Cylindrical cavities and classical rat-hole theory occurring in bulk materials

JAMES M. HILL AND GRANT M. COX

School of Mathematics and Applied Statistics, University of Wollongong, 2522, N.S.W., Australia

SUMMARY

The phenomenon of stable cylindrical cavities known as rat-holes in stockpiles and hoppers is well known but is not properly understood, and existing theory is unsatisfactory, in that it is believed not to properly incorporate actual material properties. Here the classical rat-hole theory of Jenike and his coworkers is re-examined, with a view to examining the validity of the so-called "stable rat-hole equation", which is widely used in practice. For certain plastic regimes, new exact analytical solutions are determined for two special values of the angle of internal friction. One of the exact results may be used as the basis of an approximate solution valid for small angles of internal friction. Further, these exact and approximate solutions are compared with a full numerical solution of the governing differential equations. One of the approximations used by Jenike and his coworkers is shown to be invalid.

Key words: cylindrical cavities; rat-holes; Coulomb-Mohr yield condition; plastic regimes; principal stresses; Mohr diagram

1. INTRODUCTION

The formation of stable circular and almost vertical holes in stockpiles and hoppers is a significant practical problem. This frequently occurring phenomena is sometimes referred to as "piping", and the holes themselves are known as "rat-holes". In the mineral and mining industry, once a rat-hole forms in a stockpile, it tends to remain there because the material around the hole dries out and sets as a solid material. Practicing engineers believe that the classical rat-hole theory enunciated by Jenike[1] and Jenike and Yen[2,3] does not accurately reflect actual material behavior. In addition, there is a view that classical theory does not account for the correct physics of the problem because the actual material cohesion arises from water pressure capillarity in a thin layer around the hole (see Gudehus[4]). Despite this latter point of view, we re-examine here the classical theory and in particular we examine the validity of the Jenike stable rat-hole equation (6) which is widely used by engineers in many granular material industries (see for example, McBride[5]).

Typically, a stockpile rat-hole appears as indicated in Fig. 1, where θ denotes the angle of repose, δ^* is approximately the angle of internal friction δ, and α denotes a small angle. For the idealized situation of a vertical circular rat-hole ($\alpha \equiv 0$) and with the axis as shown in Fig. 1, the mathematical problem is to solve the equilibrium equations

$$\frac{d\sigma_{rz}}{dr} + \frac{\sigma_{rz}}{r} = \rho g, \quad \frac{d\sigma_{rr}}{dr} + \frac{(\sigma_{rr} - \sigma_{\phi\phi})}{r} = 0, \tag{1}$$

subject to the boundary conditions that the surface of the hole of radius r_0 is stress free

$$\sigma_{rz} = \sigma_{rr} = 0 \quad \text{for} \quad r = r_0, \tag{2}$$

where ρ is the bulk density of the material, g is the acceleration due to gravity, $\sigma_{rr}, \sigma_{rz},$ etc. denote the stresses in a cylindrical polar coordinate system (r, ϕ, z), and which following Jenike[1] are assumed to be independent of ϕ and z. In addition, the material is assumed to satisfy the Coulomb-Mohr yield condition

$$|\tau| = c - \sigma^* \tan\delta, \tag{3}$$

where c is the cohesion and σ^* and τ denote the normal and tangential components of compressive traction, which here we assume positive in tension. Namely, we adopt the usual convention in continuum mechanics that positive forces are assumed to produce positive extensions.

Following[1,2,3] the first equation of (1) trivially integrates to give

$$\sigma_{rz} = \frac{\rho g}{2}\left(r - \frac{r_0^2}{r}\right), \tag{4}$$

and on introducing the stress-angle ψ which is defined by (10), Jenike[1] and Jenike and Yen[2,3] make use of (4) to deduce

$$\frac{d\psi}{dr} = \frac{\sin 2\psi}{(\cos 2\psi - \beta)}\left\{\frac{(1-\beta\cos 2\psi)r}{(r^2 - r_0^2)} - \frac{(1+\beta)}{2r}\right\}, \tag{5}$$

where β denotes $\sin\delta$, and which for the plastic regime A is subject to the condition $\psi = 0$ at $r = r_0$. Now on making use of $(13)_3$ and $(18)_1$ we may deduce from (4) and (5) using ℓ'Hopital's rule for $\beta \neq 1$

$$\lim_{r\to r_0}\left(\frac{d\psi}{dr}\right) = \frac{\rho g}{f_c}, \tag{6}$$

where f_c denotes the unconfined yield strength defined by (12). In principle then, we may solve (5) subject to ψ zero at $r = r_0$, and on equating the gradient of ψ at $r = r_0$ to $\rho g/f_c$, we may supposedly determine the rat-hole radius r_0. Moreover, for $1/3 < \beta < 1/2$ the classical theory then proposes that any rat-hole of radius R_0, where $R_0 < r_0$ is stable and this criteria is known as the Jenike stable rat-hole condition (see Jenike and Yen[2], page 20). Further, for $\pi/6 < \delta < \pi/2$ (namely, $1/2 < \beta < 1$) Jenike and Yen[2,3] determine $d\psi/dr$ at the "constant" ψ and r values for which both the denominator and the expression in the curly brackets of (5) both vanish. Thus from $\cos 2\psi = \beta$, $r/r_0 = (2\beta-1)^{-1/2}$, ℓ'Hopital's rule and solving a quadratic, we have

$$\left(\frac{d\psi}{dr}\right)_{r=r_0} = \frac{(2\beta-1)^{1/2}}{4r_0}\left(\frac{1+\beta}{1-\beta}\right)^{1/2}\left\{-\beta \pm (\beta^2 + 8\beta - 4)^{1/2}\right\}, \tag{7}$$

and the question arises as to how useful either estimate of the gradient actually is. In this paper, we show that both estimates are completely inaccurate and moreover that equation (6) is not an equation for the determination of r_0, but rather is a mathematical identity applying to any solution of (5) such that $\psi = 0$ at $r = r_0$.

