The force distribution at the base of sand-piles

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ABSTRACT: In 1981 it was suggested from experimental work that the peak vertical force at the base of a sand-pile does not occur directly beneath the vertex of the pile, but rather at some intermediate point so that there is a ring of maximum vertical pressure. Practising engineers have some reservation on this result and since that time numerous discrete theoretical and computational models of granular sand-piles have been proposed to explain this phenomenon, with varying degrees of success. Here for two-dimensional wedge and three-dimensional conical sand-piles we estimate the horizontal and vertical force distributions using the proper continuum mechanical theory of granular materials. For an infinite sand-pile we determine the force distributions at a certain height, and we argue that these forces should approximate those for a sand-pile of finite height resting on a horizontal surface. We exploit an adaptation of the classical Jenike radial flow solutions for granular flow in a converging wedge and show that for realistic angles of internal friction there is a solution such that the sand-pile is not entirely at yield, but has an inner dead region and an outer yield region. From this model we determine a solution that does predict the dip in the vertical force as suggested from experimental work.

1 INTRODUCTION

Throughout the world, granular materials are commonly used in many industries. These industries, such as the chemical industry that use fine powders, and the mining industry that deals with large irregularly shaped ores, frequently store granulated material in heaps. Knowledge of the stress distribution throughout the heap and particularly at the base, is therefore of importance, as this will enable us to be able to predict the amount of settlement, caking, comminution and overall deterioration of the stored material. Knowing the location of the maximum vertical pressure ensures that stock-piles can be arranged to ensure the best endurance of the stored material. Intuitively, we might expect that the maximum vertical pressure lies directly beneath the vertex of the sand-pile. However, Smid & Novosad (1981) showed experimentally that this is not the case, but rather that it occurs at some intermediate point, so that

(a). (b).

Figure 1: Coordinates for two and three-dimensional sand-piles with an inner dead region and an outer yield region with the boundary at $\theta = \gamma$. ((a) two-dimensional and (b) three-dimensional).

there is a ring of maximum vertical pressure. This result has attracted much attention, both in the popular scientific literature (see for example Watson (1991, 1996)) and has produced numerous discrete and computational models which attempt to explain this curious phenomenon (see Savage (1997) for an extensive and critical review of the literature). In this paper for two-dimensional and three-dimensional sand-piles we use the proper continuum mechanical theory of granular materials and we propose a model which is not entirely at yield, but contains an inner dead region. This model does give rise to the essential profile of the M-shaped curves from the experimental work of Smid & Novosad (1981). We comment that such models are by no means unique and that similar models incorporating inner elastic regions have been proposed by Cantelaube and Goddard (1997), Cantelaube, Didwania and Goddard (1998) and Didwania, Cantelaube and Goddard (2000). The present work differs in that the inner dead region is not prescribed to be elastic, but merely is a possible equilibrium state. We utilize the solutions first used by Jenike (1962, 1964, 1965) and Johanson (1964) which describe gravity flows of granular materials in converging wedges and cones. We note that these solutions have been re-examined more recently by Bradley (1991) and Spencer and Bradley (1996) and for convenience we follow the notation of the latter authors, with the exception that gravity in our problem is acting in the opposite direction to their problem. We also note that these authors adopt the unusual convention of the x-axis being vertical.

We consider an 'infinite' two-dimensional and three-dimensional sand-piles and set up the coordinate axis at the vertex of the sand-pile as indicated in Figure 1, with gravity shown acting in the vertically upwards direction. Our basic idea is that we attempt to determine the stress distribution in a sand-pile of infinite height and then we evaluate the horizontal and vertical forces acting along a horizontal plane at a finite height. Clearly the problem for a finite sand-pile and resting on a rigid horizontal plane is different to that examined here, but nevertheless we might expect the two force distributions to be similar. If we use the solutions for gravity flow of granular materials in converging wedges assuming that the entire sand-pile is at yield, then from Hill & Cox (2000), for the special case of an angle of internal friction equal to ninety degrees, we determine a formal exact parametric solution which coincides with the full numerical solution. In fact it is only for this special case that it is possible to determine a numerical solution which satisfies all the necessary boundary conditions. Accordingly, here we propose a sand-pile that is made up of two regions, an inner dead region and an outer yield region. For two-dimensional sand-piles we find from a full numerical solution for the outer region that the solution actually follows a known simple exact solution of the governing equations for which the stresses are linear in both x and y. From this we then deduce a possible stress distribution in the inner dead region that is also linear in both x and y and which satisfies the equilibrium equations. We note that the stress distribution in the inner region satisfies the strict inequality of the Coulomb-Mohr yield condition which means that the material in the inner region is not at yield and also that the stresses are not uniquely determined. However, we find that the resulting force distribution does have the essential profile of the M-shaped curves of the experimental curves of Smid & Novosad (1981). For three-dimensional sand-piles a similar situation applies except that in the outer plastic region we are only able to determine a numerical solution, while in the inner dead region we assume stresses involving quintic polynomial expressions in $\sin\Theta$ and $\cos\Theta$ which we subsequently show determines a stress distribution that has the essential M-shaped profile.

In the following section we briefly state the basic equations of the continuum theory of granular materials and we give expressions for the horizontal and vertical force resultants. As previously mentioned we follow the notation of Spencer & Bradley (1996), noting again that in our problem gravity acts in the opposite direction and the x-axis denotes the vertical direction. As shown in Hill & Cox (2000) additional boundary relations can be derived on the assumption that the derivative of the stress angle ψ defined by (4) at the surface of the pile remains finite. However, for a sand-pile which is entirely at yield, a numerical scheme making use of these additional relations indicates that for two-dimensional sand-piles no solution exists, and therefore we need to focus on the case when this derivative becomes infinite. Hill & Cox (2000) derive a formal exact solution of the governing equations for two-dimensional sand piles that are entirely at yield for the special case of an angle of internal friction equal to ninety degrees and it turns out that only for this special case does the formulated problem admit a solution. Accordingly in this paper we consider sand-piles that have an inner dead region and an outer yield region, and we determine solutions that exhibit the same behaviour as the experimental data shown in Smid & Novosad (1981). In the final section of the paper we show graphically numerically determined stress solutions for the model and we compare the force on the base with the experimental force distribution curves.

2 BASIC EQUATIONS OF CONTINUUM THEORY

In the following two subsections we briefly state the two and three dimensional basic equations for continuum theory.

2.1 *Two dimensional basic equations*

In cylindrical polar coordinates (r, θ, z) as defined in Figure 1, the stress components for quasi-static flow satisfy the equilibrium equations

$$
\begin{aligned}
\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} &= -\rho g \cos\theta, \\
\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{2\sigma_{r\theta}}{r} &= \rho g \sin\theta,
\end{aligned}
\tag{1}
$$

where ρ is the density, g is acceleration due to gravity and $\sigma_{rr}, \sigma_{\theta\theta}$, and $\sigma_{r\theta}$ denote the in-plane physical stress components. Following Spencer & Bradley (1996) these components can be expressed in the standard form

$$
\sigma_{rr} = -p + q\cos 2\psi, \quad \sigma_{\theta\theta} = -p - q\cos 2\psi, \quad \sigma_{r\theta} = q\sin 2\psi, \tag{2}
$$

where p and q are defined to be

$$
p = -\frac{1}{2}\left(\sigma_{rr} + \sigma_{\theta\theta}\right), \quad q = \frac{1}{2}\left\{\left(\sigma_{rr} - \sigma_{\theta\theta}\right)^2 + 4\sigma_{r\theta}^2\right\}^{1/2}, \tag{3}
$$

while ψ is defined by

$$
\tan 2\psi = \frac{2\sigma_{r\theta}}{\left(\sigma_{rr} - \sigma_{\theta\theta}\right)}, \tag{4}
$$

and physically ψ is the angle between the maximum principal stress axis and the radial direction, in the direction of increasing θ. For a cohesionless material, the Coulomb-Mohr yield condition takes the form

$$
q = \beta p, \tag{5}
$$

where $\beta = \sin\phi$ and ϕ is a material constant referred to as the angle of internal friction.

