An exact parametric solution for granular flow in a converging wedge

James M. Hill and Grant M. Cox,

Abstract. The flow of granular materials in the presence of gravity through converging wedges and cones arises in many industrial situations. For both wedges and cones, and assuming an ideal cohesionless granular material which satisfies the Coulomb-Mohr yield condition, the number of simple analytical solutions is limited and generally the governing coupled ordinary differential equations need to be solved numerically. Here we show that for plane wedge flow, an exact parametric solution may be determined for the special case of the angle of internal friction ϕ is assumed to be $\pi/2$. This is the only known exact solution of these important equations involving two arbitrary constants. A general numerical solution obtained by previous authors is shown to coincide with the special exact solution.

Mathematics Subject Classification (1991). 73B10, 73B99, 73Q05.

Keywords. Converging wedges, granular material, Coulomb-Mohr yield condition, exact parametric solution.

1. Introduction

The problem of a granular material falling under gravity but constrained to flow through either a converging wedge or a cone arises in many industrial processes and was first studied by Jenike [1, 2, 3] and Johanson [4]. These authors examine radial flow solutions for which momentum equations and the Coulomb-Mohr yield condition reduce to two highly nonlinear coupled ordinary differential equations for the determination of the stress field. One particularly simple solution of these equations is noted but it is not sufficiently general to satisfy the necessary boundary conditions and in general the coupled ordinary differential equations must be solved numerically as demonstrated by Spencer and Bradley [5] and Bradley [6]. The purpose of this note is to show that these equations admit an exact parametric solution in the special case when the angle of internal friction is $\pi/2$. This mathematically meaningful angle of internal friction physically corresponds to a granular material capable of sustaining a vertical slope. Of course such materials are not common, but nevertheless do exist. Spencer and Bradley [5] and Bradley [6] have re-examined the Jenike radial flow solutions with a view to the determination of the associated double-shearing flow field (see Spencer [7]). Here for convenience we follow the notation adopted by Spencer and Bradley [5].

In the following section we recapitulate briefly the basic equations of continuum theory for plane flow of an ideal cohesionless material which satisfies the Coulomb-Mohr yield condition. We state the coupled ordinary differential equations for the determination of the Jenike stress field and we note a particularly simple exact solution. In Section 3 we deduce certain additional relations involving boundary values from the coupled equations and the assumed boundary conditions and we give a single second order ordinary differential equation for the stress angle ψ which is defined by (2.4). In the Appendix we derive the exact parametric solution of the second order ordinary differential equation in the special case when the angle of internal friction of the material ϕ is equal to $\pi/2$ and the solution itself in terms of two arbitrary constants is given in Section 4. In Section 5 typical numerical results are presented which in particular confirm that the full numerical solution coincides with the exact solution.

2. Basic equations of continuum theory

For quasi-static plane flow under gravity through a wedge hopper as indicated in Figure 1, the stress components in a cylindrical polar coordinate system (r, θ, z) satisfy the equilibrium equations

$$\begin{aligned} \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} &= \rho g \cos\theta, \\ \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{2\sigma_{r\theta}}{r} &= -\rho g \sin\theta, \end{aligned} \tag{2.1}$$

where ρ is the density, g is the acceleration due to gravity and $\sigma_{rr}, \sigma_{\theta\theta}$ and $\sigma_{r\theta}$ denote the in-plane physical stress components. Following Spencer and Bradley [5] these components can be expressed in the standard form

$$\sigma_{rr} = -p + q\cos 2\psi, \quad \sigma_{\theta\theta} = -p - q\cos 2\psi, \quad \sigma_{r\theta} = q\sin 2\psi, \tag{2.2}$$

where p and q are defined

$$p = -\tfrac{1}{2}\left(\sigma_{rr} + \sigma_{\theta\theta}\right), \quad q = \left\{\tfrac{1}{4}\left(\sigma_{rr} - \sigma_{\theta\theta}\right)^2 + \sigma_{r\theta}^2\right\}^{1/2}, \tag{2.3}$$

while ψ is defined by

$$\tan 2\psi = \frac{2\sigma_{r\theta}}{\left(\sigma_{rr} - \sigma_{\theta\theta}\right)}, \tag{2.4}$$

and physically ψ is the angle between the maximum principal stress axis and the radial direction, in the direction of increasing θ. For a cohesionless material, the Coulomb-Mohr yield condition takes the form

$$q = p\sin\phi, \tag{2.5}$$

where ϕ is assumed to be a material constant and referred to as the angle of internal friction.