In the following section we present the necessary basic equations for the two plastic regimes A and F and in terms of the cohesion c we formulate the two basic equations for ψ and σ which are defined by (10) (see also Fig. 2). In the two subsequent sections we detail the special results applying for A and F respectively, including exact solutions applying for β zero and β unity and an approximate solution valid for small β, noting however that the case $\beta = 1$ has a well-defined mathematical meaning, but such materials do not occur in practice. In section 5 we consider the five other plastic regimes and show that results for regimes B and AB are similar to regime A, results for regimes E and EF are similar to regime F, and for the plastic regime AF we are unable to determine differential equations in terms of ψ and σ defined by (10) because in this case $\sigma_{\phi\phi}$ is indeterminant in terms of the maximum and minimum principal stresses. In section 6 we apply a simple Runge-Kutta scheme to determine a numerical solution which we relate to the various exact and approximate solutions given in previous sections. Finally, we comment that other available analytical methods can be found in Drescher[6].

2. BASIC EQUATIONS FOR PLASTIC REGIMES A AND F

Following the notation adopted in Hill and Wu[7], we assume that the three algebraic maximum, intermediate and minimum principal stresses σ_I, σ_{II}, and σ_{III} ($\sigma_I \geq \sigma_{II} \geq \sigma_{III}$), so that the Coulomb-Mohr yield condition takes the form

$$\sigma_I = 2c\left(\frac{1-\beta}{1+\beta}\right)^{1/2} + \left(\frac{1-\beta}{1+\beta}\right)\sigma_{III}, \tag{8}$$

where again c denotes the cohesion and $\beta = \sin\delta$ where δ is the angle of internal friction. The principal stresses are the eigenvalues of the stress matrix and therefore we have

$$\begin{aligned} \sigma_I &= \frac{1}{2}\left\{(\sigma_{rr}+\sigma_{zz}) + \left[(\sigma_{rr}-\sigma_{zz})^2 + 4\sigma_{rz}^2\right]^{1/2}\right\}, \\ \sigma_{III} &= \frac{1}{2}\left\{(\sigma_{rr}+\sigma_{zz}) - \left[(\sigma_{rr}-\sigma_{zz})^2 + 4\sigma_{rz}^2\right]^{1/2}\right\}, \end{aligned} \tag{9}$$

along with the intermediate principal stress σ_{II} coinciding with the hoop stress $\sigma_{\phi\phi}$. Now on introducing σ and ψ as shown in the Mohr diagram (see Fig. 2) we have

$$\sigma_{rr} - \sigma_{zz} = 2\beta\sigma\cos 2\psi, \quad \sigma_{rz} = \beta\sigma\sin 2\psi, \tag{10}$$

while from (8) and (9) we may deduce

$$\beta(\sigma_{rr}+\sigma_{zz})+\left[(\sigma_{rr}-\sigma_{zz})^2+4\sigma_{rz}^2\right]^{1/2}=(1-\beta)f_c, \tag{11}$$

where f_c denotes the unconfined yield strength which is defined by $\sigma_I = 0$ when $\sigma_{III} = -f_c$, so that from (8) we have

$$f_c = 2c\left(\frac{1+\beta}{1-\beta}\right)^{1/2}. \tag{12}$$

Now from (10) and (11) we may deduce

$$\begin{aligned}
\sigma_{rr} &= \sigma(\beta\cos 2\psi - 1) + (1-\beta)\frac{f_c}{2\beta},\\
\sigma_{zz} &= -\sigma(\beta\cos 2\psi + 1) + (1-\beta)\frac{f_c}{2\beta},\\
\sigma_{rz} &= \beta\sigma\sin 2\psi.
\end{aligned} \tag{13}$$

Moreover, from (9) and (13) we have

$$\sigma_I = -(1-\beta)\sigma + (1-\beta)\frac{f_c}{2\beta}, \quad \sigma_{III} = -(1+\beta)\sigma + (1-\beta)\frac{f_c}{2\beta}. \tag{14}$$

The yield condition (8) is usually represented by a pyramid surface in three-dimensional principal stress space with a rectangular Cartesian frame of reference $(\sigma_1, \sigma_2, \sigma_3)$ denoting a typical point (see Fig. 3), and the seven possible plastic regimes available for axially symmetric stress states arise from the general plane $\sigma_3 =$ constant. Points on the varying hexagon represent all possible plastic principal stress states given by (8). These stress states are given in tabular form in Table 1. For further details we refer the reader to either Hill and Wu[7] or Cox, Eason and Hopkins[8]. For the plastic regimes A and F we have respectively

$$\begin{aligned}
A: &\quad \sigma_I = \left(\frac{1-\beta}{1+\beta}\right)(f_c+\sigma_{\phi\phi}) \qquad (\sigma_I > \sigma_{\phi\phi} = \sigma_{III}),\\
F: &\quad \sigma_{\phi\phi} = \left(\frac{1-\beta}{1+\beta}\right)(f_c+\sigma_{III}) \quad (\sigma_I = \sigma_{\phi\phi} > \sigma_{III}),
\end{aligned} \tag{15}$$

so that for the plastic regime A we have

$$\sigma_{\phi\phi} = -(1+\beta)\sigma + (1-\beta)\frac{f_c}{2\beta}, \tag{16}$$

while for the plastic regime F we obtain

$$\sigma_{\phi\phi} = -(1-\beta)\sigma + (1-\beta)\frac{f_c}{2\beta}. \tag{17}$$

Now from the stress free boundary conditions (2) and the relations $(10)_2$ and $(13)_1$, the following possibilities for $r = r_0$ arise. For $\beta \neq 0$ or 1 we have

