Some exact velocity profiles for granular flow in converging hoppers

Grant M. Cox and James M. Hill

Abstract. Gravity flow of granular materials through hoppers occurs in many industrial processes. For an ideal cohesionless granular material, which satisfies the Coulomb-Mohr yield condition, the number of known analytical solutions is limited. However, for the special case of the angle of internal friction δ equal to ninety degrees, there exist exact parametric solutions for the governing coupled ordinary differential equations for both two-dimensional wedges and three-dimensional cones, both of which involve two arbitrary constants of integration. These solutions are the only known analytical solutions of this generality. Here, we utilize the double-shearing theory of granular materials to determine the velocity field corresponding to these exact parametric solutions for the two problems of gravity flow through converging wedge and conical hoppers. An independent numerical solution for other angles of internal friction is shown to coincide with the analytical solution.

Mathematics Subject Classification (2000). 73B10, 73B99, 73Q05.

Keywords. Converging hoppers, granular material, Coulomb-Mohr yield condition, exact parametric solutions, velocity profiles.

1 Introduction

The flow of granular materials through converging hoppers under the presence of gravity occurs throughout the world in many industries. Despite the common occurrence of such phenomena, disruptions to the flow of material occurs frequently due to the lack of a proper understanding of the complex nature of granular materials. Here, we consider the problem of granular flow under gravity through two-dimensional converging wedges and three-dimensional cones. Jenike [1, 2, 3] and Johanson [4] first

Granular material	Measured values of $\beta = \sin\delta$			
Coal	0.939	0.958	0.973	0.985
Alumina cake		0.941		
Waste rock		0.974		
Silica		0.979		

Table 1: Measured values of $\beta = \sin\delta$ for certain granular materials, where δ is the angle of internal friction.

studied these problems by looking for radial stress field solutions where the equilibrium equations and the Coulomb-Mohr yield condition reduce to two highly nonlinear coupled ordinary differential equations. Bradley [5] and Spencer and Bradley [6] have re-examined the Jenike radial flow solutions with a view to determining the associated double-shearing velocity flow field for fully developed steady flow, where the granular material is assumed to be incompressible.

Previously, only a simple exact solution of the governing nonlinear coupled ordinary differential equations with a single arbitrary constant, as determined by Sokolovsky [7], was known along with some isolated corresponding three-dimensional solutions given by Spencer and Bradley [6]. Recently, these equations have been shown to admit exact parametric solutions for the special cases of $\beta = \pm 1$, where $\beta = \sin\delta$ and δ is the angle of internal friction (see Hill and Cox [8] and Cox and Hill [9]). These solutions are the first exact solutions of the highly nonlinear coupled ordinary differential equations which involve two arbitrary constants. While of course the special case of $\beta = -1$ corresponds to a non-physical material, there are however many granular materials which do indeed exhibit large angles of internal friction such as those shown in Table 1, and the case $\beta = 1$ is physically plausible and constitutes an ideal limiting theory.

In this paper, we utilize the non-dilatant double-shearing theory of granular materials to determine the corresponding two and three-dimensional associated velocity fields for the exact parametric solutions determined by Hill and Cox [8] and Cox and

Hill [9] for the special case of $\delta = \pi/2$. Here, for convenience and for purposes of comparison, we follow the notation adopted by Spencer and Bradley [6], including adopting the unusual convention of the x-axis being vertical.

In the following section, we briefly state the basic equations for the determination of the stress field using the continuum mechanical theory of granular materials for quasi-static flow of an ideal cohesionless material which satisfies the Coulomb-Mohr yield condition, and for both two-dimensional plane strain flow and three-dimensional axially symmetric flow. Following Spencer and Bradley [6], we also state the basic equations of the non-dilatant double shearing theory of granular materials for the determination of the associated velocity field for fully developed steady flow. In Section 3, we state the two and three-dimensional exact parametric solutions for the special case of $\delta = \pi/2$ and determine the corresponding velocity fields. Independent numerical solutions for any angle of internal friction are presented in Section 4. For $\delta = \pi/2$, these coincide with the exact parametric solutions. Some brief concluding remarks made in Section 5.

