta = 1$  for the determination of the stress distribution in a sand-pile, these arbitrary constants must be chosen so that  $\psi(\theta)$  satisfies

$$\psi(0) = 0, \quad \psi(\alpha) = -\frac{1}{2}\pi. \quad (27)$$

From (25) we see that the first of these conditions is satisfied provided we associate the parameter value  $\omega \equiv 0$  with  $\theta = 0$  and take the integral  $I(\omega)$  to be defined by

$$I(\omega) = \int_0^\omega t^{-1/2} e^{t/2} dt. \quad (28)$$

Now from the derivation given in [26] or directly from (25) we may verify

$$I(\omega) = C_2 \tan(\psi + \theta), \quad (29)$$

and therefore in particular from (25)<sub>1</sub> and (27)<sub>2</sub> we have at  $\theta = \alpha$ ,

$$\tan \alpha = C_2 \left\{ 2e^{\omega_0/2} \omega_0^{-1/2} + C_2 \cot \alpha \right\}^{-1}, \quad (30)$$

where we use  $\omega_0$  to denote the value of the parameter corresponding to  $\theta = \alpha$  and we have used  $\tan(\alpha - \pi/2) = -\cot \alpha$ . From (30) it is clear that  $\omega_0 = -\infty$ .

Accordingly, we change the parameter from  $\omega$  to  $-\lambda$ , make the substitution  $t = -s$  in the integral (28) so that the exact parametric solution (25) now becomes

$$\begin{aligned} \tan \theta &= -C_3 \left\{ 2e^{-\lambda/2} \lambda^{-1/2} + J(\lambda) \right\}^{-1}, \\ \tan \psi(\theta) &= C_3^{-1} J(\lambda) \left\{ 1 + \frac{1}{2} \lambda^{1/2} e^{\lambda/2} J(\lambda) \right\} + \frac{1}{2} C_3 \lambda^{1/2} e^{\lambda/2}, \\ F(\theta) &= -\frac{1}{4} \lambda^{-1/2} e^{\lambda/2} [C_3^2 + J(\lambda)^2] \left\{ C_3^2 + \left[ 2e^{-\lambda/2} \lambda^{-1/2} + J(\lambda) \right]^2 \right\}^{-1/2}, \end{aligned} \quad (31)$$

where  $C_2 = iC_3$  and the integral  $J(\lambda)$  is defined by

$$J(\lambda) = \int_0^\lambda s^{-1/2} e^{-s/2} ds = 2^{3/2} \int_0^{(\lambda/2)^{1/2}} e^{-x^2} dx = (2\pi)^{1/2} \operatorname{erf} \left\{ (\lambda/2)^{1/2} \right\}, \quad (32)$$

where erf is the error function. Now since  $\lambda = \infty$  is the parameter value corresponding to  $\theta = \alpha$ , we require an estimate of  $J(\lambda)$  as  $\lambda \rightarrow \infty$ . Using the standard asymptotic expansion for the complementary error function (see for example [27], p. 482) we deduce

$$J(\lambda) = (2\pi)^{1/2} \left\{ 1 - \operatorname{erfc} \left[ (\lambda/2)^{1/2} \right] \right\} = (2\pi)^{1/2} - \frac{2e^{-\lambda/2}}{\lambda^{1/2}} \left\{ 1 - \frac{1}{\lambda} + O\left(\frac{1}{\lambda^2}\right) \right\}, \quad (33)$$

and from this equation and (31) we find

$$\begin{aligned} \tan \theta &= -C_3 \left\{ (2\pi)^{1/2} + \frac{2e^{-\lambda/2}}{\lambda^{3/2}} \left[ 1 + O\left(\frac{1}{\lambda}\right) \right] \right\}^{-1}, \\ \tan \psi(\theta) &= \left( \frac{\pi}{C_3} + \frac{C_3}{2} \right) \lambda^{1/2} e^{\lambda/2} + O(1), \\ F(\theta) &= -\frac{e^{\lambda/2}}{4\lambda^{1/2}} \left\{ (C_3^2 + 2\pi)^{1/2} - 4 \frac{e^{-\lambda/2}}{\lambda^{1/2}} \left( \frac{2\pi}{C_3^2 + 2\pi} \right)^{1/2} \left[ 1 + O\left(\frac{1}{\lambda}\right) \right] \right\}. \end{aligned} \quad (34)$$