Following Jenike (1964) and Spencer & Bradley (1996) we examine solutions of the form

$$
\psi = \psi(\theta), \quad q = -\rho g r F(\theta), \tag{6}
$$

and from the above equations we may deduce

$$
\begin{aligned}
\frac{\mathrm{d}F}{\mathrm{d}\theta} &= \frac{F\sin 2\psi + \beta\sin(\theta + 2\psi)}{\beta + \cos 2\psi}, \\
\frac{\mathrm{d}\psi}{\mathrm{d}\theta} + 1 &= \frac{F\left(\beta^{-1} - \beta\right) + \cos\theta + \beta\cos(\theta + 2\psi)}{2F(\beta + \cos 2\psi)}.
\end{aligned}
\tag{7}
$$

In order to determine the appropriate boundary conditions, we assume zero stress along the sand-pile slope, that is $\sigma_{r\theta} = \sigma_{\theta\theta} = 0$ at $\theta = \alpha$, which gives rise to the condition

$$
F(\alpha) = 0, \tag{8}
$$

where α denotes the semi-vertex angle. Secondly, we observe from the equilibrium equations (1) that if σ_{rr} and $\sigma_{\theta\theta}$ are assumed to be even functions of θ, then $\sigma_{r\theta}$must be an odd function or skew-symmetric and therefore $\sigma_{r\theta}$ must vanish at the origin giving rise to the second boundary condition

$$\psi(0) = 0. \tag{9}$$

Thus, we need to solve (7) subject to (8) and (9). However, in general this can only be attempted numerically. We note that for the special case of $\beta = 1$ a possible alternative boundary condition to (8) is $\psi(\alpha) = \pm\pi/2$, which is sufficient in the special case to guarantee zero stress on the surface, and also indicates possible non-uniqueness in the decomposition (2). Alternatively, if we eliminate F from (7), we may deduce the following second order differential equation for $\psi(\theta)$,

$$\begin{aligned}
&(\beta + \cos 2\psi)[\cos\theta + \beta\cos(2\psi + \theta)]\psi'' \\
&= 2(\psi' + 1)\left\{\sin 2\psi[\cos\theta + \beta\cos(2\psi + \theta)]\psi'\right. \\
&\left.-2\beta(\beta + \cos 2\psi)\sin(2\psi + \theta)\psi' - (3\beta^2 + 2\beta\cos 2\psi - 1)\sin(2\psi + \theta)\right\}.
\end{aligned} \tag{10}$$

From Hill & Cox (2000) the vertical and horizontal force distributions acting along a plane $x = \text{constant} = h$ are

$$\sigma_x = \rho g h \frac{F(\theta)}{\cos\theta}\left\{\frac{1}{\beta} - \cos 2[\theta + \psi(\theta)]\right\}, \quad \sigma_y = -\rho g h \frac{F(\theta)}{\cos\theta}\sin 2[\theta + \psi(\theta)], \tag{11}$$

respectively. We also note that (7) admits the special exact solution

$$\psi(\theta) = -\theta + \psi_0, \quad F(\theta) = -\frac{\beta}{(1 - \beta^2)}\{\cos\theta + \beta\cos(2\psi_0 - \theta)\}, \tag{12}$$

where ψ_0 is a constant, but in general, we observe that (12) can only satisfy one of the boundary conditions (8) or (9).

Finally, as shown in Hill & Cox (2000), equations (7), (8), and (9) for the special case of $\beta = 1$ admit the following exact parametric solution

$$\begin{aligned}
\tan\theta &= \frac{(2\pi)^{1/2}\tan\alpha}{2e^{-\lambda/2}\lambda^{-1/2} + J(\lambda)}, \\
\tan\psi(\theta) &= \frac{-J(\lambda)}{(2\pi)^{1/2}\tan\alpha}\left\{1 + \frac{\lambda^{1/2}}{2}e^{\lambda/2}J(\lambda)\right\} - \left(\frac{\pi\lambda}{2}\right)^{1/2} e^{\lambda/2}\tan\alpha, \\
F(\theta) &= -\frac{e^{\lambda/2}\left[2\pi\tan^2\alpha + J(\lambda)^2\right]}{4\lambda^{1/2}\left\{2\pi\tan^2\alpha + \left[2e^{-\lambda/2}\lambda^{-1/2} + J(\lambda)\right]^2\right\}},
\end{aligned} \tag{13}$$

where the parameter λ is such that, $0 \leq \lambda \leq \infty$ and $J(\lambda)$ is defined by

$$J(\lambda) = (2\pi)^{1/2}\text{erf}(\lambda/2)^{1/2}. \tag{14}$$

As described in Hill & Cox (2000) for a two-dimensional sand-pile entirely at yield, a bonafide solution of the formulated problem (7), (8) and (9) appears only to exist for this special angle of internal friction. For all other values of β at least one of the stated conditions appears not to be satisfied. Accordingly, we are lead to propose a model which incorporates an inner dead region.

2.2 *Three-dimensional basic equations*

For quasi-static axially symmetric flow, the stress components in a spherical polar coordinate system (R, Θ, Φ) satisfy the equilibrium equations

$$\begin{aligned} \frac{\partial \sigma_{RR}}{\partial R} + \frac{1}{R}\frac{\partial \sigma_{R\Theta}}{\partial \Theta} + \frac{1}{R}(2\sigma_{RR} - \sigma_{\Theta\Theta} - \sigma_{\Phi\Phi} + \sigma_{R\Theta}\cot\Theta) = -\rho g \cos\Theta, \\ \frac{\partial \sigma_{R\Theta}}{\partial R} + \frac{1}{R}\frac{\partial \sigma_{\Theta\Theta}}{\partial \Theta} + \frac{1}{R}(\sigma_{\Theta\Theta} - \sigma_{\Phi\Phi})\cot\Theta + \frac{3}{R}\sigma_{R\Theta} = \rho g \sin\Theta, \end{aligned} \tag{15}$$

where ρ is the density, g is the acceleration due to gravity and $\sigma_{RR}, \sigma_{\Theta\Theta}, \sigma_{\Phi\Phi}$ and $\sigma_{R\Theta}$ denote the physical components of stress. Assuming the 'Haar-von Karman' region for which two principal stresses coincide, we may deduce

$$\begin{aligned} \sigma_{RR} &= -p + q\cos 2\Psi, & \sigma_{\Theta\Theta} &= -p - q\cos 2\Psi, \\ \sigma_{R\Theta} &= q \sin 2\Psi, & \sigma_{\Phi\Phi} &= -p - q, \end{aligned} \tag{16}$$

where

$$p = -\tfrac{1}{2}(\sigma_{RR} + \sigma_{\Theta\Theta}), \quad q = \left\{\tfrac{1}{4}(\sigma_{RR} - \sigma_{\Theta\Theta})^2 + \sigma_{R\Theta}^2\right\}^{1/2}, \tag{17}$$

while Ψ is defined by

$$\tan 2\Psi = \frac{2\sigma_{R\Theta}}{(\sigma_{RR} - \sigma_{\Theta\Theta})}, \tag{18}$$

and physically Ψ is the angle between the direction of the maximum principal stress and the radial direction, measured in the direction of Θ increasing.