Following Jenike [2] and Spencer and Bradley [5] we examine solutions of the form

$$\psi = \psi(\theta), \quad q = \rho g r F(\theta), \tag{2.6}$$

and from the above equations we may deduce

$$\begin{gathered}\frac{\mathrm{d}F}{\mathrm{d}\theta} = \frac{F\sin 2\psi + \sin\phi\sin(2\psi + \theta)}{\sin\phi + \cos 2\psi}, \\ \frac{\mathrm{d}\psi}{\mathrm{d}\theta} + 1 = \frac{F\cot\phi\cos\phi + \cos\theta + \sin\phi\cos(2\psi + \theta)}{2F(\sin\phi + \cos 2\psi)}.\end{gathered} \tag{2.7}$$

Now for a symmetrical stress distribution we have the condition

$$\psi(0) = 0, \tag{2.8}$$

and at the wall $\theta = \alpha$ we assume a Coulomb friction condition, so that

$$\sigma_{r\theta} = -\sigma_{\theta\theta} \tan \mu, \quad \text{at} \quad \theta = \alpha,$$

where μ is the angle of wall friction and α denotes the semi-vertex angle. Thus, from (2.2) and (2.5) we find

$$\sin[2\psi(\alpha) - \mu] = \frac{\sin \mu}{\sin \phi}, \tag{2.9}$$

which is meaningful provided $\mu \leq \phi$. If $\mu \geq \phi$ then the wall is 'perfectly rough' and the material slips on itself at the wall. In this case we find

$$\psi(\alpha) = \frac{\phi}{2} + \frac{\pi}{4}, \tag{2.10}$$

and we observe that for ψ positive, this value of ψ provides the first singularity of both equations of (2.7) in the sense that this value of $\psi(\alpha)$ satisfies $\cos 2\psi = -\sin \phi$. Thus we need to solve (2.7) subject to (2.8) and either (2.9) or (2.10), depending on the value of μ. In general this must be achieved numerically (Spencer and Bradley [5]), and some results are given in the final section of the paper. We note here that a special exact solution of (2.7) is

$$\psi(\theta) = -\theta + \psi^*, \quad F(\theta) = -\frac{\sin \phi}{\cos^2 \phi} \left[\cos \theta + \sin \phi \cos(\theta - 2\psi^*) \right], \tag{2.11}$$

for some constant ψ^*.

3. Mathematical analysis

From the coupled ordinary differential equations and the boundary conditions at $\theta = 0$ and $\theta = \alpha$, we can determine certain additional relations which apply on these boundaries. First at $\theta = 0$, we have from (2.7) and (2.8)

$$F'(0) = 0, \quad \psi'(0) = \frac{1}{2} \left\{ \operatorname{cosec}\phi - 3 + \frac{1}{F(0)} \right\}, \tag{3.1}$$

where primes throughout the section denotes differentiation with respect to θ. Now at $\theta = \alpha$ we have the two possible boundary conditions (2.9) and (2.10) depending on the value of μ. For $\mu \leq \phi$, we have from (2.7) and (2.9)

$$
\begin{aligned}
F'(\alpha) &= \frac{F(\alpha)\sin\mu + \sin\phi\sin(\alpha+\mu)}{\sqrt{\sin^2\phi - \sin^2\mu}} - \frac{\sin\phi\sin\alpha\cos\mu}{\cos^2\phi\sqrt{\sin^2\phi - \sin^2\mu}} + \frac{\sin\phi\sin\alpha}{\cos^2\phi}, \\
\psi'(\alpha) &= -\frac{3}{2} + \frac{\cos\mu}{2\sqrt{\sin^2\phi - \sin^2\mu}} + \frac{\sin\phi\cos(\alpha+\mu)}{2F(\alpha)\sqrt{\sin^2\phi - \sin^2\mu}},
\end{aligned}
\tag{3.2}
$$

where we have taken the positive square root, due to the fact that the negative square root is non-physical. We note from (3.2) that for $\phi = \mu$, both $F'(\alpha)$ and $\psi'(\alpha)$ become infinite and that the numerical results indicate that these remain infinite for $\mu > \phi$.