2 Basic equations

In the following two subsections we state briefly the basic equations for the proper continuum mechanical theory of granular materials for the two problems of gravity flow through hoppers for two-dimensional wedges and three-dimensional cones.

2.1 Two-dimensional basic equations

In terms of cylindrical polar coordinates (r, θ, z) as defined by Figure 1(a), the non-zero Cauchy stress components for quasi-static plane strain flow in wedge shaped hoppers, under the presence of gravity, satisfy the equilibrium equations

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = \rho g \cos\theta,$$
$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{2\sigma_{r\theta}}{r} = -\rho g \sin\theta, \tag{2.1}$$

where ρ is the bulk density, assumed constant, g is acceleration due to gravity, and $\sigma_{rr}, \sigma_{r\theta}$ and $\sigma_{\theta\theta}$ denote the usual in-plane Cauchy stress components, which are assumed to be positive in tension. Namely, the usual convention in continuum mechanics is adopted that positive forces are assumed to produce positive extensions. Following Spencer and Bradley [6] these components can be expressed in the standard form

$$\sigma_{rr} = -p + q\cos 2\psi, \quad \sigma_{\theta\theta} = -p - q\cos 2\psi, \quad \sigma_{r\theta} = q\sin 2\psi, \tag{2.2}$$

where p and q are the positive quantities defined by

$$p = -\frac{1}{2}(\sigma_{rr} + \sigma_{\theta\theta}), \quad q = \frac{1}{2}\left\{(\sigma_{rr} - \sigma_{\theta\theta})^2 + 4\sigma_{r\theta}^2\right\}^{1/2}, \tag{2.3}$$

while the stress angle ψ satisfies

$$\tan 2\psi = \frac{2\sigma_{r\theta}}{\sigma_{rr} - \sigma_{\theta\theta}}, \tag{2.4}$$

and physically ψ is the angle between the direction of the maximum principal stress axis and the r direction, in the direction of increasing θ as shown in Figure 1(a). For a cohesionless material, the Coulomb-Mohr yield condition takes the form

$$q = p\sin\delta, \tag{2.5}$$

where δ is a material constant referred to as the angle of internal friction.

Following Jenike [2] and Spencer and Bradley [6] we look for a wedge field solution of the form

$$\psi = \psi(\theta), \quad q = \rho g r F(\theta), \tag{2.6}$$

so that from the above equations we may deduce the following governing nonlinear equations

$$\frac{dF}{d\theta} = \frac{F\sin 2\psi + \sin\delta\sin(\theta + 2\psi)}{\sin\delta + \cos 2\psi},$$
$$\frac{d\psi}{d\theta} + 1 = \frac{F\cos\delta\cot\delta + \cos\theta + \sin\delta\cos(\theta + 2\psi)}{2F(\sin\delta + \cos 2\psi)}. \tag{2.7}$$

We note from $(2.6)_2$ that since q is a positive quantity, then F must also be positive for all θ. We also note that if we eliminate F from (2.7) then we may deduce the following single second order ordinary differential equation for $\psi(\theta)$,

$$(\sin\delta + \cos 2\psi)[\cos\theta + \sin\delta\cos(\theta + 2\psi)]\psi''$$
$$= 2(1 + \psi')\{\sin 2\psi[\cos\theta + \sin\delta\cos(\theta + 2\psi)]\psi' \tag{2.8}$$
$$-2\sin\delta(\sin\delta + \cos 2\psi)\sin(\theta + 2\psi)\psi' - (3\sin^2\delta + 2\sin\delta\cos 2\psi - 1)\sin(\theta + 2\psi)\},$$

where throughout the paper primes denote differentiation with respect to θ. This latter formulation formed the basis for the exact analysis for $\delta = \pi/2$ given in [8], and it is also the basis for an independent numerical scheme.