$$\begin{aligned} &\psi = 0, \qquad \sigma = \frac{f_c}{2\beta}, \qquad (\beta \neq 0, 1) \\ &\psi = \frac{\pi}{2}, \quad \sigma = \frac{f_c}{2\beta}\left(\frac{1-\beta}{1+\beta}\right), \quad (\beta \neq 0) \end{aligned} \tag{18}$$

while for β tending to zero we have

$$\begin{aligned} &\psi = 0, \quad \sigma = \frac{f_c}{2\beta}, \quad (\beta \to 0) \\ &\psi = \frac{\pi}{2}, \quad \sigma = \frac{f_c}{2\beta}, \quad (\beta \to 0) \end{aligned} \tag{19}$$

and for $\beta = 1$ we have

$$\begin{aligned} &\psi = 0, \quad \sigma = \text{anything}, \quad (\beta = 1) \\ &\psi = \text{anything}, \qquad \sigma = 0. \qquad (\beta = 1) \end{aligned} \tag{20}$$

Further, from the differential equation $(1)_2$ and the relation which is obtained from (4) and $(13)_3$, namely

$$\sigma \sin 2\psi = \lambda\left(r - \frac{r_0^2}{r}\right), \tag{21}$$

where the constant λ denotes $\rho g/2\beta$, we may deduce the following differential equations

$$\frac{d\psi}{dr} = \frac{\sin 2\psi}{(\cos 2\psi - \beta)} \left\{ \frac{(1-\beta\cos 2\psi)r}{(r^2 - r_0^2)} - \frac{(1+\varepsilon\beta)}{2r} \right\},$$

$$\frac{d\sigma}{dr}\left\{\left[\sigma^2 - \lambda^2\left(r - \frac{r_0^2}{r}\right)^2\right]^{1/2} - \beta\sigma\right\} \tag{22}$$

$$-\frac{\beta\sigma}{r}\left\{\varepsilon\left[\sigma^2 - \lambda^2\left(r - \frac{r_0^2}{r}\right)^2\right]^{1/2} + \sigma\right\} = -2\beta\lambda^2\left(r - \frac{r_0^2}{r}\right),$$

where $\varepsilon = +1$ for the plastic regime A and $\varepsilon = -1$ for the plastic regime F. In view of the relation (21), for each plastic regime, these two differential equations are not independent. In the following two sections we present some simple exact and approximate analytical solutions of these equations which have not been given previously. The exact solutions apply for the special cases β tending to zero and β unity, where β denotes $\sin\delta$. The special case $\beta = 0$ may be viewed as a limiting situation of the yield condition (3), that is $|\tau| = c$. The solutions presented in the following section are exact solutions of (22), except that for β zero, the solution is a limiting case and appropriate care must be made in terms of the interpretation of such formulae.

3. EXACT AND APPROXIMATE SOLUTIONS FOR THE PLASTIC REGIME A

In this section for the plastic regime A we derive the special solutions of (22) subject to the conditions (18) $-$ (20). We first make some transformations of the ψ equation, namely (5) or $(22)_1$ with $\varepsilon = +1$. On making the transformation

$$u = (1 + \cos 2\psi)^{-1}, \tag{23}$$

we find that equation (5) becomes

$$[1 - (1+\beta)u]\frac{du}{dr} = \frac{(2u-1)}{r(r^2 - r_0^2)}\left[(1+\beta)(r^2 + r_0^2)u - 2\beta r^2\right], \tag{24}$$

which is an Abel equation of the second kind (see for example Murphy[9], page 25). Now on making the further transformation

$$u = \frac{1}{2} + \frac{v}{2(1+\beta)(r - r_0^2/r)^2}, \tag{25}$$

equation (24) eventually becomes

$$\xi' \left\{ \frac{[v - (1-\beta)\xi^2]}{4} \frac{dv}{d\xi} + \xi v \right\} = 2\beta\xi v, \tag{26}$$

where primes denotes differentiation with respect to r and ξ denotes $(r - r_0^2/r)$. From (26) we see that the special cases $\beta = 0$ and $\beta = 1$ can be readily integrated. In addition, equation (26) is linear in β and therefore suitable to deduce an approximate solution of the form

$$v(r) = v_0(r) + \beta V_0(r) + O(\beta^2). \tag{27}$$

If v_{approx} denotes any estimate for v, then from (23) and (25) we may deduce an approximation for ψ given by

$$\tan^2\psi = \frac{v_{\text{approx}}}{(1+\beta)(r - r_0^2/r)^2}. \tag{28}$$

3.1 *Exact and approximate solutions for small* β

From (26) and (27) we have

$$\begin{gathered} \frac{1}{4}\left(v_0 - \xi^2\right)\frac{dv_0}{d\xi} + \xi v_0 = 0, \\ \xi' \left\{ \frac{1}{4}\left(V_0 + \xi^2\right)\frac{dv_0}{d\xi} + \frac{1}{4}\left(v_0 - \xi^2\right)\frac{dV_0}{d\xi} + \xi V_0 \right\} = 2\xi v_0, \end{gathered} \tag{29}$$

and the first equation may be readily solved to yield

$$v_0(\xi) = C - \xi^2 \pm \left(C^2 - 2C\xi^2\right)^{1/2}, \tag{30}$$

where C denotes the constant of integration and we observe that the solution remains valid only for $\xi \leq (C/2)^{1/2}$. Outside this range the solution $v_0 \equiv 0$ is adopted. Now as described in the Appendix, on making use of (30) we may eventually deduce the following expression for the solution of $(29)_2$,