We note that, if we eliminate $F(\theta)$ from (2.7), we may deduce the following second order differential equation for $\psi(\theta)$,

$$
\begin{aligned}
&(\sin\phi + \cos 2\psi)[\cos\theta + \sin\phi\cos(2\psi+\theta)]\psi'' \\
&= 2(\psi'+1)\Big\{ \sin 2\psi[\cos\theta + \sin\phi\cos(2\psi+\theta)]\psi' \\
&\qquad -2\sin\phi(\sin\phi + \cos 2\psi)\sin(2\psi+\theta)\psi' \\
&\qquad -(3\sin^2\phi + 2\sin\phi\cos 2\psi - 1)\sin(2\psi+\theta) \Big\}.
\end{aligned}
\tag{3.3}
$$

The formal exact parametric solution of this equation for $\phi = \pi/2$ is derived in the Appendix and which applies for $\mu < \phi$. In the following section, we state this solution and determine the constants of integration which satisfy the two boundary conditions given by (2.8) and (2.9). In Section 5, we briefly present the numerical solution of the system of coupled ordinary differential equations (2.7) subject to the boundary conditions (2.8) and (2.9).

4. Exact solution for the special case $\phi = \pi/2$

As shown in the Appendix, equation (3.3) for the special case of $\phi = \pi/2$ admits the following exact parametric solution for $\psi(\theta)$

$$\begin{aligned} \tan\psi &= \frac{I(\omega)}{C_2}\left\{1-\frac{\omega^{1/2}}{2}e^{-\omega/2}I(\omega)\right\}-\frac{C_2}{2}\omega^{1/2}e^{-\omega/2}, \\ \tan\theta &= C_2\left\{2e^{\omega/2}\omega^{-1/2}-I(\omega)\right\}^{-1}, \end{aligned} \tag{4.1}$$

where the integral $I(\omega)$ is defined by

$$I(\omega)=\int^{\omega}t^{-1/2}e^{t/2}dt+C_1, \tag{4.2}$$

where C_1 and C_2 denote two arbitrary constants of integration.

In the special case of $\phi=\pi/2$, we have from (2.8), (2.9) and $(3.2)_2$

$$\psi(0)=0,\quad \psi(\alpha)=\mu,\quad \psi'(\alpha)=-1+\frac{\cos(\alpha+\mu)}{2F(\alpha)\cos\mu}, \tag{4.3}$$

and obviously in this case we have $\mu\leq\phi$. We are now required to determine the constants C_1 and C_2 such that the boundary conditions $(4.3)_1$ and $(4.3)_2$ are satisfied. From (4.1) and (4.2) it is clear that $\psi(0)=0$ provided we identify the parameter value $\omega\equiv 0$ corresponding to $\theta=0$ so that we choose C_1 such that

$$I(\omega)=\int_0^{\omega}t^{-1/2}e^{t/2}dt, \tag{4.4}$$

and in the following we assume this to be the case. Now, to determine C_2 we identify the parameter value $\omega\equiv\omega_0$ to correspond with $\theta=\alpha$ for some ω_0. Then from $(4.3)_2$ we find that (4.1) becomes

$$\begin{aligned} \tan\mu &= \frac{I(\omega_0)}{C_2}\left\{1-\frac{\omega_0^{1/2}}{2}e^{-\omega_0/2}I(\omega_0)\right\}-\frac{C_2}{2}\omega_0^{1/2}e^{-\omega_0/2}, \\ \tan\alpha &= C_2\left\{2e^{\omega_0/2}\omega_0^{-1/2}-I(\omega_0)\right\}^{-1}, \end{aligned} \tag{4.5}$$

and from $(4.5)_2$ we find

$$C_2=\tan\alpha\left\{\frac{2e^{\omega_0/2}}{\omega_0^{1/2}}-I(\omega_0)\right\}, \tag{4.6}$$

which upon substituting into $(4.5)_1$ gives

$$\frac{\sin\alpha}{\cos\mu}\sin(\alpha+\mu)=\frac{\omega_0^{1/2}}{2}e^{-\omega_0/2}I(\omega_0), \tag{4.7}$$

which is a transcendental equation for ω_0. Thus, C_2 is determined from (4.6) where ω_0 is a solution of (4.7).