Now, due to the geometry of the hopper, we assume that the stress distribution is symmetrical around the vertical axis. As a result, we observe from the equilibrium equations (2.1) that σ_{rr} and $\sigma_{\theta\theta}$ must be even functions of θ, while $\sigma_{r\theta}$ must be an odd function or skew-symmetric. Thus, to ensure continuity $\sigma_{r\theta}$ must vanish at the origin giving rise to the boundary condition

$$\psi(0) = 0. \tag{2.9}$$

To determine the second stress boundary condition, following Spencer and Bradley [6] we assume a Coulomb friction condition at the wall of the hopper at $\theta = \alpha$, such that

$$\sigma_{r\theta} = -\sigma_{\theta\theta}\tan\mu, \quad \text{at} \quad \theta = \alpha, \tag{2.10}$$

where μ is the angle of wall friction and α denotes the semi-vertex angle. Thus, from (2.2) and (2.5) we find

$$\sin[2\psi(\alpha) - \mu] = \frac{\sin \mu}{\sin \delta}, \tag{2.11}$$

which is meaningful provided $\mu \leq \delta$. If $\mu \geq \delta$ then the wall is perfectly rough and the material slips on itself at the wall, and in this case we find

$$\psi(\alpha) = \frac{\delta}{2} + \frac{\pi}{4}. \tag{2.12}$$

We observe that for ψ positive, this value of ψ provides the first singularity of both equations of (2.7) in the sense that it satisfies $\cos 2\psi = -\sin \delta$. We note that (2.11) and (2.12) coincide when $\mu = \delta$, and that if $\delta = \pi/2$, then only the boundary condition (2.11) can apply.

The above equations are generally accepted as a reasonable basis for the determination of the plane strain stress profile for a free flowing (cohesionless) granular material. However, to prescribe equations for the determination of the associated velocity profile is far more controversial. Here, we assume the non-dilatant double-shearing theory as enunciated by Spencer [10, 11]. This theory assumes that the non-zero steady components of velocity $u(r, \theta)$ and $v(r, \theta)$ in the r and θ directions respectively, satisfy the following equations

$$\begin{gathered}\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} = 0, \\ \left(\frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}\right)\cos 2\psi - \left(\frac{\partial u}{\partial r} - \frac{u}{r} - \frac{1}{r}\frac{\partial v}{\partial \theta}\right)\sin 2\psi = \sin \delta \left(\frac{\partial v}{\partial r} - \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r} - 2\Omega\right),\end{gathered} \tag{2.13}$$

where the quantity Ω is given by

$$\Omega = u\frac{\partial \psi}{\partial r} + \frac{v}{r}\frac{\partial \psi}{\partial \theta}. \tag{2.14}$$

Physically speaking, equation $(2.13)_1$ corresponds to the assumption that the flow is isochoric, while $(2.13)_2$ expresses the condition that the flow arises due to simultaneous shearing on the two families of surfaces on which the critical shear stress is mobilized. Together with the appropriate velocity boundary conditions, the above equations represent a complete description for the dynamical behaviour of non-dilatant cohesionless granular flow through a two-dimensional converging wedge shaped hopper under the presence of gravity.

2.2 Three-dimensional basic equations

In terms of spherical polar coordinates (R, Θ, Φ) as defined in Figure 1(b), the non-zero Cauchy stress components for quasi-static axially symmetric gravity flow in cone shaped hoppers satisfy the equilibrium equations

$$\begin{aligned} \frac{\partial \sigma_{RR}}{\partial R} + \frac{1}{R}\frac{\partial \sigma_{R\Theta}}{\partial \Theta} + \frac{1}{R}(2\sigma_{RR} - \sigma_{\Theta\Theta} - \sigma_{\Phi\Phi} + \sigma_{R\Theta}\cot\Theta) = \rho g \cos\Theta, \\ \frac{\partial \sigma_{R\Theta}}{\partial R} + \frac{1}{R}\frac{\partial \sigma_{\Theta\Theta}}{\partial \Theta} + \frac{1}{R}(\sigma_{\Theta\Theta} - \sigma_{\Phi\Phi})\cot\Theta + \frac{3}{R}\sigma_{R\Theta} = -\rho g \sin\Theta, \end{aligned} \tag{2.15}$$

where ρ is the bulk density, assumed constant, g is acceleration due to gravity, and $\sigma_{RR}, \sigma_{R\Theta}, \sigma_{\Theta\Theta}$ and $\sigma_{\Phi\Phi}$ denote the physical components of the Cauchy stress, which are again assumed to be positive in tension. Again, following Spencer and Bradley [6] these components can be expressed in the standard form