$$V_0(\xi) = -\frac{\left[(C^2-2C\xi^2)^{1/2}-C\right]^2}{2C\,(C^2-2C\xi^2)^{1/2}}\left[\,\left(C^2-2C\xi^2\right)^{1/2}+C\ln\left\{\xi^2\left[\xi+(\xi^2+4r_0^2)^{1/2}\right]\right\}\right.$$
$$\left.+2\left(C^2+8Cr_0^2\right)^{1/2}\{F(\gamma,\nu)-E(\gamma,\nu)\}+2\xi\left(C^2-2C\xi^2\right)^{1/2}\left(\xi^2+4r_0^2\right)^{-1/2}\,\right], \tag{31}$$

where $F(\gamma,\nu)$ and $E(\gamma,\nu)$ are elliptic functions of the first and second kind respectively and are defined in the Appendix, with γ and ν defined by

$$\gamma = \sin^{-1}\left\{\frac{\xi}{C}\left(\frac{C^2+8Cr_0^2}{\xi^2+4r_0^2}\right)^{1/2}\right\},\quad \nu = \frac{C}{(C^2+8Cr_0^2)^{1/2}}, \tag{32}$$

and we have assumed the minus sign in (30). We note that the solution (31) also remains valid only for $\xi < (C/2)^{1/2}$ and outside this range, we adopt the solution of $V_0 \equiv 0$, which is a bonafide solution of $(29)_2$ provided $v_0 \equiv 0$ in this region.

Now either making use of (25) and (30) or by direct integration of (24) with β zero we may deduce the following exact solution applying for zero angle of internal friction, thus

$$\cos^2\psi = \frac{1}{2}\left\{1 \mp \left(1-k^2\xi^2\right)^{1/2}\right\}, \tag{33}$$

where k is related to C by the equation $k^2 = 2/C$. We observe from (33) that the two possible conditions at $r = r_0$, $\psi = 0$ or $\psi = \pi/2$ arising in (19), are both feasible and correspond to the plus and minus signs respectively. Further, from the relation (21) we obtain

$$\sigma = \frac{\rho g}{2\beta}\frac{\xi}{\left[\left\{1\pm(1-k^2\xi^2)^{1/2}\right\}\left\{1\mp(1-k^2\xi^2)^{1/2}\right\}\right]^{1/2}},$$

from which we may deduce σ constant and given by

$$\sigma = \frac{\rho g}{2\beta k}, \tag{34}$$

and therefore from both the conditions (19) at $r = r_0$, the constant $k = \rho g/f_c$. Clearly, this exact solution of the differential equation (5) for the case β zero can be interpreted as an asymptotic solution which may be made rigorous by simply rescaling σ with respect to β. This special case is instructive because it makes it clear that the arbitrary constant k (or C) is not necessarily determined by the condition on ψ at $r = r_0$, but rather in this case by the condition on σ at $r = r_0$. Thus, in summary the exact solution for β tending to zero is given by

$$\cos^2 \psi = \frac{1}{2} \left\{ 1 \mp \left(1 - \left(\frac{\rho g \xi}{f_c} \right)^2 \right)^{1/2} \right\}, \quad \sigma = \frac{f_c}{2\beta}, \tag{35}$$

where the $\mp$ corresponds to $\psi = \pi/2$ or $\psi = 0$ respectively. From these results and the relations (13) and (16) we have

$$\sigma_{rr} = -f_c \sin^2 \psi = -\frac{f_c}{2} \left\{ 1 \pm \left(1 - \left(\frac{\rho g \xi}{f_c} \right)^2 \right)^{1/2} \right\},$$

$$\sigma_{zz} = -f_c \cos^2 \psi = -\frac{f_c}{2} \left\{ 1 \mp \left(1 - \left(\frac{\rho g \xi}{f_c} \right)^2 \right)^{1/2} \right\}, \tag{36}$$

$$\sigma_{rz} = f_c \sin \psi \cos \psi = \frac{\rho g}{2} \xi,$$

and $\sigma_{\phi\phi} = -f_c$. We observe that this is a well-defined solution of (1) and (2) only for the positive case of $(35)_1$, even though the negative case of $(35)_1$ does give rise to $\psi = \pi/2$, but does not lead to σ_{rr} vanishing at $r = r_0$.

3.2 *Exact solution for β unity*

From (26) we have for β unity

$$\xi' \left\{ \frac{v}{4} \frac{dv}{d\xi} + \xi v \right\} = 2\xi v, \tag{37}$$

which upon recalling that $\xi = (r - r_0^2/r)$ and assuming $v \neq 0$, simplifies considerably to give

$$\frac{dv}{dr} = \frac{4\xi^2}{r}. \tag{38}$$

Now (38) integrates easily to give

$$v(r) = 2 \left\{ r^2 - r_0^4/r^2 - 4r_0^2 \log (r/r_0) \right\}, \tag{39}$$

and we note that the constant of integration is zero for boundary condition $(20)_1$.