Following Jenike (1964) and Spencer & Bradley (1996) we examine solutions of the form

$$\Psi = \Psi(\Theta), \quad q = -\rho g R G(\Theta), \tag{19}$$

and from the above equations we can deduce

$$\begin{aligned} \frac{\mathrm{d}G}{\mathrm{d}\Theta} &= \frac{2G\sin\Psi\{\cos\Psi - \beta\,\mathrm{cosec}\Theta\sin(\Theta+\Psi)\} + \beta\sin(\Theta + 2\Psi)}{\beta + \cos 2\Psi}, \\ \frac{\mathrm{d}\Psi}{\mathrm{d}\Theta} + 1 &= \frac{G\{\beta^{-1} - 2\beta - 1 - (1+\beta)\mathrm{cosec}\Theta\sin(\Theta + 2\Psi)\} + \cos\Theta + \beta\cos(\Theta + 2\Psi)}{2G(\beta + \cos 2\Psi)}. \end{aligned} \tag{20}$$

Now for a symmetrical stress distribution and for zero stress along the sand-pile slope, we require the following conditions

$$\Psi(0) = 0, \quad G(\alpha) = 0, \tag{21}$$

where α denotes the semi-vertex angle. We observe from the equilibrium equations (15) that if σ_{RR}, $\sigma_{\Theta\Theta}$, and $\sigma_{\Phi\Phi}$ are assumed to be even functions of Θ, then $\sigma_{R\Theta}$ is necessarily an odd function or skew-symmetric and therefore $\sigma_{R\Theta}$ vanishes at the origin, and hence condition $(21)_1$. Thus we need to solve (20) subject to (21). In general this can only be attempted numerically.

Similarly from the previous section, we determine expressions for the vertical and horizontal force distributions along the plane Z = constant = h where $R = h \sec \Theta$ to be

$$\sigma_Z = \rho g h \frac{G(\Theta)}{\cos \Theta} \left\{ \frac{1}{\beta} - \cos 2[\Psi(\Theta) + \Theta] \right\}, \quad \sigma_r = -\rho g h \frac{G(\Theta)}{\cos \Theta} \sin 2[\Psi(\Theta) + \Theta], \tag{22}$$

where in this context r and Z denote the cylindrical polar radius and height as indicated in Figure 1.

3 TWO-DIMENSIONAL SOLUTION STRUCTURE INCORPORATING AN INNER DEAD REGION

In this section we assume that there is an inner dead region for $0 \leq \theta \leq \gamma$, and an outer yield region for $\gamma \leq \theta \leq \alpha$, where $\theta = \gamma$ is the boundary between the two regions as shown in Figure 1. In the following two subsections we examine the two regions within the sand-pile.

3.1 *Outer yield region*

In the outer yield region of the sand-pile, the stresses are at equilibrium and also at yield. This means that the stresses satisfy the equilibrium equations (1) and the Coulomb-Mohr yield condition (5), and therefore the governing equations (7). Again, we assume that there is zero stress on the slope of the sand-pile which provides the boundary condition for $F(\alpha)$ of (8), and upon assuming that $\psi'(\alpha)$ is finite we see from Hill & Cox (2000) that $\psi(\alpha)$ is given by

$$\psi(\alpha) = -\frac{1}{2} \left\{ \pi + \alpha - \cos^{-1} \left(\frac{\cos \alpha}{\beta} \right) \right\}, \tag{23}$$

and remarkably, that $\psi'(\alpha)$ is given by the simple result

$$\psi'(\alpha) = -1. \tag{24}$$

This gives us two boundary conditions at $\theta = \alpha$ from which we can numerically solve (7) backwards towards $\theta = \gamma$.

In the final section we find that the numerical solution of (7) subject to (8) and (23) remarkably turns out to be the simple exact solution (12), where the constant ψ_0 is determined from $(12)_1$ and (23). This means we know the exact form of the solution in the outer yield region, and from (2), (5), (6) and (12) we find that the cylindrical polar stresses $\sigma_{rr}, \sigma_{\theta\theta}$, and $\sigma_{r\theta}$ become

$$\sigma_{rr} = -\frac{\rho g r \left[1 - \beta \cos 2(\psi_0 - \theta)\right] \left[\cos\theta + \beta \cos(2\psi_0 - \theta)\right]}{(1-\beta^2)},$$

$$\sigma_{\theta\theta} = -\frac{\rho g r \left[1 + \beta \cos 2(\psi_0 - \theta)\right] \left[\cos\theta + \beta \cos(2\psi_0 - \theta)\right]}{(1-\beta^2)}, \tag{25}$$

$$\sigma_{r\theta} = \frac{\rho g r \beta \sin 2(\psi_0 - \theta) \left[\cos\theta + \beta \cos(2\psi_0 - \theta)\right]}{(1-\beta^2)},$$

which from (11) or Hunter (1983, page 102), gives

$$\sigma_{xx} = -\frac{\rho g \left[1 - \beta \cos 2\psi_0\right] \left[(1 + \beta \cos 2\psi_0)x + \beta y \sin 2\psi_0\right]}{(1-\beta^2)},$$

$$\sigma_{yy} = -\frac{\rho g \left[1 + \beta \cos 2\psi_0\right] \left[(1 + \beta \cos 2\psi_0)x + \beta y \sin 2\psi_0\right]}{(1-\beta^2)}, \tag{26}$$

$$\sigma_{xy} = -\frac{\rho g \beta \sin 2\psi_0 \left[(1 + \beta \cos 2\psi_0)x + \beta y \sin 2\psi_0\right]}{(1-\beta^2)}.$$

In the following subsection we assume that the stresses are not at yield in an inner dead region, but satisfy the equilibrium equations and remain continuous across the boundary at $\theta = \gamma$, where γ is yet to be determined.

3.2 *Inner dead region*

In the inner dead region of the sand-pile, we assume that the stresses satisfy the equilibrium equations, but not the equality of the Coulomb-Mohr yield condition (5). The stresses are not uniquely determined but must satisfy the strict inequality

$$q < \beta p, \tag{27}$$

where p and q are defined by (3). Now, we need to make an assumption about the form of the stresses. From (26) we see that the stresses in Cartesian coordinates in the outer yield region are linear in both x and y, and therefore we assume that the stresses in the inner dead region are also linear in both x and y so that

$$\sigma_{xx} = -\rho g(Ax + By), \quad \sigma_{yy} = -\rho g(Cx + Dy), \quad \sigma_{xy} = \rho g(Ex + Gy), \tag{28}$$

where A, B, C, D, E, and G denote constants. We note that the equilibrium equations (1) in Cartesian coordinates become

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = -\rho g, \quad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0, \tag{29}$$

so that the stresses defined by (28) satisfy

$$E = D, \quad G = A - 1. \tag{30}$$

Across the the boundary $\theta = \gamma$ between the two regions, we require that both $\sigma_{r\theta}$ and $\sigma_{\theta\theta}$ remain continuous. Therefore, from (28), (30), and Hunter (1983) we find

$$\begin{aligned} \sigma_{rr} &= -\rho g r \left\{ \left[A\cos^2\theta + C\sin^2\theta - D\sin 2\theta \right]\cos\theta \right. \\ &\qquad \left. + \left[B\cos^2\theta + D\sin^2\theta - (A-1)\sin 2\theta \right]\sin\theta \right\}, \\ \sigma_{\theta\theta} &= -\rho g r \left\{ \left[A\sin^2\theta + C\cos^2\theta + D\sin 2\theta \right]\cos\theta \right. \\ &\qquad \left. + \left[B\sin^2\theta + D\cos^2\theta + (A-1)\sin 2\theta \right]\sin\theta \right\}, \\ \sigma_{r\theta} &= \rho g r \left\{ \left[(A-C)\sin\theta\cos\theta + D\cos 2\theta \right]\cos\theta \right. \\ &\qquad \left. + \left[(B-D)\sin\theta\cos\theta + (A-1)\cos 2\theta \right]\sin\theta \right\}, \end{aligned} \tag{31}$$

and upon equating $(25)_2$ with $(31)_2$ and $(25)_3$ with $(31)_3$ at $\theta = \gamma$, we determine expressions for C and D in terms of A and B, namely

$$\begin{aligned} C &= \frac{\left[\cos\gamma + \beta\cos(2\psi_0 - \gamma)\right]\left[1 - 3\sin^2\gamma + \beta\cos 2\psi_0 - \beta\sin\gamma\sin(2\psi_0 - \gamma)\right]}{(1-\beta^2)\cos^3\gamma} \\ &\quad + 3A\tan^2\gamma + 2B\tan^3\gamma - \tan^2\gamma, \\ D &= \frac{\left[\cos\gamma + \beta\cos(2\psi_0 - \gamma)\right]\left[\sin\gamma + \beta\sin(2\psi_0 - \gamma)\right]}{(1-\beta^2)\cos^2\gamma} \\ &\quad -2A\tan\gamma - B\tan^2\gamma + \tan\gamma. \end{aligned} \tag{32}$$