From (34)<sub>1</sub> we may deduce that

$$C_3 = -(2\pi)^{1/2} \tan \alpha, \quad (35)$$

and that

$$\tan \psi(\theta) = -\left(\frac{\pi\lambda}{2}\right)^{1/2} \frac{e^{\lambda/2}}{\sin \alpha \cos \alpha} + O(1), \quad F(\theta) = -\left(\frac{\pi}{2\lambda}\right)^{1/2} \frac{e^{\lambda/2}}{2 \cos \alpha} + O\left(\frac{1}{\lambda}\right), \quad (36)$$

which confirms  $\psi(\alpha) = -\pi/2$  as  $\lambda \rightarrow \infty$ , but note that  $F(\alpha) \rightarrow \infty$  rather than zero. In the case with  $\beta = 1$  we have satisfied the alternative conditions  $\psi(0) = 0$  and  $\psi(\alpha) = -\pi/2$ , which apply only for this special case. From the asymptotic expressions (36) we may confirm that  $\sigma_{r\theta}$  and  $\sigma_{\theta\theta}$  both tend to zero.

Thus for a two-dimensional pile with  $\phi = 90^\circ$ , an exact parametric solution is given by (31) for  $0 \leq \lambda \leq \infty$  with  $J(\lambda)$  defined by (32) and the constant  $C_3$  defined by (35).

Hence, the special case of  $\beta = 1$  admits the following exact parametric solution for  $\psi(\theta)$

$$\begin{aligned} \tan \theta &= \frac{(2\pi)^{1/2} \tan \alpha}{2e^{-\lambda/2} \lambda^{-1/2} + J(\lambda)}, \\ \tan \psi(\theta) &= \frac{-J(\lambda)}{(2\pi)^{1/2} \tan \alpha} \left\{ 1 + \frac{\lambda^{1/2}}{2} e^{\lambda/2} J(\lambda) \right\} - \left(\frac{\pi\lambda}{2}\right)^{1/2} e^{\lambda/2} \tan \alpha, \\ F(\theta) &= -\frac{e^{\lambda/2} [2\pi \tan^2 \alpha + J(\lambda)^2]}{4\lambda^{1/2} \left\{ 2\pi \tan^2 \alpha + [2e^{-\lambda/2} \lambda^{-1/2} + J(\lambda)]^2 \right\}}, \end{aligned} \quad (37)$$

where  $0 \leq \lambda \leq \infty$  and the integral  $J(\lambda)$  is defined by (32). As described in Section 6, numerical results indicate that a bona fide solution of the formulated problem only exists for this special angle of internal friction. For all other values of  $\beta$  at least one of the stated conditions, or the derived boundary relations, is not satisfied.

## 5. Alternative formulation

In this section we present an alternative formulation of the problem which highlights the incompatibility of the yield condition and the vanishing of the stress on the surface. The

alternative formulation of the problem can be obtained by assuming

$$\sigma_{rr} = rL(\theta), \quad \sigma_{r\theta} = rM(\theta), \quad \sigma_{\theta\theta} = rN(\theta), \quad (38)$$

for certain functions of  $\theta$ ,  $L(\theta)$ ,  $M(\theta)$ , and  $N(\theta)$ . From this equation and (1) we obtain

$$M' + 2L - N = -\rho g \cos \theta, \quad N' + 3M = \rho g \sin \theta, \quad (39)$$