Now we need to determine a relation for $F(\theta)$ in terms of the parameter ω. On differentiating (A15) and $(4.1)_2$ we may deduce

$$\psi'(\theta) + 1 = \frac{\omega \cos^2(\psi + \theta)}{\sin^2 \theta} = \frac{\cos(\psi + \theta)}{2F \cos \psi}, \tag{4.8}$$

where the latter equality follows from the differential equation $(2.7)_2$ with $\phi = \pi/2$. From (4.8) we obtain

$$F(\theta) = \frac{\sec \psi \sec(\psi + \theta)}{2\omega \ \mathrm{cosec}^2\theta} = \frac{\left\{1 + \left(\dfrac{y - x}{1 + xy}\right)^2\right\}^{1/2} (1 + y^2)^{1/2}}{2\omega(1 + x^{-2})} = \frac{x^2(1 + y^2)}{2\omega(1 + xy)(1 + x^2)^{1/2}}, \tag{4.9}$$

and in terms of x and y defined by (A3). On simplifying (4.9) we may deduce the following expression for $F(\theta)$ in terms of the parameter ω,

$$F(\theta) = \frac{e^{-\omega/2}[C_2^2 + I(\omega)^2]}{4\omega^{1/2} \left\{C_2^2 + [2e^{\omega/2}\omega^{-1/2} - I(\omega)]^2\right\}^{1/2}}, \tag{4.10}$$

where C_2 is given by (4.6). We also note that from (4.8) we may conclude $\psi'(0)+1 = (2F(0))^{-1}$ which is entirely consistent with $(3.1)_2$ in the special case $\phi = \pi/2$.

5. Numerical results

The numerical results shown in Figures 2 and 3 were obtained using an iterative scheme to determine successive numerical solutions which converge to the solution which satisfies the appropriate boundary conditions. For $\mu \leq \phi$, we have used both a Shooting Method, employing a Runge-Kutta scheme for the system of coupled first order ordinary differential equations (2.7), and a Finite-Difference Method for the second order ordinary differential equation (3.3) subject to (2.8) and (2.9) and both methods give same results. Figures 2 and 3 show the variation of $\psi(\theta)$ and $F(\theta)$ respectively for three values of $\phi > \mu$, assuming an average bulk density $\rho = 1567$ kg/m^3 and an angle of wall friction of $\mu = \pi/12$. We note that the solutions shown in Figures 2 and 3 satisfy the relations (3.1) and (3.2) on the boundaries $\theta = 0$ and $\theta = \alpha$. For $\phi = \pi/2$ the general numerical solution and that determined by the exact parametric solution give the same curve with absolute errors as shown in Figures 4 and 5.

Acknowledgements

JMH is grateful to the Australian Research Council for providing a Senior Research Fellowship.

References

[1] A. W. Jenike, Gravity flow of solids. *Trans. Inst. Chem. Engrs.* **40** (1962), 264-271.

[2] A. W. Jenike, Steady gravity flow of frictional-cohesive solids in converging channels. *J. Appl. Mech.* **31** (1964), 5-11.

[3] A. W. Jenike, Gravity flow of frictional-cohesive solids - Convergence to radial stress fields. *J. Appl. Mech.* **32** (1965), 205-207.

[4] J. R. Johanson, Stress and velocity fields in the gravity flow of bulk solids. *J. Appl. Mech.* **31** (1964), 499-506.

[5] A. J. M. Spencer and N. J. Bradley, Gravity flow of granular materials in converging wedges and cones. Proc. 8th Int. Sym. Continuum Modes and Discrete Systems, Ed. K.Z. Markov, World Scientific 1996, June 11-16 1995, Varna, Bulgaria, 581-590.

[6] N. J. Bradley, Gravity flows of granular materials. Ph.D. Thesis, University of Nottingham (1991).

[7] A. J. M. Spencer, Mechanics of Solids: the Rodney Hill Anniversary Volume, Eds. H. G. Hopkins and M. J. Sewell, 607, Pergamon Press, Oxford, 1982.

Appendix: Derivation of the exact parametric solution (4.1) and (4.2) of equation (3.3) for the special case of $\phi = \pi/2$.