Now in order to determine A and B, we assume that σ_{xx} and σ_{xy} are continuous at $\theta = \gamma$, and hence, from comparing $(26)_1$ with $(28)_1$ we find

$$B = \frac{\left[1 - \beta\cos 2\psi_0\right]\left[\cos\gamma + \beta\cos(2\psi_0 - \gamma)\right]}{(1-\beta^2)\sin\gamma} - A\cot\gamma, \tag{33}$$

and then from comparing $(26)_3$ with $(28)_3$ at $\theta = \gamma$, noting (30) and (32), we find

$$A = \frac{\left[1 - \beta\cos 2\psi_0\right]\left[\cos\gamma + \beta\cos(2\psi_0 - \gamma)\right]}{(1-\beta^2)\cos\gamma} - B\tan\gamma, \tag{34}$$

and we note that the two equations (33) and (34) coincide. In order to ensure that $\sigma_{r\theta}$ is an odd function or skew-symmetric we choose B such that $\sigma_{r\theta} = 0$ at $\theta = 0$, and then from (32) and (34) we find

$$\begin{aligned} A &= \frac{\left[1 + \beta\cos 2\psi_0\right]}{(1-\beta^2)}\left[1 + \frac{\beta\sin(2\psi_0 - \gamma)}{\sin\gamma}\right], \\ B &= \frac{\left[\sin\gamma - \beta\sin(2\psi_0 + \gamma)\right]\left[\cos\gamma + \beta\cos(2\psi_0 - \gamma)\right]}{(1-\beta^2)\sin^2\gamma} - \cot\gamma, \\ C &= \frac{\left[1 + \beta\cos 2\psi_0\right]}{(1-\beta^2)}\left[1 + \frac{\beta\cos(2\psi_0 - \gamma)}{\cos\gamma}\right], \end{aligned} \tag{35}$$

(a). (b). (c).

Figure 2: Typical variation of $\sigma_{rr}, \sigma_{r\theta}$, and $\sigma_{\theta\theta}$ for two-dimensional sand-pile with an inner dead region and an outer yield region for $\beta = \cos\alpha$ and $\gamma = 0.382$. ((a) σ_{rr}, (b) $\sigma_{r\theta}$, and (c) $\sigma_{\theta\theta}$).

and $D = 0$. We note that assuming the stresses $\sigma_{\theta\theta}, \sigma_{r\theta}, \sigma_{xx}$, and σ_{xy} are continuous across the boundary at $\theta = \gamma$, ensures that both σ_{rr} and σ_{yy} are also continuous at $\theta = \gamma$. However, from Figure 2 we see that while the derivatives of $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$ are continuous across the boundary, the derivative of σ_{rr} is discontinuous. We note that if we alternatively determine B so that the derivative of σ_{rr} is continuous across the boundary at $\theta = \gamma$, then we obtain a stress distribution that has $\sigma_{r\theta}$ being an even function. However, as previously stated we require $\sigma_{r\theta}$ to be an odd function to satisfy the equilibrium equations (1), because both σ_{rr} and $\sigma_{\theta\theta}$ are even functions with respect to θ. This means that if we assume all the stresses and their derivatives are continuous across the boundary $\theta = \gamma$ then the full equality of the yield condition (5) holds in the dead region. In other words, there is no dead region as the inner region is also at yield. Thus, we need to assume that the derivative of σ_{rr} remains discontinuous across the boundary at $\theta = \gamma$.

Now, we attempt to determine a value of γ, which is not unique but which is such that the stresses satisfy the strict inequality (27) throughout the entire inner dead region. Numerical results indicate that if the strict inequality (27) is satisfied at $\theta = 0$, then it is always satisfied throughout the entire inner dead region. This is shown in Figure 3, where for $\gamma = 0.37$ and $\gamma = 0.39$ we have plotted the LHS (——) of the inequality (27) versus the RHS ($\cdots$) of the inequality (27). The value $\gamma = 0.37$ is such that the inequality (27) is not satisfied at $\theta = 0$ and elsewhere, whereas for $\gamma = 0.39$, we see that the inequality (27) is satisfied at $\theta = 0$ and also throughout the entire inner dead region which means that the value $\gamma = 0.37$ is too small but $\gamma = 0.39$ is sufficiently large to ensure the inequality is satisfied throughout the entire dead region. Therefore, upon considering the inequality (27) at $\theta = 0$, where we have used the Cartesian stresses (28) with constants A, B, C, D, E, and G defined by (30) and (35), we may deduce the inequality

$$\sin^2(2\psi_0 - 2\gamma) < [\sin 2\gamma + \beta \sin 2\psi_0]^2, \qquad (36)$$

Figure 3: Variation of the LHS (——) of the inequality (27) versus the RHS ($\cdots$) of the inequality (27) for the two values of $\gamma = 0.37$ and $\gamma = 0.39$ showing that the value $\gamma = 0.37$ is too small whereas the value 0.39 is sufficient.

and upon assuming that $\sin(2\psi_0 - 2\gamma) < 0$, we obtain the inequality

$$\gamma > \frac{1}{2}\left[\psi_0 + \pi - \cos^{-1}(\beta \cos\psi_0)\right]. \tag{37}$$

However, of course (37) does not give a unique value of γ, but only determines a range for γ which ensures that the inequality (27) is satisfied. Here we adopt the smallest value of γ which satisfies (37). In the following section, we display graphically the stress distributions for the three proposed models for a two-dimensional sand-pile.

4 THREE-DIMENSIONAL SOLUTION STRUCTURE INCORPORATING AN INNER DEAD REGION

In this section, for three-dimensional sand-piles, we assume that there exists an inner dead region for $0 \leq \Theta \leq \gamma$, and an outer yield region for $\gamma \leq \Theta \leq \alpha$, where $\Theta = \gamma$ is the boundary between the two regions as shown in Figure 1. In the following two subsections we provide details for the two regions.

4.1 *Outer yield region*

In the outer yield region of the three-dimensional sand-pile, the stresses are at equilibrium and also at yield. This means that the stresses satisfy the equilibrium equations (15) and the Coulomb-Mohr yield condition (5), and therefore the governing equations (20). Again, we assume that there is zero stress on the slope of the sand-pile which provides the boundary condition for $G(\alpha)$ of $(21)_2$, and upon assuming that $\Psi'(\alpha)$ is finite we see that $\Psi(\alpha)$ is given by

$$\Psi(\alpha) = -\frac{1}{2}\left\{\pi + \alpha - \cos^{-1}\left(\frac{\cos\alpha}{\beta}\right)\right\}, \tag{38}$$

which gives us two boundary conditions at $\Theta = \alpha$ from which we can numerically solve (20) backwards towards $\Theta = \gamma$.

In the following section we find that the numerical solution of (20) subject to $(21)_2$ and (38) does not remain continuous throughout the entire region $0 \leq \Theta \leq \alpha$. Instead, the solution encounters a singularity, in the sense that the denominators of (20) become zero. This means there is a value of Θ which Ψ satisfies the equation

$$\cos 2\Psi + \beta = 0, \tag{39}$$

and we denote this value by $\Theta = \gamma$. In other words, the location of the boundary between the inner dead region and the outer yield region is where the numerical solution first encounters the singularity defined by (39), when (20) is solved backwards from $\Theta = \alpha$, and this defines the angle γ.