while the yield condition (5) gives

$$\{(L - N)^2 + 4M^2\}^{1/2} = -\beta(L + N), \quad (40)$$

which on squaring becomes

$$(1 - \beta^2)(L^2 + N^2) - 2LN(1 + \beta^2) + 4M^2 = 0. \quad (41)$$

Now from (40) it is clear that we require  $L + N < 0$ , while (41) is only sensible provided the product  $LN > 0$ . Accordingly, the functions  $L(\theta)$  and  $N(\theta)$  are both negative. The skew-symmetry of  $\sigma_{r\theta}$  and the vanishing of the stress on the surface of the sand-pile means that we require to solve the differential equations (39) subject to

$$M(0) = 0, \quad M(\alpha) = 0, \quad N(\alpha) = 0. \quad (42)$$

We make the following observations. Firstly, the condition  $M(0) = 0$  and the differential equation (39)<sub>2</sub> are sufficient to ensure  $N'(0) = 0$ . Secondly for  $\beta \neq 1$ , the yield condition (40) or (41) is not compatible with the two boundary conditions (42)<sub>2</sub> and (42)<sub>3</sub> unless  $L(\alpha) = 0$ , namely  $\sigma_{rr} = 0$  along  $\theta = \alpha$ . In general this does not occur and this incompatibility constitutes the major factor prohibiting the determination of a numerical solution.

On solving (41) as a quadratic we obtain

$$L(\theta) = (1 - \beta^2)^{-1} \left\{ (1 + \beta^2)N(\theta) - 2[\beta^2 N(\theta)^2 - (1 - \beta^2)M(\theta)^2]^{1/2} \right\}, \quad (43)$$

where we have chosen the negative sign so that  $L(\theta)$  remains finite as  $\beta \rightarrow 1$ . From (39) and (43) we deduce two nonlinear coupled equations for the functions  $M(\theta)$  and  $N(\theta)$ ,

$$M' + \left( \frac{1 + 3\beta^2}{1 - \beta^2} \right) N = -\rho g \cos \theta + \frac{4}{(1 - \beta^2)} [\beta^2 N^2 - (1 - \beta^2)M^2]^{1/2}, \quad (44)$$

$$N' + 3M = \rho g \sin \theta,$$

which we require to solve subject to the ‘three’ conditions (42). We note for the solution of (44) to remain valid that  $M(\theta)$  and  $N(\theta)$  must satisfy  $|M| \leq -N \tan \phi$ , where  $N(\theta)$  is known to be negative. However, numerical results for  $\beta \neq 1$  indicate that it is not possible to determine solutions of (44) which satisfy all three conditions (42), as might be anticipated.

We have attempted to determine a numerical solution of (44) subject to (42), where the three boundary conditions are determined from the assumptions of skew-symmetry of  $\sigma_{r\theta}$  and zero stress on the surface of the sand-pile. The zero stress conditions are generally incompatible with the yield condition and it is not possible to determine a solution for this over-determined system. However, we may determine a numerical solution of (44) subject to only the two boundary conditions (42)<sub>2</sub> and (42)<sub>3</sub>, but only for the special case of the angle of internal friction equal to the angle of repose, that is  $\beta = \cos \alpha$ . This is because from (44)<sub>1</sub>, we require  $|M| \leq -N \tan \phi$  but the numerical results for  $\beta > \cos \alpha$  indicate that this inequality is not satisfied over the entire range  $0 \leq \theta \leq \alpha$ .