$$\sigma_{RR} = -p + q\cos 2\Psi, \quad \sigma_{\Theta\Theta} = -p - q\cos 2\Psi, \quad \sigma_{R\Theta} = q\sin 2\Psi, \tag{2.16}$$

where the positive quantities p and q are defined as

$$p = -\frac{1}{2}(\sigma_{RR} + \sigma_{\Theta\Theta}), \quad q = \frac{1}{2}\left\{(\sigma_{RR} - \sigma_{\Theta\Theta})^2 + 4\sigma_{R\Theta}^2\right\}^{1/2}, \tag{2.17}$$

while the stress angle Ψ is given by

$$\tan 2\Psi = \frac{2\sigma_{R\Theta}}{\sigma_{RR} - \sigma_{\Theta\Theta}}, \tag{2.18}$$

and physically Ψ is the angle between the direction of the maximum principal stress axis and the R direction, in the direction of increasing Θ as shown in Figure 1(b).

Now, we need to make an assumption about the hoop stress in order to determine an expression for $\sigma_{\Phi\Phi}$ in terms of p, q and Ψ. Cox *et al* [12] state that the plastic regimes which agree with the Haar-von Karman hypothesis are likely to be of greatest significance in the solution of problems of interest. The heuristic Haar-von Karman principle states, under an axially symmetric condition, that the hoop stress is equal to either the maximum or minimum principal stress. This gives rise to the notion of the Haar-von Karman regimes, and in particular, either $\sigma_I = \sigma_{\Phi\Phi} = \sigma_{II} > \sigma_{III}$ or $\sigma_I > \sigma_{II} = \sigma_{\Phi\Phi} = \sigma_{III}$, where σ_I, σ_{II} and σ_{III} denote the maximum, intermediate and minimum principal stresses respectively. Thus, we may deduce

$$\sigma_{\Phi\Phi} = -p + \epsilon q, \tag{2.19}$$

where ϵ has the value 1 if the hoop stress is the maximum principal stress and -1 if the hoop stress is the minimum principal stress. We note that Spencer and Bradley [6] take the hoop stress to be equal to the minimum principal stress. When the hoop stress is equal to the minimum principal stress, Cox and Hill [9] show that an exact parametric solution can only be determined for the case of the angle of internal friction equal to minus ninety degrees, which is evidently non-physical. However, when the hoop stress is equal to the maximum principal stress, we find from Cox and Hill [9] that an exact parametric solution can be determined for the special case of the angle of internal friction equal to ninety degrees. Thus, we assume that the hoop stress is equal to the maximum principal stress, or in other words we suppose $\epsilon = 1$. For a cohesionless material, the Coulomb-Mohr yield condition again becomes (2.5).

Following Jenike [2] and Spencer and Bradley [6] we look for a wedge field solution of the form

$$\Psi = \Psi(\Theta), \quad q = \rho g R G(\Theta), \tag{2.20}$$

so that from the above equations we may deduce the following governing nonlinear equations

$$\frac{dG}{d\Theta} = \frac{2G\cos\Psi\{\sin\Psi - \sin\delta \mathrm{cosec}\Theta\cos(\Theta+\Psi)\} + \sin\delta\sin(\Theta+2\Psi)}{\sin\delta + \cos 2\Psi},$$

$$\frac{d\Psi}{d\Theta} + 1 \tag{2.21}$$

$$= \frac{G(1-\sin\delta)\{\mathrm{cosec}\delta(1+2\sin\delta) - \mathrm{cosec}\Theta\sin(\Theta+2\Psi)\} + \cos\Theta + \sin\delta\cos(\Theta+2\Psi)}{2G(\sin\delta+\cos 2\Psi)}.$$