Now from (23), (25) with $\beta = 1$ and (39) we may deduce the exact solution of (5) for the case $\beta = 1$, thus

$$\cos^2 \psi = \frac{\left(r - r_0^2/r\right)^2}{2\left[r^2 - r_0^2 - 2r_0^2 \log\left(r/r_0\right)\right]}, \tag{40}$$

and from (21) we have

$$\sigma = \frac{\rho g}{2} \frac{\left[r^2 - r_0^2 - 2r_0^2 \log\left(r/r_0\right)\right]}{\left[r^2 - r_0^4/r^2 - 4r_0^2 \log\left(r/r_0\right)\right]^{1/2}}. \tag{41}$$

We observe from (13) and (16) that in this case we have

$$\sigma_{rr} = -2\sigma \sin^2 \psi = -\frac{\rho g}{2} \left\{ r^2 - \frac{r_0^4}{r^2} - 4r_0^2 \log\left(\frac{r}{r_0}\right) \right\}^{1/2},$$

$$\sigma_{zz} = -2\sigma \cos^2 \psi = -\frac{\rho g}{2} \left\{ r^2 - \frac{r_0^4}{r^2} - 4r_0^2 \log\left(\frac{r}{r_0}\right) \right\}^{-1/2} \left(r - \frac{r_0^2}{r} \right)^2, \tag{42}$$

$$\sigma_{rz} = 2\sigma \sin\psi \cos\psi = \frac{\rho g}{2} \left(r - \frac{r_0^2}{r} \right),$$

and $\sigma_{\phi\phi} = -2\sigma$, where σ is given by (41) and that this is a well-defined solution of (1) and (2). As noted in the numerical results it is not possible to utilize this exact solution for $\beta = 1$ as the basis of an approximate solution valid for β close to unity because the solution characteristics are quite different for $\beta = 1$ and for $\beta \neq 1$, in the sense that ψ has an infinite gradient at $r = r_0$ for $\beta = 1$, while for $\beta \neq 1$ the gradient is finite.

4. SOLUTIONS FOR THE PLASTIC REGIME F

In this section we give the corresponding formulae for the plastic regime F. On making the transformation

$$u = (1 - \cos 2\psi)^{-1}, \tag{43}$$

we find that equation $(22)_1$ with $\varepsilon = -1$ becomes

$$\left[1 - (1-\beta)u\right] \frac{du}{dr} = \frac{(2u-1)}{r(r^2 - r_0^2)} \left[(1-\beta)\left(r^2 + r_0^2\right) u + 2\beta r^2\right], \tag{44}$$

which is precisely equation (24) except that β in (24) is replaced by $-\beta$. Thus, from the transformation

$$u = \frac{1}{2} + \frac{v}{2(1-\beta)(r - r_0^2/r)^2}, \quad (\beta \neq 1) \tag{45}$$

we obtain

$$\xi' \left\{ \frac{[v - (1+\beta)\xi^2]}{4} \frac{dv}{d\xi} + \xi v \right\} = -2\beta \xi v, \tag{46}$$

where as before primes denote differentiation with respect to r and ξ denotes $(r - r_0^2/r)$, and we may deduce an approximate solution v_{approx} of the form (27) simply replacing β by $-\beta$ and in place of (28) we have for $\beta \neq 1$

$$\cot^2 \psi = \frac{v_{\text{approx}}}{(1-\beta)(r - r_0^2/r)^2}. \tag{47}$$

For β tending to zero we may use the approximate solution (27) with β replaced with $-\beta$ and for the leading term we have from the results of the previous section

$$\sin^2 \psi = \frac{1}{2} \left\{ 1 \pm \left(1 - \left(\frac{\rho g \xi}{f_c} \right)^2 \right)^{1/2} \right\}, \quad \sigma = \frac{f_c}{2\beta}, \tag{48}$$

where here the $\pm$ corresponds to $\psi = \pi/2$ and $\psi = 0$ respectively. From the relations (13) and (17) we have

$$\begin{aligned} \sigma_{rr} &= -f_c \sin^2 \psi = -\frac{f_c}{2} \left\{ 1 \pm \left(1 - \left(\frac{\rho g \xi}{f_c} \right)^2 \right)^{1/2} \right\}, \\ \sigma_{zz} &= -f_c \cos^2 \psi = -\frac{f_c}{2} \left\{ 1 \mp \left(1 - \left(\frac{\rho g \xi}{f_c} \right)^2 \right)^{1/2} \right\}, \\ \sigma_{rz} &= f_c \sin \psi \cos \psi = \frac{\rho g}{2} \xi, \end{aligned} \tag{49}$$

and again $\sigma_{\phi\phi} = -f_c$. In this case this is clearly a well-defined solution of (1) and (2) only for $\psi = 0$, namely taking the minus sign in (48).

For β unity the transformation (45) is invalid and we have directly from equation (43) and (44) with $\beta = 1$

$$\cot^2 \psi = C \left(r^2 - r_0^2 \right)^2, \tag{50}$$

where C denotes an arbitrary constant. Further, from (21) we may deduce

$$\sigma = \frac{\rho g}{4} \frac{\left[1 + C\left(r^2 - r_0^2\right)^2\right]}{r C^{1/2}}. \tag{51}$$

Thus, from (13) and (17) we obtain

$$\begin{aligned}
\sigma_{rr} &= -2\sigma \sin^2 \psi = -\frac{\rho g}{2 r C^{1/2}}, \\
\sigma_{zz} &= -2\sigma \cos^2 \psi = -\frac{\rho g}{2r} C^{1/2} \left(r^2 - r_0^2\right)^2, \\
\sigma_{rz} &= 2\sigma \sin\psi \cos\psi = \frac{\rho g}{2} \left(r - \frac{r_0^2}{r}\right),
\end{aligned} \tag{52}$$

along with $\sigma_{\phi\phi} = 0$. It is clear from these expressions that there is no finite value of the arbitrary constant C which produces a solution of (1) and (2) such that σ_{rr} vanishes along $r = r_0$. Thus an approximate solution of the from (27) is not meaningful in this context.

5. OTHER PLASTIC REGIMES

In this section we show that the plastic regimes AB and B are similar to regime A, while EF and E are similar to regime F. Finally, we examine the regime AF for which we are unable to determine the hoop stress in terms of the variables σ and ψ defined by (10), and hence we cannot solve the equilibrium equation $(1)_2$.