In order to obtain a continuous solution throughout the entire sand-pile, we need to know the value of the outer yield stresses at the boundary $\Theta = \gamma$. We know that γ is determined numerically, and that $\Psi(\gamma)$ is defined by (39), so that if we assume that $\Psi(\gamma)$ is the first negative root of (39), then

$$\Psi(\gamma) = -\frac{\phi}{2} - \frac{\pi}{4}, \tag{40}$$

recalling that $\beta = \sin\phi$. Now, assuming that $G'(\gamma)$ and $\Psi'(\gamma)$ are finite, we see that (20) at $\Theta = \gamma$, yields

$$\begin{aligned} G(\gamma)\sin 2\Psi(\gamma) - 2\sin\phi G(\gamma)\mathrm{cosec}\gamma \sin\Psi(\gamma)\sin[\gamma + \Psi(\gamma)] \\ + \sin\phi\sin[\gamma + 2\Psi(\gamma)] = 0, \\ (1+\sin\phi)G(\gamma)\left\{2 - \mathrm{cosec}\phi - \mathrm{cosec}\gamma\sin[\gamma + 2\Psi(\gamma)]\right\} \\ + \cos\gamma + \sin\phi\cos[\gamma + 2\Psi(\gamma)] = 0, \end{aligned} \tag{41}$$

and from (40), we find that both $(41)_1$ and $(41)_2$ give

$$G(\gamma) = \frac{\sin\phi\sin\gamma\cos(\gamma-\phi)}{\sin\phi\sin(\gamma-\phi) - \sin(\gamma+\phi)}. \tag{42}$$

Accordingly, γ is analytical indeterminant from this approach. Now, from (5), (16), (19), (40), and (42) we determine expressions for the stresses $\sigma_{RR}, \sigma_{R\Theta}, \sigma_{\Theta\Theta}$, and $\sigma_{\Phi\Phi}$ at $\Theta = \gamma$ to be

$$\begin{aligned} \sigma_{RR}(R,\gamma) &= \rho g R\frac{(1+\sin^2\phi)\sin\gamma\cos(\gamma-\phi)}{\sin\phi\sin(\gamma-\phi) - \sin(\gamma+\phi)}, \\ \sigma_{\Theta\Theta}(R,\gamma) &= \rho g R\frac{\cos^2\phi\sin\gamma\cos(\gamma-\phi)}{\sin\phi\sin(\gamma-\phi) - \sin(\gamma+\phi)}, \\ \sigma_{R\Theta}(R,\gamma) &= \rho g R\frac{\sin\phi\cos\phi\sin\gamma\cos(\gamma-\phi)}{\sin\phi\sin(\gamma-\phi) - \sin(\gamma+\phi)}, \\ \sigma_{\Phi\Phi}(R,\gamma) &= \rho g R\frac{(1+\sin\phi)\sin\gamma\cos(\gamma-\phi)}{\sin\phi\sin(\gamma-\phi) - \sin(\gamma+\phi)}, \end{aligned} \tag{43}$$

and (43) enables us to ensure that the solution is continuous across the boundary at $\Theta = \gamma$. We also find using (15) and (43) that the derivatives of $\sigma_{R\Theta}$ and $\sigma_{\Theta\Theta}$ with respect to Θ at $\Theta = \gamma$, will be continuous across the boundary between the inner dead region and the outer yield region. To ensure that all stresses have continuous derivatives across the boundary, we need to determine the value of the derivatives of the stresses in the outer yield region at the boundary. We first need to determine the finite values of $\Psi'(\gamma)$ and $G'(\gamma)$ in order to determine the derivatives of the stresses defined by (16). At $\Theta = \gamma$, we have determined that $\Psi(\gamma)$ is given by (40) and $G(\gamma)$ is given by (42), which means that the numerators and the denominators of both equations (20) vanish and therefore from ℓ'Hopitals rule, (20) gives

$$\begin{aligned} & G'\left[2\Psi'\cos\phi + \mathrm{cosec}\gamma\sin(\gamma+\phi) - \sin\phi\,\mathrm{cosec}\gamma\sin(\gamma-\phi)\right] \\ & = \sin\phi\sin(\gamma-\phi)(1+2\Psi') - 2G\Psi'\sin\phi\,\mathrm{cosec}\gamma[\sin\gamma - \cos(\gamma-\phi)] + (1+\sin\phi)G\sin\phi\,\mathrm{cosec}^2\gamma, \\ & (1+\sin\phi)G'\left[2 - \mathrm{cosec}\phi - \mathrm{cosec}\gamma\cos(\gamma-\phi)\right] \\ & = -4G\Psi'^2\cos\phi - 2\Psi'[2G\cos\phi + (1+\sin\phi)G\,\mathrm{cosec}\gamma\sin(\gamma-\phi) - \sin\phi\cos(\gamma-\phi)] \\ & -(1+\sin\phi)G\cos\phi\,\mathrm{cosec}^2\gamma - \cos\phi\sin(\gamma-\phi), \end{aligned} \tag{44}$$

where G is defined by (42). Upon eliminating G' from (44), we find that Ψ' satisfies the cubic equation

$$\begin{aligned} & -8G\cos^2\phi\Psi'^3 - 4\cos\phi\Psi'^2[4G\cos\phi - \sin\phi\cos(\gamma-\phi)] \\ & +2\Psi'\left[(1+\sin\phi)G\,\mathrm{cosec}^2\gamma\left\{2\cos^2\gamma - 4\cos\phi\sin\gamma\cos\gamma - 5\cos^2\phi\right.\right. \\ & \left. +(1-\sin\phi)(2\cos^2\gamma + 3\sin\phi + 5\sin\gamma\cos(\gamma-\phi))\right\} \\ & \left. +(1+\sin\phi)\cos\gamma - (1-\sin\phi)\cos\phi\cos(\gamma-\phi)\right] = 0, \end{aligned} \tag{45}$$

and obviously, one solution is $\Psi'(\gamma) = 0$. The non-trivial solutions are given as the roots of a quadratic, thus

$$\begin{aligned} \Psi'(\gamma) = \frac{-1}{4G\cos\phi}\Big\{ & 4G\cos\phi - \sin\phi\cos(\gamma-\phi) \pm \Big[2(1+\sin\phi)G^2\mathrm{cosec}^2\gamma\big[6\sin\phi\sin^2\gamma \\ & +2\cos\phi\sin\gamma\cos\gamma - 2(1-\sin\phi+3\sin^2\phi) - 10\sin\phi\cos\gamma\sin(\gamma-\phi)\big] \\ & -4(1+\sin\phi)G\sin\phi\sin(\gamma-\phi) + \sin^2\phi\cos^2(\gamma-\phi)\Big]^{1/2}\Big\}, \end{aligned} \tag{46}$$

where the numerical results presented in the following section indicate that $\Psi'(\gamma)$ takes the + sign in (46). Now, upon substituting (46) into $(44)_2$ and using MAPLE, we find that $G'(\gamma)$ is given by

$$
\begin{aligned}
G'(\gamma) = &-\{2(1+\sin\phi)G\mathrm{cosec}^2\gamma\,[\cos\phi\sin\gamma\cos\gamma - 3\sin\phi\cos\gamma[\cos\gamma+\sin(\gamma-\phi)] \\
&+4\sin\phi(1-\sin\phi) - 2\sin(\gamma-\phi)\sin(\gamma+\phi)] + \sin(\gamma+\phi) \\
&+(1-2\sin\phi-2\sin^2\phi)\sin(\gamma-\phi) + (1+\sin\phi)\sin\phi\mathrm{cosec}\gamma\sin(\gamma-\phi)\cos(\gamma-\phi)\Big\} \\
&/\{2\cos\phi(1+\sin\phi)[2-\mathrm{cosec}\phi-\mathrm{cosec}\gamma\cos(\gamma-\phi)]\}\,. \\
&\pm\frac{2\cos\phi-(1+\sin\phi)\mathrm{cosec}\gamma\sin(\gamma-\phi)}{2\cos\phi(1+\sin\phi)[2-\mathrm{cosec}\phi-\mathrm{cosec}\gamma\cos(\gamma-\phi)]}\Big[\sin^2\phi\cos^2(\gamma-\phi) \\
&+2(1+\sin\phi)G^2\mathrm{cosec}^2\gamma\Big[6\sin\phi\sin^2\gamma + 2\cos\phi\sin\gamma\cos\gamma - 2(1-\sin\phi+3\sin^2\phi) \\
&-10\sin\phi\cos\gamma\sin(\gamma-\phi)] - 4(1+\sin\phi)G\sin\phi\sin(\gamma-\phi)\Big]^{1/2},
\end{aligned}
\tag{47}
$$