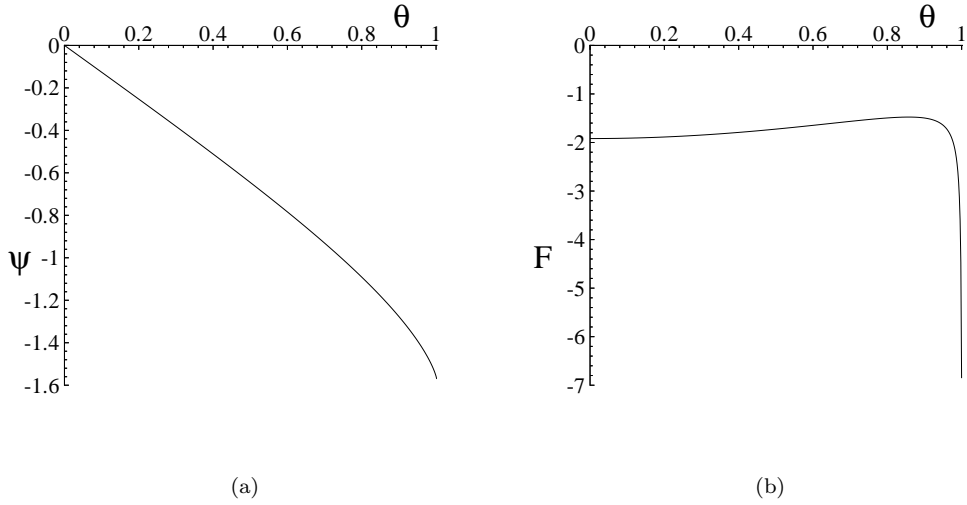
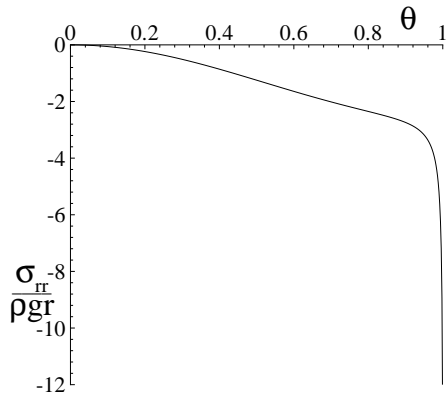


Figure 3: Variation of (a)  $\psi(\theta)$  and (b)  $F(\theta)$  for two-dimensional sand piles for the exact parametric solution for  $\beta = 1$  as given by (37) using an angle of repose of  $32.6^\circ$  ( $\alpha = 1.0018$ ).

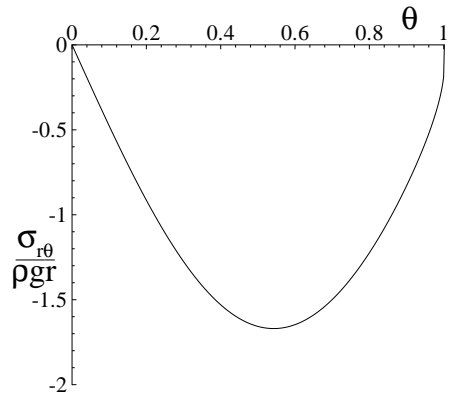
## 6. Numerical results and conclusions

We have considered a continuum model as a possible procedure to solve the problem of determining the force distribution at the base of a two-dimensional sand-pile. We have made use of the Jenike solutions for radial flow in a converging wedge, but with gravity acting in the opposite direction. For the stress angle  $\psi$  negative and for the special case of an angle of internal friction equal to  $90^\circ$ , we have determined an exact analytical solution as given by (37) and shown graphically in Figure 3. We note that the particular material constants used for the exact solution coincide with the experimental results in Figure 1 given by Smid and Novosad [1], which were obtained at various stages during the pouring of a three-dimensional heap. When the heap was at height  $h$ , the horizontal and vertical stresses were measured at the base of the heap. The material used was sand for which the angle of repose was  $32.6^\circ$  and the average bulk density was determined as  $\rho = 1567\text{kg/m}^3$ , and for the sake of definiteness these values have been adopted here.