We note from $(2.20)_2$ that since q is a positive quantity, then G must also be positive for all Θ. We also note that if we eliminate G from (2.21) then we may deduce the following single second order ordinary differential equation for $\Psi(\Theta)$,

$$2(\sin\delta+\cos 2\Psi)[\cos\Theta + \sin\delta\cos(\Theta+2\Psi)]\Psi''$$

$$= 2(1+\Psi')\left\{ \mathrm{cosec}\Theta \left[\sin\delta(\sin\delta+\cos 2\Psi)[1+\cos\{2(\Theta+\Psi)\}] \right.\right.$$

$$-[\cos\Theta+\sin\delta\cos(\Theta+2\Psi)][2\sin\Theta\sin 2\Psi + (1-\sin\delta)\cos(\Theta+2\Psi)]$$

$$\left. +2(1-\sin\delta)\sin(\Theta+2\Psi)[(1+2\sin\delta)\sin\Theta - \sin\delta\sin(\Theta+2\Psi)] \right] \tag{2.22}$$

$$\left. +2(1+\Psi')\left[\sin 2\Psi[\cos\Theta+\sin\delta\cos(\Theta+2\Psi)] - 2\sin\delta\sin(\Theta+2\Psi)(\sin\delta+\cos 2\Psi)\right] \right\}$$

$$+(1-\sin\delta)\mathrm{cosec}^2\Theta \left\{ \sin\{2(\Theta+\Psi)\}[\cos\Theta+\sin\delta\cos(\Theta+2\Psi)] \right.$$

$$\left. -[1+\cos\{2(\Theta+\Psi)\}][(1+2\sin\delta)\sin\Theta - \sin\delta\sin(\Theta+2\Psi)] \right\},$$

where throughout the paper primes denote differentiation with respect to Θ. Again we note that this formulation formed the basis for the exact analysis for $\delta = \pi/2$ given in [9], and it is also the basis for an independent numerical scheme.

Now, the stress distribution is symmetrical about the vertical axis, and we observe from the equilibrium equations (2.15) that $\sigma_{RR}, \sigma_{\Theta\Theta}$ and $\sigma_{\Phi\Phi}$ must be even functions of Θ, while $\sigma_{R\Theta}$ must be an odd function or skew-symmetric. Thus, to ensure continuity $\sigma_{R\Theta}$ vanishes at the origin, so we require the boundary condition

$$\Psi(0) = 0. \tag{2.23}$$

To determine the second stress boundary condition, following Spencer and Bradley [6] we assume a Coulomb friction condition at the wall of the hopper at $\Theta = \alpha$, such that

$$\sigma_{R\Theta} = -\sigma_{\Theta\Theta} \tan\mu, \quad \text{at} \quad \Theta = \alpha, \tag{2.24}$$

where μ is the angle of wall friction. Thus, from (2.5) and (2.16) we find

$$\sin[2\Psi(\alpha) - \mu] = \frac{\sin\mu}{\sin\delta}, \tag{2.25}$$

which is again only meaningful provided $\mu \leq \delta$. If $\mu \geq \delta$ then the wall is perfectly rough and the material slips on itself at the wall, and in this case we find

$$\Psi(\alpha) = \frac{\delta}{2} + \frac{\pi}{4}. \tag{2.26}$$

We again observe that for Ψ positive, this value of Ψ provides the first singularity of both equations of (2.21) in the sense that it satisfies $\cos 2\Psi = -\sin\delta$. We again note that (2.25) and (2.26) coincide when $\mu = \delta$, and that if $\delta = \pi/2$, then only the boundary condition (2.25) can apply.

For the determination of the associated velocity profile, we again assume the non-dilatant double-shearing theory as enunciated by Spencer [10, 11]. This theory assumes that the non-zero steady components of velocity $U(R, \Theta)$ and $V(R, \Theta)$ in the R and Θ directions respectively, satisfy the following equations

$$\frac{\partial U}{\partial R} + \frac{1}{R}\frac{\partial V}{\partial \Theta} + \frac{2U}{R} + \cot\Theta \frac{V}{R} = 0,$$

$$\left(\frac{1}{R}\frac{\partial U}{\partial \Theta} + \frac{\partial V}{\partial R} - \frac{V}{R}\right)\cos 2\Psi - \left(\frac{\partial U}{\partial R} - \frac{U}{R} - \frac{1}{R}\frac{\partial V}{\partial \Theta}\right)\sin 2\Psi = \sin\delta\left(\frac{\partial V}{\partial R} - \frac{1}{R}\frac{\partial U}{\partial \Theta} - \frac{V}{R} - 2\Omega\right), \tag{2.27}$$

where the quantity Ω is given by

$$\Omega = U\frac{\partial \Psi}{\partial R} + \frac{V}{R}\frac{\partial \Psi}{\partial \Theta}. \tag{2.28}$$

3 Exact solutions for the special case of $\delta = \pi/2$

In this section, we briefly state the two and three-dimensional exact parametric solutions given in [8] and [9] for the special case of an angle of internal friction equal to ninety degrees and for gravity flow through a converging hopper. We utilize these known solutions to determine the corresponding velocity fields applying to flow thorugh converging hoppers, according to the non-dilatant double-shearing theory of granular materials.