where the $\pm$ sign in (47) corresponds to the $\pm$ sign in (46), respectively. Now, from (5), (16), and (17) we find that the derivatives of the stresses in the outer yield region at the boundary $\Theta = \gamma$ are given by

$$
\begin{aligned}
\frac{\partial\sigma_{RR}}{\partial\Theta} &= \rho g R\,\{G'(\gamma)[\mathrm{cosec}\phi+\sin\phi] - 2\cos\phi G(\gamma)\Psi'(\gamma)\}\,, \\
\frac{\partial\sigma_{\Theta\Theta}}{\partial\Theta} &= \rho g R\,\{G'(\gamma)[\mathrm{cosec}\phi-\sin\phi] + 2\cos\phi G(\gamma)\Psi'(\gamma)\}\,, \\
\frac{\partial\sigma_{R\Theta}}{\partial\Theta} &= \rho g R\,\{\cos\phi G'(\gamma) + 2\sin\phi G(\gamma)\Psi'(\gamma)\}\,, \\
\frac{\partial\sigma_{\Phi\Phi}}{\partial\Theta} &= \rho g R G'(\gamma)(1+\mathrm{cosec}\phi),
\end{aligned}
\tag{48}
$$

where $G(\gamma)$ is given by (42), $\Psi'(\gamma)$ is given by (46), and $G'(\gamma)$ is given by (47). In the following subsection we assume that the stresses are not at yield in an inner dead region, but satisfy only the equilibrium equations and remain continuous across the boundary at $\Theta = \gamma$.

4.2 *Inner dead region*

In the inner dead region of the sand-pile, we assume that the stresses satisfy only the equilibrium equations (15), but not the equality of the Coulomb-Mohr yield condition (5). Accordingly, the stresses are not uniquely determined but must satisfy the strict inequality (27) and we need to make an assumption about the form of the stresses. We assume that the non-zero spherical polar stresses are quintic expressions involving

$\sin\Theta$ and $\cos\Theta$ of the form

$$
\begin{aligned}
\sigma_{RR} &= \rho g R \Big\{ c_1 \sin^5\Theta + c_2 \sin^4\Theta\cos\Theta + c_3 \sin^3\Theta\cos^2\Theta \\
&\qquad + c_4 \sin^2\Theta\cos^3\Theta + c_5 \sin\Theta\cos^4\Theta + c_6\cos^5\Theta \Big\}, \\
\sigma_{\Theta\Theta} &= \rho g R \Big\{ c_7 \sin^5\Theta + c_8 \sin^4\Theta\cos\Theta + c_9 \sin^3\Theta\cos^2\Theta \\
&\qquad + c_{10} \sin^2\Theta\cos^3\Theta + c_{11} \sin\Theta\cos^4\Theta + c_{12}\cos^5\Theta \Big\}, \\
\sigma_{R\Theta} &= \rho g R \Big\{ c_{13} \sin^5\Theta + c_{14} \sin^4\Theta\cos\Theta + c_{15} \sin^3\Theta\cos^2\Theta \\
&\qquad + c_{16} \sin^2\Theta\cos^3\Theta + c_{17} \sin\Theta\cos^4\Theta + c_{18}\cos^5\Theta \Big\}, \\
\sigma_{\Phi\Phi} &= \rho g R \Big\{ c_{19} \sin^5\Theta + c_{20} \sin^4\Theta\cos\Theta + c_{21} \sin^3\Theta\cos^2\Theta \\
&\qquad + c_{22} \sin^2\Theta\cos^3\Theta + c_{23} \sin\Theta\cos^4\Theta + c_{24}\cos^5\Theta \Big\},
\end{aligned}
\tag{49}
$$

where c_i is a constant, for $i = 1, \dots, 24$. Now, upon substituting (49) into the equilibrium equations (15), we find

$$
\begin{aligned}
&(3c_1 - c_7 - c_{14} - c_{19})\sin^6\Theta + (3c_2 - c_8 + 6c_{13} - 2c_{15} - c_{20} + 1)\sin^5\Theta\cos\Theta \\
&+(3c_3 - c_9 + 5c_{14} - 3c_{16} - c_{21})\sin^4\Theta\cos^2\Theta \\
&+(3c_4 - c_{10} + 4c_{15} - 4c_{17} - c_{22} + 2)\sin^3\Theta\cos^3\Theta \\
&+(3c_5 - c_{11} + 3c_{16} - 5c_{18} - c_{23})\sin\Theta^2\cos^4\Theta \\
&+(3c_6 - c_{12} + 2c_{17} - c_{24} + 1)\sin\Theta\cos^5\Theta + c_{18}\cos^6\Theta = 0, \\
\\
&(-c_8 + 4c_{13} - 1)\sin^6\Theta + (6c_7 - 2c_9 + 4c_{14} - c_{19})\sin^5\Theta\cos\Theta \\
&+(5c_8 - 3c_{10} + 4c_{15} - c_{20} - 2)\sin^4\Theta\cos^2\Theta + (4c_9 - 4c_{11} + 4c_{16} - c_{21})\sin^3\Theta\cos^3\Theta \\
&+(3c_{10} - 5c_{12} + 4c_{17} - c_{22} - 1)\sin\Theta^2\cos^4\Theta + (2c_{11} + 4c_{18} - c_{23})\cos\Theta\cos^5\Theta \\
&+(c_{12} - c_{24})\cos^6\Theta = 0,
\end{aligned}
\tag{50}
$$

which is certainly satisfied provided each of the coefficients vanish and from which we may deduce

$$\begin{aligned}
&c_6 = -\frac{1}{12}(2c_2 + 3c_4 - 3c_8 - 2c_{10} - 3c_{12} + 10), \quad c_9 = \frac{1}{3}(3c_1 + 3c_3 + 7c_5 - 7c_7 - 3c_{11}),\\
&c_{13} = \frac{1}{4}(c_8 + 1), \quad c_{14} = \frac{1}{15}(15c_1 + 10c_3 + 14c_5 - 35c_7 - 6c_{11}),\\
&c_{15} = \frac{1}{4}(2c_2 - 3c_8 + 2c_{10} + 3), \quad c_{16} = c_{11} - c_5,\\
&c_{17} = \frac{1}{8}(2c_2 + 3c_4 - 3c_8 - 2c_{10} + 5c_{12} + 6), \quad c_{18} = 0,\\
&c_{19} = \frac{1}{15}(30c_1 - 6c_3 - 14c_5 + 20c_7 + 6c_{11}), \quad c_{20} = 2c_2 + 2c_8 - c_{10} + 1,\\
&c_{21} = \frac{1}{3}(12c_1 + 12c_3 + 16c_5 - 28c_7 - 12c_{11}), \quad c_{22} = \frac{1}{2}(2c_2 + 3c_4 - 3c_8 + 4c_{10} - 5c_{12} + 4),\\
&c_{23} = 2c_{11}, \quad c_{24} = c_{12}.
\end{aligned} \tag{51}$$

Now, in order to determine the remaining unknown constants, we firstly assume that the stresses are continuous throughout the entire sand-pile. This means that we need to ensure that the stress solution from the outer yield region into the inner dead region remains continuous at the boundary $\Theta = \gamma$. Therefore, we equate (43) with (49) at