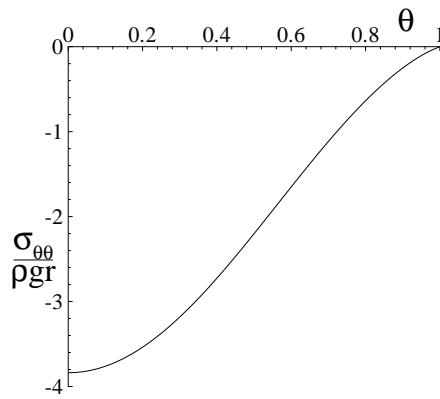
An independent numerical solution for the special case of  $\beta = 1$  which coincides with the exact analytical solution has been determined using the second order differential equation for  $\psi$ , namely (21) with  $\psi(0) = 0$  and  $\psi(\alpha) = -\pi/2$  and using the Nonlinear Finite-Difference Method given by Burden and Faries ([28], p. 601). Further, for  $\beta < 1$  and by carefully delineating the four cases of  $\psi$  positive and negative and  $\psi'(\alpha)$  finite and infinite, numerical results indicate that in all of the cases it is not possible to determine a solution which satisfies all the required conditions. Figure 4 shows the variation in  $\sigma_{rr}$ ,  $\sigma_{r\theta}$  and  $\sigma_{\theta\theta}$  for the exact parametric solution for  $\beta = 1$ , noting that both  $\sigma_{r\theta}$  and  $\sigma_{\theta\theta}$  are zero at  $\theta = \alpha$ . Many materials such as Coal and Silica do exhibit large angles of internal friction such as  $80^\circ$  and  $78.34^\circ$  respectively, but the exact analytical solution for  $\phi = 90^\circ$  for the horizontal and vertical stresses, as shown in Figure 5, do not exhibit the experimentally determined profile obtained by Smid and Novosad [1]. However, the authors believe that this solution may well be the only solution to the problem which satisfies all the necessary requirements.



(a)



(b)



(c)

Figure 4: Variation of (a)  $\sigma_{rr}$ , (b)  $\sigma_{r\theta}$  and (c)  $\sigma_{\theta\theta}$  for two-dimensional sand-piles for the exact parametric solution for  $\beta = 1$  as given by (37) using an angle of repose of  $32.6^\circ$  ( $\alpha = 1.0018$ ).

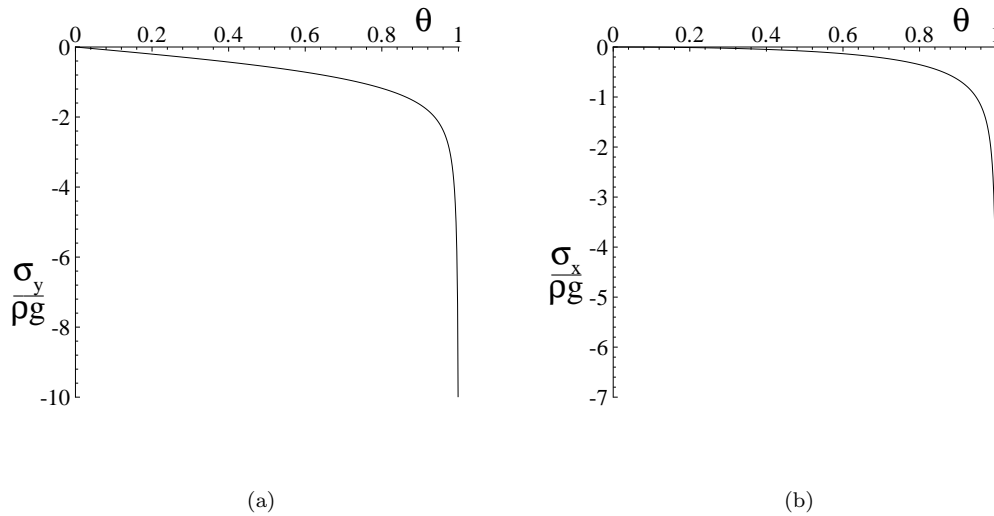


Figure 5: Horizontal (a) and vertical (b) force distributions for a two-dimensional sand-pile for the exact parametric solution for  $\beta = 1$  as given by (37) using an angle of repose of  $32.6^\circ$  ( $\alpha = 1.0018$ ).

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