3.1 Two-dimensional exact solution

The two-dimensional exact parametric solution for the special case of $\delta = \pi/2$ was first derived in Hill and Cox [8] for the determination of the stress distribution in a hopper, and later utilized in Hill and Cox [13] to solve the problem of determining the exact stress distribution beneath a granular heap and in Cox *et al* [14] to find the stress distribution in a sloping rat-hole. Here, following Spencer and Bradley [6], we utilize the non-dilatant double-shearing theory to determine the corresponding velocity field for converging wedge flow. Hill and Cox [8] show that equation (2.8) for the special case of $\delta = \pi/2$ admits the following exact parametric solution for $\psi(\theta)$,

$$\begin{aligned} \tan\psi &= \frac{I(\omega)}{C_2}\left\{1 - \frac{\omega^{1/2}}{2}e^{-\omega/2}I(\omega)\right\} - \frac{C_2}{2}\omega^{1/2}e^{-\omega/2}, \\ \tan\theta &= \frac{C_2}{2\omega^{-1/2}e^{\omega/2} - I(\omega)}, \end{aligned} \tag{3.1}$$

where the integral $I(\omega)$ is defined by

$$I(\omega) = \int_0^\omega t^{-1/2}e^{t/2}dt, \tag{3.2}$$

and the constant of integration C_2 is given by

$$C_2 = \left\{2\omega_0^{-1/2} e^{\omega_0/2} - I(\omega_0)\right\} \tan\alpha, \tag{3.3}$$

where ω_0 satisfies the transcendental equation

$$\frac{\sin\alpha}{\cos\mu} \sin(\alpha+\mu) = \frac{\omega_0^{1/2}}{2} e^{-\omega_0/2} I(\omega_0), \tag{3.4}$$

with the parameter range $0 \leq \omega \leq \omega_0$ corresponding directly to $0 \leq \theta \leq \alpha$. We note that this solution satisfies the boundary conditions (2.9) and (2.11). We also note from Hill and Cox [8] that F is given by

$$F = \frac{1}{4} \frac{\omega^{-1/2} e^{-\omega/2} [C_2^2 + I^2(\omega)]}{\{C_2^2 + [2\omega^{-1/2} e^{\omega/2} - I(\omega)]^2\}^{1/2}}. \tag{3.5}$$

Now, following Jenike [2] and Spencer and Bradley [6], for converging wedge flow we assume a velocity field of the form

$$u = \frac{u(\theta)}{r}, \quad v = 0, \tag{3.6}$$

so that $(2.13)_1$ is satisfied and $(2.13)_2$ becomes

$$\frac{du}{d\theta}(\cos 2\psi + \sin\delta) + 2u \sin 2\psi = 0. \tag{3.7}$$

Clearly, for the special case of $\delta = \pi/2$, (3.7) becomes simply

$$\frac{du}{d\theta} = -2u \tan\psi, \tag{3.8}$$

or,

$$\frac{du}{u} = -2 \tan\psi \frac{d\theta}{d\omega} d\omega. \tag{3.9}$$

Thus, from (3.1) we find that (3.9) can be expressed in the form

$$\frac{du}{u} = \frac{d\omega}{\omega} - \frac{2\omega^{-3/2} e^{\omega/2} [2\omega^{-1/2} e^{\omega/2} - I(\omega)]}{\{C_2^2 + [2\omega^{-1/2} e^{\omega/2} - I(\omega)]^2\}} d\omega \tag{3.10}$$

which can be integrated to yield

$$u = C_3 \omega \left\{C_2^2 + [2\omega^{-1/2} e^{\omega/2} - I(\omega)]^2\right\}, \tag{3.11}$$

where C_3 is a constant of integration. If we assume the boundary condition that $u = u_0$ at $\theta = 0$, then from (3.11) we find that $C_3 = u_0/4$.