$\Theta = \gamma$, and using MAPLE we find

$$c_1 = -\frac{H}{6\sin^5\gamma\cos\gamma}\Bigg\{ 50\cos\phi\cos^2\gamma\cos(\gamma-\phi) + 23\cos\gamma(\sin^2\phi - \cos^2\gamma)$$

$$-7\sin\phi\cos\phi\sin\gamma - 7\cos\gamma(\cos^2\gamma + \sin\phi) + \cos^2\phi\cos\gamma\Bigg\} + \frac{5c_2\cos\gamma(3\cos^2\gamma - 2)}{24\sin^3\gamma}$$

$$+\frac{5c_4\cos^3\gamma}{16\sin^3\gamma} - \frac{c_8(3\cos^4\gamma - 2\cos^2\gamma + 14)}{48\sin^3\gamma\cos\gamma} + \frac{c_{10}\cos\gamma(9\cos^2\gamma + 14)}{24\sin^3\gamma}$$

$$+\frac{c_{11}\cos^2\gamma(3\cos^2\gamma + 4)}{3\sin^4\gamma} + \frac{7c_{12}\cos^3\gamma(3\cos^2\gamma + 5)}{16\sin^5\gamma} + \frac{15\cos^4\gamma + 15\cos^2\gamma - 7}{24\sin^3\gamma\cos\gamma},$$

$$c_3 = \frac{H}{6\sin^4\gamma\cos^3\gamma}\Bigg\{ 28\sin\phi\cos\gamma\cos(\gamma-\phi) + 32\cos\phi\sin\gamma\cos^2\gamma\cos(\gamma-\phi) \quad (52)$$

$$-7\sin\phi\sin\gamma\cos\gamma - 6\cos^2\gamma\cos(\gamma-\phi)\sin(\gamma+\phi) + 7\cos(\gamma+\phi)\sin(\gamma-\phi)$$

$$-3\left(1+\sin\phi + 2\sin^2\phi\right)\Bigg\} - \frac{c_2(8\cos^4\gamma - 17\cos^2\gamma + 14)}{24\sin^3\gamma\cos\gamma} + \frac{c_4\cos\gamma(16\cos^2\gamma - 21)}{16\sin^3\gamma}$$

$$+\frac{c_8(11\cos^4\gamma - 10\cos^2\gamma + 14)}{48\sin^3\gamma\cos^3\gamma} + \frac{c_{10}(20\cos^4\gamma - \cos^2\gamma - 14)}{24\sin^3\gamma\cos\gamma} - \frac{2c_{11}(3\cos^2\gamma + 2)}{3\sin^2\gamma}$$

$$-\frac{5c_{12}\cos\gamma(8\cos^2\gamma + 7)}{16\sin^3\gamma} + \frac{20\cos^6\gamma - 3\cos^4\gamma - 19\cos^2\gamma + 7}{24\sin^3\gamma\cos^3\gamma},$$

and also

$$c_5 = -\frac{H}{2\sin^2\gamma\cos^3\gamma}\Bigg\{ 4\sin\phi\cos\gamma\cos(\gamma-\phi) + 2\sin\phi\cos\phi\cos^2\gamma - \sin\phi\sin\gamma\cos\gamma$$

$$+\cos(\gamma+\phi)\sin(\gamma-\phi)\Bigg\} + \frac{c_2(\cos^2\gamma + 2)}{8\sin\gamma\cos\gamma} + \frac{9c_4\cos\gamma}{16\sin\gamma} - \frac{c_8(5\cos^4\gamma + 2\cos^2\gamma + 2)}{16\sin\gamma\cos^3\gamma}$$

$$-\frac{c_{10}(5\cos^2\gamma - 2)}{8\sin\gamma\cos\gamma} + c_{11} + \frac{15c_{12}\cos\gamma}{16\sin\gamma} + \frac{5\cos^4\gamma + \cos^2\gamma - 1}{8\sin\gamma\cos^3\gamma}, \quad (53)$$

$$c_7 = -\frac{H}{2\sin^5\gamma}\Bigg\{ 4\cos\phi\cos\gamma\cos(\gamma-\phi) - 2\cos^2\phi\sin^2\gamma - (1+\sin\phi)\cos^2\gamma\Bigg\} - \frac{c_8\cos\gamma}{2\sin\gamma}$$

$$+\frac{c_{10}\cos^3\gamma}{2\sin^3\gamma} + \frac{c_{11}\cos^4\gamma}{\sin^4\gamma} + \frac{3c_{12}\cos^5\gamma}{2\sin^5\gamma} + \frac{\cos\gamma}{2\sin^3\gamma},$$

where $H = G\mathrm{cosec}\phi$. We note that (52) and (53) ensures that the stresses are continuous at the boundary between the inner dead region and the outer yield region at $\Theta = \gamma$. Secondly, we assume that the derivatives of the stresses at the boundary are continuous. As previously stated, if the stresses are continuous and satisfy the equilibrium equations (15) throughout the entire sand-pile, then we see from (15) that the derivatives of $\sigma_{R\Theta}$ and $\sigma_{\Theta\Theta}$ must also be continuous throughout the entire sand-pile, and hence, we only need to examine the continuity of the derivatives of σ_{RR} and $\sigma_{\Phi\Phi}$. Therefore, from (48) and (49) and using MAPLE, we find

$$
\begin{aligned}
c_4 = & \frac{H}{9\sin^5\gamma\cos^2\gamma}\Big\{ -8\cos^2\phi\sin\gamma\cos\gamma(82\cos^4\gamma - 74\cos^2\gamma + 19) \\
& +48\Psi'(\gamma)\sin\phi\cos\phi\sin^6\gamma - 4\sin\phi\cos\phi\sin^2\gamma(146\cos^4\gamma - 37\cos^2\gamma - 28) \\
& -\sin\phi\sin\gamma\cos\gamma(19\cos^4\gamma - 218\cos^2\gamma + 118) + \sin\gamma\cos\gamma(629\cos^4\gamma - 862\cos^2\gamma + 314)\Big\} \\
& -\frac{2c_2(5\cos^2\gamma - 2)}{9\cos^2\gamma} + \frac{c_{10}(25\cos^4\gamma + 16\cos^2\gamma - 14)}{9\sin^2\gamma\cos^2\gamma} + \frac{8c_{11}(3\cos^4\gamma + 2\cos^2\gamma - 2)}{3\sin^3\gamma\cos\gamma} \\
& +\frac{5c_{12}(8\cos^4\gamma + 8\cos^2\gamma - 7)}{3\sin^4\gamma} + \frac{(2\cos^2\gamma + 7)(5\cos^2\gamma - 2)}{9\sin^2\gamma\cos^2\gamma} \\
& +\frac{G'(\gamma)\left[24\cos^2\phi\sin^4\gamma + (1-\sin\phi)(\cos^4\gamma + 22\cos^2\gamma - 14) - 50\cos^4 + 52\cos^2\gamma - 20\right]}{9\sin\phi\sin^3\gamma\cos^2\gamma}, \\
c_8 = & -\frac{H}{\sin\phi\sin^4\gamma}\Big\{ 4\sin^2\phi\cos\phi\sin\gamma(10\cos^2\gamma - 1) + 8\sin\phi\cos^2\phi\cos\gamma(5\cos^2\gamma - 2) \\
& -\sin^2\phi\cos\gamma(5\cos^2\gamma + 4) - \sin\phi\cos\gamma(29\cos^2\gamma - 20)\Big\} + \frac{3c_{10}\cos^2\gamma}{\sin^2\gamma} \\
& +\frac{8c_{11}\cos^3\gamma}{\sin^3\gamma} + \frac{15c_{12}\cos^4\gamma}{\sin^4\gamma} - \frac{\cos^2\gamma(1+\sin\phi)G'(\gamma)}{\sin\phi\sin^3\gamma} - \frac{(1-4\cos^2\gamma)}{\sin^2\gamma},
\end{aligned}
\tag{54}
$$