For regimes AB and B the hoop stress is always the minimum principal stress. Therefore, from the Mohr diagram (Fig. 2) we find the expression for the hoop stress to be given by

$$\sigma_{\phi\phi} = -(1+\beta)\sigma + (1-\beta)\frac{f_c}{2\beta}, \tag{53}$$

noting that this is the same as for regime A. Now as the results (13) and (14) hold for all seven plastic regimes, then from $(1)_2$, (13), (21), and (53) we find the differential equations (22) with $\varepsilon = +1$ hold for both regimes AB and B. This suggests that there is little difference between regimes A, B, and AB. The three stress states are defined by

$$\begin{aligned} A: &\quad \sigma_I > \sigma_{II} = \sigma_{\phi\phi}, \\ B: &\quad \sigma_I = \sigma_{II} > \sigma_{\phi\phi}, \\ AB: &\quad \sigma_I > \sigma_{II} > \sigma_{\phi\phi}, \end{aligned} \tag{54}$$

where σ_I, σ_{II}, and $\sigma_{\phi\phi}$ are the maximum, intermediate, and minimum principal stresses respectively. From (54) we can see that the main difference is the location of the intermediate principal stress σ_{II}. We note that for regimes A and B that the intermediate stress σ_{II} can be determined, whereas for regime AB it is indeterminant.

For regimes EF and E the hoop stress is always the maximum principal stress. Therefore, from the Mohr diagram (Fig. 2) we find the expression for the hoop stress to be given by

$$\sigma_{\phi\phi} = -(1-\beta)\sigma + (1-\beta)\frac{f_c}{2\beta}, \tag{55}$$

noting that this is the same as for regime F. Now again, since the results (13) and (14) hold for all seven plastic regimes, then from $(1)_2$, (13), (21), and (55) we find the differential equations (22) with $\varepsilon = -1$ hold for both regimes EF and E. Again this suggests that there is little difference between the regimes E, F, and EF, for which the three stress states are given by

$$\begin{aligned} F: &\quad \sigma_{\phi\phi} = \sigma_{II} > \sigma_{III}, \\ E: &\quad \sigma_{\phi\phi} > \sigma_{II} = \sigma_{III}, \\ EF: &\quad \sigma_{\phi\phi} > \sigma_{II} > \sigma_{III}, \end{aligned} \tag{56}$$

where $\sigma_{\phi\phi}, \sigma_{II}$, and σ_{III} are the maximum, intermediate, and minimum principal stresses respectively. From (56) it is clear that the difference lies in the location of the intermediate principal stress σ_{II} and we note that for regimes F and E that the intermediate stress σ_{II} can be determined, whereas for regime EF it is indeterminant.

For the plastic regime AF we have the stress state

$$AF: \quad \sigma_I > \sigma_{\phi\phi} > \sigma_{III}, \tag{57}$$

and in this case we are unable to determine an expression for the hoop stress $\sigma_{\phi\phi}$ in terms of the variables σ and ψ defined by (10) and we are therefore unable to solve the equilibrium equation $(1)_2$.

6. NUMERICAL RESULTS

Fig. 4 shows the numerically determined variation of ψ (———) with position for two values of the angle of internal friction ($\beta = 0.2$ and $\beta = 0.5$) as compared to the approximate analytical expressions (− − −) obtained from (27) and (28). The numerical curves are obtained by solving $(22)_2$ for σ with $\varepsilon = +1$ using a fourth order Runge-Kutta numerical scheme together with the condition $\sigma = f_c/2\beta$ at $r = r_0$ and then ψ is determined by (21). The constant r_0 is taken to be unity. We observe the close agreement for small r and that the two results diverge with increasing r. Similarly, Fig. 5 shows the variation of σ for both the numerical (———) and approximate (− − −) results for the same two values of the angle of internal friction. We observe from Fig. 5 that for small r the approximate solution for σ does not agree closely with the numerical solution, even though the approximate solution for ψ does. This is because the approximate solution for σ was obtained from (21) using the approximate solution for ψ, and we can see that for small r, a small change in ψ will cause a large change in σ. For the special cases of β zero and β unity the ψ variation with position is shown in Fig. 6 and we note there is exact coincidence with the analytical results obtained from $(35)_1$ and (40) respectively and the full numerical solution.

We observe that $\beta = 1$ and only $\beta = 1$ has an infinite gradient at $r = r_0$ and therefore we cannot use this exact solution as the basis of an approximate solution. We note that the infinite gradient for $\beta = 1$ also follows directly from the differential equation (5) with (21) using the conditions $\psi = 0$ and $\sigma = 0$ at $r = r_0$. In order that the numerical solution follows the analytical solution for $\beta = 1$, namely (41), both ψ and σ must vanish at $r = r_0$ and we adopt these values for the numerical scheme. Now for the numerical solution we need to determine the value for $d\sigma/dr$

at $r = r_0$, so we consider $(22)_2$ with $\varepsilon = +1$ and $\beta = 1$. We find $(d\sigma/dr)_{r=r_0}$ becomes indeterminant as both numerator and denominator vanish, and if we use ℓ'Hopital's rule then the result becomes complicated. However, an easier approach is to substitute (21) into $(22)_2$ for $\lambda(r - r_0^2/r)$, and recall that $\psi = 0$ at $r = r_0$ to obtain

$$\left(\frac{d\sigma}{dr}\right)_{r=r_0} = \lim_{r \to r_0} \frac{\beta}{1-\beta} \frac{2\sigma}{r}, \tag{58}$$

which is valid provided $\beta \neq 1$. If $\beta = 1$ then at $r = r_0$ we have two possibilities, either $(d\sigma/dr)_{r=r_0}$ is infinite if $\sigma \neq 0$ at $r = r_0$ or $(d\sigma/dr)_{r=r_0}$ indeterminant as both the numerator and denominator vanish if $\sigma = 0$ at $r = r_0$. From the fact that $(d\psi/dr)_{r=r_0}$ is infinite for $\beta = 1$, it follows from (21) that $(d\sigma/dr)_{r=r_0}$ is also infinite for $\beta = 1$. Thus, for $\beta = 1$ and any value of σ at $r = r_0$, we have $(d\sigma/dr)_{r=r_0}$ as positive infinity. Accordingly from (58) we see that we cannot use the solution for β unity to approximate a solution for β close to unity, because their slopes are not close at $r = r_0$.