If $\phi = \pi/2$, equation (3.3) simplifies to give

$$\cos\psi\cos(\psi+\theta)\psi'' = -2(\psi'+1)\left\{\ \sin(\psi+\theta)\cos\psi + \cos(\psi+\theta)\sin\psi\ \right\}, \qquad \text{(A1)}$$

which can be rearranged to yield

$$[\sec^2(\psi+\theta)(\psi'+1)]' + 2\tan\psi[\sec^2(\psi+\theta)(\psi'+1)] = 0. \qquad \text{(A2)}$$

Thus if we make the transformations

$$x = \tan\theta, \quad y = \tan(\psi+\theta), \qquad \text{(A3)}$$

so that

$$\tan\psi = \frac{y-x}{1+xy}, \qquad \text{(A4)}$$

then equation (A2) can be shown to simplify to yield

$$\frac{d^2y}{dx^2} + \frac{2y}{(1+xy)}\frac{dy}{dx} = 0. \qquad \text{(A5)}$$

Now this equation remains invariant under the stretching group of transformations

$$x_1 = \lambda x, \quad y_1 = \lambda^{-1}y, \qquad \text{(A6)}$$

and therefore we introduce the new variable $z = xy$ and equation (A5) becomes on making the Euler transformation $s = \log x$,

$$\frac{d^2z}{ds^2} - 3\frac{dz}{ds} + 2z + \frac{2z}{(1+z)}\left(\frac{dz}{ds} - z\right) = 0. \qquad \text{(A7)}$$

Now equation (A7) can be reduced to an Abel equation of the second kind by making the substitution $u = dz/ds$ and taking z as the independent variable. However, although a first order differential equation, (A7) appears not to be integrable by this procedure. Alternatively, if we introduce ω such that

$$\omega = \frac{dz}{ds} - z, \qquad \text{(A8)}$$

then equation (A7) is equivalent to

$$\frac{dz}{ds} = z + \omega, \quad \frac{d\omega}{ds} = \frac{2\omega}{(z+1)}. \tag{A9}$$

On eliminating z from these equations and introducing

$$v = \frac{1}{\omega}\frac{d\omega}{ds}, \tag{A10}$$

we may readily deduce the standard first order differential equation

$$\frac{dv}{d\omega} + \frac{1}{2}\left(1 - \frac{1}{\omega}\right) v = -\frac{1}{\omega}. \tag{A11}$$

This equation readily integrates to give

$$v = 2 - \omega^{1/2} e^{-\omega/2} I(\omega), \tag{A12}$$

where $I(\omega)$ is the integral defined by (4.2). From (A10) and (A12) we may perform a second integration to obtain

$$2\omega^{-1/2} e^{\omega/2} - I(\omega) = \frac{C_2}{x}, \tag{A13}$$

from which we may readily deduce $(4.1)_2$. Equation $(4.1)_1$ follows from the fact that $v = 2/(z+1)$ and therefore

$$z = \frac{2}{v} - 1 = \frac{\omega^{1/2}}{2} e^{-\omega/2} I(\omega) \left\{ 1 - \frac{\omega^{1/2}}{2} e^{-\omega/2} I(\omega) \right\}^{-1}. \tag{A14}$$

But $z = xy$ and we may deduce

$$y = \tan(\psi + \theta) = \frac{I(\omega)}{C_2}, \tag{A15}$$

and $(4.1)_1$ now follows from this equation and (A4), and we have established the formal parametric solution given by (4.1) and (4.2).

Figure Captions

Figure 1. Coordinates for two-dimensional flow in a converging wedge.

Figure 2. Variation of $\psi(\theta)$ for three values of ϕ for which $\mu \leq \phi$, where the angle of wall friction μ is $\pi/12$.

Figure 3. Variation of $F(\theta)$ for three values of ϕ for which for $\mu \leq \phi$, where the angle of wall friction μ is $\pi/12$.

Figure 4. Absolute error between the exact parametric solution (4.1) and the numerical solution for $\psi(\theta)$ for $\phi = \pi/2$.

Figure 5. Absolute error between the exact parametric solution (4.1) and (4.10) and the numerical solution for $F(\theta)$ for $\phi = \pi/2$.