where $G'(\gamma)$ is given by (47), $\Psi'(\gamma)$ is given by (46), and $H = G\mathrm{cosec}\phi$. Therefore, we have now ensured that the derivatives of the stresses throughout the entire sand-pile will remain continuous across the boundary at $\Theta = \gamma$. Now, recall that we are looking for a symmetrical stress distribution where $\sigma_{RR}, \sigma_{\Theta\Theta}$, and $\sigma_{\Phi\Phi}$ are even functions and $\sigma_{R\Theta}$ is an odd function. This means that the derivatives of $\sigma_{RR}, \sigma_{\Theta\Theta}$, and $\sigma_{\Phi\Phi}$ directly beneath the vertex of the sand-pile must be zero, and the value of $\sigma_{R\Theta}$ directly beneath the vertex must also be zero. From $(49)_3$, and noting from (51) that $c_{18} = 0$, we see that $\sigma_{R\Theta}$ must be zero directly beneath the vertex at $\Theta = 0$. From $(15)_2$ and (49), we see at $\Theta = 0$ that for $\partial\sigma_{\Theta\Theta}/\partial\Theta$ to be zero, we require $c_{23} = 2c_{11}$ which is also required directly from satisfying the equilibrium equations (15) as given in (51). However, upon differentiating $(49)_2$ with respect to Θ, we see for $\partial\sigma_{\Theta\Theta}/\partial\Theta(R, 0) = 0$, that $c_{11} = 0$, and hence $c_{23} = 0$. For $\partial\sigma_{\Phi\Phi}/\partial\Theta$, we find from differentiating $(49)_4$ with respect to Θ that as $c_{23} = 0$, then $\partial\sigma_{\Phi\Phi}/\partial\Theta(R, 0) = 0$. Finally, for $\partial\sigma_{RR}/\partial\Theta$ we find from differentiating

$(49)_1$ with respect to Θ that c_2 is given by

$$c_2 = -\frac{H}{2\sin^4\gamma}\Big\{ 12\Psi'(\gamma)\sin\phi\cos\phi\sin^3\gamma + 6\cos^2\phi\cos\gamma(19\cos^2\gamma - 7)$$
$$+32\sin\phi\cos\phi\sin\gamma(3\cos^2\gamma+1) + \sin\phi\cos\gamma(11\cos^2\gamma - 38)$$
$$-\cos\gamma(121\cos^2\gamma - 94)\Big\} + \frac{c_{10}(5\cos^2\gamma+4)}{2\sin^2\gamma} + \frac{15c_{12}\cos^2\gamma(\cos^2\gamma+2)}{2\sin^4\gamma} \tag{55}$$
$$-\frac{\sin\phi\sin\gamma(5\cos^2\gamma-4) - G'(\gamma)[\sin^2\gamma(6\sin^2\phi - \sin\phi + 5) - 3(1+\sin\phi)]}{2\sin\phi\sin^3\gamma}.$$

Therefore, it remains only to specify constants c_{10} and c_{12}. The former is arbitrarily chosen so as to satisfy the Coulomb-Mohr inequality while the latter is completely arbitrary and the value $c_{12} = -0.08$ is adopted.

5 NUMERICAL RESULTS

The experimental results of Smid & Novosad (1981), were obtained at various stages during the pouring of a three-dimensional heap. When the heap was at height h, the horizontal and vertical stresses were measured at the base of the heap. The material used was sand for which the angle of repose was 32.6° and the average bulk density was determined to be $\rho = 1567\text{kg/m}^3$. From Burden & Faries (1993), we use a Shooting Method incorporating a Runge-Kutta scheme of order 4, to do all the numerical solutions presented in this paper.

In section 3 we examine a two-dimensional sand-pile with an inner dead region and an outer yield region, where $\theta = \gamma$ is the boundary between the two regions. We note that remarkably the numerical solution for the outer yield region always gives identically the special solution of (12) where γ is determined to be the smallest value that satisfies the inequality (37). As described in section 3, we are consequently able to analytically determine the stresses in the outer and inner regions. Figure 2 shows the variation of $\sigma_{rr}, \sigma_{r\theta}$, and $\sigma_{\theta\theta}$ with respect to θ for $\beta = \cos\alpha$, $\gamma = 0.382$, and $h = 0.575$m. We note from Figure 2 that both the derivatives of $\sigma_{r\theta}$ and $\sigma_{\theta\theta}$ with respect to θ remain continuous across the boundary between the inner dead region and the outer yield region at $\theta = \gamma$, whereas the derivative of σ_{rr} is discontinuous. This means that the derivatives of horizontal and vertical forces will also be discontinuous as shown in Figure 4. However, from Figure 4 we see that the horizontal and vertical forces do indeed possess the qualitative features of the experimental results presented in Smid & Novosad. In fact, we may identify the location of the maximum vertical stress to be the boundary between the inner dead region and the outer yield region at $\theta = \gamma$.

In section 4 we examine a three-dimensional sand-pile with an inner dead region and an outer yield region, where $\Theta = \gamma$ is the boundary between the two regions. Unlike the two-dimensional sand-pile, the numerical solution in the outer yield region does not seem to follow a known exact special solution. This inevitably means that γ must be determined numerically and for $\beta = 0.69$ we find that $\gamma = 0.3$. As detailed

(a). (b).

Figure 4: Variation of the horizontal and vertical force distributions for a two-dimensional sand-pile with an inner dead region and an outer yield region for stresses from Figure 2.

in section 4, the stresses in the outer yield region are given by (49), and the stress distribution throughout the entire sand-pile is shown in Figure 5. We note that all the stresses and their derivatives remain continuous across the boundary at $\Theta = \gamma$, and $\sigma_{RR}, \sigma_{\Theta\Theta}$, and $\sigma_{\Phi\Phi}$ are even functions of Θ, while $\sigma_{R\Theta}$ is an odd function. This means that unlike the two-dimensional sand-pile, the horizontal and vertical stresses and their derivatives also remain continuous as shown in Figure 6. We note that the horizontal and vertical stresses in Figure 6 also have the M-shaped profile.

6 CONCLUSIONS

We have proposed a possible model to solve the problem of determining the force distribution at the base of two-dimensional and three-dimensional sand-piles. We have made use of the Jenike solutions for radial flow in converging wedge or cone shaped hoppers, but with gravity acting in the opposite direction. For a two-dimensional sand-pile entirely at yield and for the special case of an angle of internal friction equal to ninety degrees Hill & Cox (2000) have determined an exact analytical solution which coincides with a full numerical solution of the problem. However, for more realistic angles of internal friction, numerical results indicate that it is not possible to determine such a solution which satisfies all the required conditions. While many materials such as Coal and Silica do exhibit large angles of internal friction such as 80 degrees and 78.34 degrees respectively, the exact analytical solution for an angle of internal friction equal to ninety degrees does not exhibit the experimentally determined profile obtained by Smid & Novosad (1981). In this paper for both two and three-dimensional piles we have assumed that the sand-pile has an inner dead region and an outer yield region. For the two-dimensional sand-pile, by numerically solving the outer yield region we have determined that the solution follows the special exact solution of (12), which

(a). (b).

(c). (d).

Figure 5: Variation of $\sigma_{RR}, \sigma_{\Theta\Theta}, \sigma_{R\Theta}$, and $\sigma_{\Phi\Phi}$ for three-dimensional sand-piles with inner dead region and outer yield region using an angle of repose of 32.6° ($\alpha = 1.0018$) for $\beta = 0.69$ ((a) σ_{RR}, (b) $\sigma_{\Theta\Theta}$, (c) $\sigma_{R\Theta}$, and (d) $\sigma_{\Phi\Phi}$).

(a). (b).

Figure 6: Variation of the horizontal and vertical stresses in a three-dimensional sand-pile. ((a) horizontal and (b) vertical).

means that we may analytically determine the stress profile throughout the entire sand-pile, with stresses which are linear in both x and y. This model does exhibit the experimentally determined M-shaped profile as obtained by Smid & Novosad (1981), where the location of the maximum vertical pressure is at the boundary between the two regions at $\theta = \gamma$, where γ is determined as the smallest value satisfying the strict inequality (37). For three-dimensional sand-piles a similar situation applies except that the numerical solution in the outer yield region appears not to coincide with a simple analytical solution one possible solution for the polar stresses in the dead region involves quintic expressions of $\sin\Theta$ and $\cos\Theta$ which also give rise to the observed M-shaped profile.

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