Fig. 7 shows the variation of the numerically determined value of $(d\psi/dr)_{r=r_0}$ (—— ——) with β for $0 \leq \beta \leq 1$ and compared with the positive estimate obtained from (7) $(- \cdot - \cdot)$ for the range $1/2 \leq \beta \leq 1$. There are clearly large discrepancies between the Jenike estimate and the actual values of the gradient. Fig. 7 also shows the variation of the analytical estimate of $(d\psi/dr)_{r=r_0}$ $(- - -)$ which is obtained from (27), (28), (30) and (31) and is compared with the purely numerical result. It is clear that this provides a very accurate approximation for $(d\psi/dr)_{r=r_0}$ in the entire range $0 \leq \beta \leq 1$. This close agreement occurs because for small r the analytical approximation is very close to the purely numerical result and these results diverge for increasing r. Finally, Fig. 8 shows the overall variation of the numerically determined stresses $\sigma_{rr}, \sigma_{rz}, \sigma_{zz}$, and $\sigma_{\phi\phi}$ as a function of position for the angle of internal friction $\delta = \pi/6 \quad (\beta = 1/2)$.

7. CONCLUSIONS

We have re-examined the classical rat-hole theory of Jenike and his coworkers and for the plastic regime A we have determined some new analytical results, both exact and approximate. The exact results apply to the limiting angles of internal friction, namely δ zero and $\delta = \pi/2$, while the approximate solution is valid for small angles of internal friction and is obtained by using the exact result for δ zero as the leading term. In addition, we have determined an independent fully numerical solution which we have compared to the approximate analytical result developed here as well as the estimate of $(d\psi/dr)_{r=r_0}$ used by Jenike. We conclude that the numerical results obtained for regime A are accurate and compare well with the analytical approximation for the various boundary conditions. However, the approximation made by Jenike and his coworkers does not give accurate results, and in fact is invalid. We have also showed that the so-called "stable rat-hole equation" is simply a mathematical identity valid for any solution of $(22)_1$ with $\varepsilon = +1$ for which $\psi = 0$ at $r = r_0$.

We have also noted that from the solution for regime A, we can determine the solutions for regimes B and AB, except that we cannot determine the intermediate principal stress for regime AB as it is indeterminant in terms of the Mohr diagram variables σ and ψ which are defined by (10). Similarly, from the solution for regime F, we can determine the solutions for regimes E and EF, again with the exception that we cannot determine the intermediate principal stress for regime EF as it is indeterminant in terms of σ and ψ.

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APPENDIX : Solution details for the first order ordinary differential equation $(29)_2$

In this Appendix we present the details for the determination of an approximate solution of (26) for small β. We note that while we can solve $(29)_1$ for v_0, we cannot determine the constant of integration C from the boundary condition for $v_0(\xi)$, but as described in section 3, this constant is determined from the condition on σ at $r = r_0$. On solving $(29)_2$ as a standard first order linear differential equation for V_0 and determining in the process several integrals that we need to integrate using MAPLE, we obtain a solution in terms of elliptic integrals. On making the substitution $v_0 = \xi^2 w_0$ in $(29)_1$ we may readily deduce a standard separable first order ordinary differential which readily integrates to yield the solution (30). We observe from (30) that the solution is valid only for $\xi \leq (C/2)^{1/2}$ and for $\xi > (C/2)^{1/2}$ we assume that the appropriate solution of $(29)_1$ is $v_0 \equiv 0$, which is indeed a solution.

Now from (29) and (30) assuming the minus sign in (30), we obtain

$$\frac{dV_0}{d\xi} - \frac{\left[1 + (1 - k^2\xi^2)^{1/2}\right]^2}{\xi\,(1 - k^2\xi^2)} V_0 = -\frac{4\xi}{\xi'} \frac{\left[1 - (1 - k^2\xi^2)^{1/2}\right]}{(1 - k^2\xi^2)^{1/2}} + \frac{k^2\xi^3}{(1 - k^2\xi^2)},$$

where $k^2 = 2/C$, which is a first order linear differential equation for V_0 in terms of ξ and ξ', for which we find the integrating factor $R(\xi)$ to be

$$R(\xi) = \frac{-k^2\,(k^2\xi^2 - 1)^{1/2}}{\left[(1 - k^2\xi^2)^{1/2} - 1\right]^2},$$

and after integrating, we can determine the result

$$\frac{-k^2 V_0\,(k^2\xi^2 - 1)^{1/2}}{\left[(1 - k^2\xi^2)^{1/2} - 1\right]^2} = I_1 - I_2, \tag{59}$$

where

$$I_1 = \int \frac{4k^2\xi\,(k^2\xi^2 - 1)^{1/2}}{\xi'\,(1 - k^2\xi^2)^{1/2}\left[1 - (1 - k^2\xi^2)^{1/2}\right]} d\xi,$$

and

$$I_2 = \int \frac{k^4\xi^3\,(k^2\xi^2 - 1)^{1/2}}{(1 - k^2\xi^2)\left[1 - (1 - k^2\xi^2)^{1/2}\right]^2} d\xi.$$