3.2 Three-dimensional exact solution

The three-dimensional exact parametric solution for the special case of $\delta = \pi/2$ was first derived in Cox and Hill [9] for determining the stress distributions for the two distinct problems of axially symmetric hopper flow and the determination of the stress beneath a granular heap, and later utilized in Cox *et al* [14] to find the stress distribution in a sloping rat-hole. Again, following Spencer and Bradley [6], we utilize the non-dilatant double-shearing theory to determine the corresponding velocity field for converging axially symmetric conical flow. Hill and Cox [9] show that equation (2.22) for the special case of $\delta = \pi/2$ admits the following exact parametric solution for $\Psi(\Theta)$,

$$\tan\Psi = \frac{I(\omega)}{C_2}\left\{1 - \frac{\omega^{1/3}}{3}e^{-\omega/3}I(\omega)\right\} - \frac{C_2}{3}\omega^{1/3}e^{-\omega/3},$$
$$\tan\Theta = \frac{C_2}{3\omega^{-1/3}e^{\omega/3} - I(\omega)}, \tag{3.12}$$

where the integral $I(\omega)$ is defined by

$$I(\omega) = \int_0^{\omega} t^{-1/3}e^{t/3}dt, \tag{3.13}$$

and the constant of integration C_2 is given by

$$C_2 = \left\{3\omega_0^{-1/3}e^{\omega_0/3} - I(\omega_0)\right\}\tan\alpha, \tag{3.14}$$

where ω_0 satisfies the transcendental equation

$$\frac{\sin\alpha}{\cos\mu}\sin(\alpha+\mu) = \frac{\omega_0^{1/3}}{3}e^{-\omega_0/3}I(\omega_0), \tag{3.15}$$

with the parameter range $0 \le \omega \le \omega_0$ corresponding directly to $0 \le \Theta \le \alpha$. We note that this solution satisfies the boundary conditions (2.23) and (2.25). We also note from Hill and Cox [9] that G is given by

$$G = \frac{1}{6}\frac{\omega^{-2/3}e^{-\omega/3}[C_2^2 + I^2(\omega)]}{\{C_2^2 + [3\omega^{-1/3}e^{\omega/3} - I(\omega)]^2\}^{1/2}}. \tag{3.16}$$

Now, following Jenike [2] and Spencer and Bradley [6], for converging axially symmetric conical flow we assume a velocity field of the form

$$U = \frac{U(\Theta)}{R^2}, \quad V = 0, \tag{3.17}$$

so that $(2.27)_1$ is satisfied and $(2.27)_2$ becomes

$$\frac{dU}{d\Theta}(\cos 2\Psi + \sin\delta) + 3U\sin 2\Psi = 0. \tag{3.18}$$

Clearly, for the special case of $\delta = \pi/2$, (3.18) becomes simply

$$\frac{dU}{d\Theta} = -3U\tan\Psi, \tag{3.19}$$

or,

$$\frac{dU}{U} = -3\tan\Psi\frac{d\Theta}{d\omega}d\omega. \tag{3.20}$$

Thus, from (3.12) we find that (3.20) can be expressed in the form

$$\frac{dU}{U} = \frac{d\omega}{\omega} - \frac{3\omega^{-4/3}e^{\omega/3}[3\omega^{-1/3}e^{\omega/3} - I(\omega)]}{\{C_2^2 + [3\omega^{-1/3}e^{\omega/3} - I(\omega)]^2\}}d\omega \tag{3.21}$$

which can be integrated to yield

$$U = C_3\omega\left\{C_2^2 + [3\omega^{-1/3}e^{\omega/3} - I(\omega)]^2\right\}^{3/2}, \tag{3.22}$$

where C_3 is a constant of integration. If we assume the boundary condition that $U = U_0$ at $\Theta = 0$, then from (3.22) we find that $C_3 = U_0/27$.

4 Numerical results

In this section, we illustrate graphically the two and three-dimensional exact parametric velocity profiles, (3.11) and (3.22), which apply for the special case of an angle of internal friction $\delta = \pi/2$. We compare these profiles with a full independent numerical solution, and present velocity profiles for two other values of the angle of internal friction, namely $\delta = \pi/3$ and $\delta = \pi/6$.