Now, in order to integrate I_1 we need to rewrite ξ' in terms of ξ and to do this we recall that $\xi = (r - r_0^2/r)$, which can be solved for r to give

$$r = \frac{1}{2}\left[\xi + \left(\xi^2 + 4r_0^2\right)^{1/2}\right],$$

so that from $\xi' = 1 + r_o^2/r^2$ we can determine the relation

$$\xi' = \frac{2\left(\xi^2 + 4r_0^2\right)^{1/2}}{\left[\xi + \left(\xi^2 + 4r_0^2\right)^{1/2}\right]}.$$

Thus I_1 can be rewritten as

$$I_1 = I_3 + I_4 = \int \frac{2k^2\xi^2\left(k^2\xi^2 - 1\right)^{1/2}}{\left(1 - k^2\xi^2\right)^{1/2}\left(\xi^2 + 4r_0^2\right)^{1/2}\left[1 - \left(1 - k^2\xi^2\right)^{1/2}\right]}d\xi$$

$$+ \int \frac{2k^2\xi\left(k^2\xi^2 - 1\right)^{1/2}}{\left(1 - k^2\xi^2\right)^{1/2}\left[1 - \left(1 - k^2\xi^2\right)^{1/2}\right]}d\xi,$$

where we can also rewrite I_3 as

$$I_3 = I_5 + I_6 = \int \frac{2i}{\left(\xi^2 + 4r_0^2\right)^{1/2}}d\xi + \int \frac{2i\left(1 - k^2\xi^2\right)^{1/2}}{\left(\xi^2 + 4r_0^2\right)^{1/2}}d\xi.$$

Now, from MAPLE we find

$$I_2 = 2\tan^{-1}\left\{\left(k^2\xi^2 - 1\right)^{-1/2}\right\} - \frac{2\log\xi\left[-\left(k^2\xi^2 - 1\right)^2\right]^{1/2}}{\left(k^2\xi^2 - 1\right)} + \left(k^2\xi^2 - 1\right)^{1/2},$$

$$I_4 = 2\tan^{-1}\left\{\left(k^2\xi^2 - 1\right)^{-1/2}\right\} + \frac{2\log\xi\left[-\left(k^2\xi^2 - 1\right)^2\right]^{1/2}}{\left(k^2\xi^2 - 1\right)} + 2\left(k^2\xi^2 - 1\right)^{1/2}, \tag{60}$$

$$I_5 = 2i\log\left[\xi + \left(\xi^2 + 4r_0^2k^2\right)^{1/2}\right],$$

and from Gradshteyn and Ryzhik[10] (page 326), we find that I_6 can be written as

$$I_6 = 2i\left(1 + 4k^2r_0^2\right)^{1/2}\left\{F(\gamma,\nu) - E(\gamma,\nu)\right\} + 2i\xi\left(\frac{1 - k^2\xi^2}{4r_0^2 + \xi^2}\right)^{1/2},$$

where γ and ν are defined as

$$\gamma = \sin^{-1}\left\{\xi\left(\frac{1 + 4k^2r_0^2}{4r_0^2 + \xi^2}\right)^{1/2}\right\}, \quad \nu = \frac{1}{\left(1 + 4k^2r_0^2\right)^{1/2}},$$

and $F(\gamma,\nu)$ and $E(\gamma,\nu)$ are elliptic integrals of the first and second kind respectively. From either Gradshteyn and Ryzhik[10] or Abramowitz and Stegun[11] we find that the elliptic integral of the first kind $F(\gamma,\nu)$ is defined as

$$F(\gamma,\nu)=\int_0^{\sin\gamma}\frac{1}{\left[\left(1-t^2\right)\left(1-\nu^2t^2\right)\right]^{1/2}}dt,$$

and the elliptic integral of the second kind $E(\gamma,\nu)$ is defined as

$$E(\gamma,\nu)=\int_0^{\sin\gamma}\frac{\left[1-\nu^2t^2\right]^{1/2}}{\left[1-t^2\right]^{1/2}}dt.$$

Thus from (60) and (59) we find we obtain equation (31),where $C=2/k^2$ and γ and ν are defined by (32). We note that from the boundary condition (19), the constant of integration for V_0 is zero.

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Figure Captions

FIG. 1. Schematic drawing showing a rat-hole in a stockpile.

FIG. 2. Mohr diagram showing the decomposition into the variables σ and ψ.

FIG. 3. Yield pyramid surface in principal stress space $(\sigma_1, \sigma_2, \sigma_3)$.

FIG. 4. Variation of the numerical ψ (——) with position for two angles of internal friction ($\beta = 0.2$ and $\beta = 0.5$) compared with the analytical result $(- - -)$ as determined from (27) and (28).

FIG. 5. Variation of the numerical σ (——) with position for two angles of internal friction ($\beta = 0.2$ and $\beta = 0.5$) compared with the analytical result $(- - -)$ as determined from (21), (27) and (28).

FIG. 6. Variation of the analytical ψ for the two special values, β zero and β unity.

FIG. 7. Variation of the numerically determined value of $(d\psi/dr)_{r=r_0}$ (———) with β for $0 \leq \beta \leq 1$ and compared with the estimate obtained from (7) $(- \cdot -\cdot)$ over the range $1/2 \leq \beta \leq 1$ and compared with the analytical estimate of $(d\psi/dr)_{r=r_0}$ $(- - -)$ with β for $0 \leq \beta \leq 1$ determined from (27) and (28).

FIG. 8. Variation of numerically determined stresses $\sigma_{rr}, \sigma_{rz}, \sigma_{zz}$ and $\sigma_{\phi\phi}$ with position for $\beta = 1/2$.

Table Caption

TABLE. 1. Plastic regimes for axially symmetric deformations.