Figure 2(a) shows the numerical variation of $\psi(\theta)$ for the three values of the angle of internal friction. These numerical solutions were obtained following Hill and Cox [8], where a Shooting Method, employing a fourth order Runge-Kutta scheme, was applied to the governing stress equations (2.7) subject to the boundary conditions (2.9) and (2.11), and following Hill and Cox [8] the constants μ, α and u_0 are assumed to be given by $\mu = \pi/12$, $\alpha = 287\pi/900$ and $u_0 = 1$. The corresponding velocity profiles are then numerically determined from (3.7) using a simple fourth order Runge-Kutta scheme, and the results are shown in Figure 2(b). We note that the exact parametric velocity profile (3.11) coincides precisely with the full numerical solution for $\delta = \pi/2$.

Figure 3(a) shows the numerical variation of $\Psi(\Theta)$ for the three values of the angle of internal friction. These numerical solutions were also obtained following Hill and Cox [9], where a Shooting Method, employing a fourth order Runge-Kutta scheme, was applied to the governing stress equations (2.21) subject to the boundary conditions (2.23) and (2.25), and the constants μ, α and U_0 are again assumed to be given by $\mu = \pi/12$, $\alpha = 287\pi/900$ and $U_0 = 1$. The corresponding velocity profiles are then numerically determined from (3.18) using a simple fourth order Runge-Kutta scheme, and the results are shown in Figure 3(b). We again note that the exact parametric velocity profile (3.22) coincides precisely with the full numerical solution for $\delta = \pi/2$.

5 Conclusions

In a previous paper appearing in this journal (Hill and Cox [8]) the present authors deduced the first exact analytical solution of the stress equations (2.7) for plane strain quasi-static flows of a Coulomb-Mohr granular solid applying to materials for which the angle of internal friction equals ninety degrees. In a practical context there are many such materials with high angles of internal friction such as those given in Table 1, and the value $\delta = \pi/2$ provides an idealized limiting theory of granular behaviour. In a subsequent paper (Cox and Hill [9]) the authors have given the corresponding analytical solution for the stress equations for axially symmetric flows. Here, for both exact solutions we have determined the corresponding velocity profiles, assuming the non-dilatant double-shearing theory as proposed by Spencer [10, 11] and applying to the two problems of converging granular flow in wedge and conical hoppers. We find that these profiles can also be integrated exactly giving rise to the explicit expressions (3.11) and (3.22) for the two velocity fields, and that both agree with an independent numerical scheme. Our results are in complete qualitative agreement with the extensive numerical results given by Bradley [5]. Surprisingly, for the parameter values chosen here, they indicate that for the conical hopper, the predicated velocity profiles do not vary much with the angle of internal friction of the material and therefore the solution corresponding to $\delta = \pi/2$ is a useful estimate for all angles of internal friction.

Acknowledgements

This work is supported by the Australian Research Council, both through the Large Grant Scheme and for providing a Senior Research Fellowship for JMH. This support is gratefully acknowledged. The authors are also grateful for the data given in Table 1 which was provided by Ms. Wendy Halford, Centre for Bulk Solids and Particulate Technologies, University of Wollongong.

References

[1] A. W. Jenike, Gravity flow of solids. *Trans. Inst. Chem. Engrs.* **40** (1962), 264-271.

[2] A. W. Jenike, Steady gravity flow of frictional-cohesive solids in converging channels. *J. Appl. Mech.* **31** (1964), 5-11.

[3] A. W. Jenike, Gravity flow of frictional-cohesive solids - Convergence to radial stress fields. *J. Appl. Mech.* **32** (1965), 205-207.

[4] J. R. Johanson, Stress and velocity fields in the gravity flow of bulk solids. *J. Appl. Mech.* **31** (1964), 499-506.

[5] N. J. Bradley, Gravity flows of granular materials. Ph.D. Thesis, University of Nottingham (1991).

[6] A. J. M. Spencer and N. J. Bradley, Gravity flow of granular materials in converging wedges and cones. Proc. 8th Int. Sym. Continuum Modes and Discrete Systems, Ed. K.Z. Markov, World Scientific 1996, June 11-16 1995, Varna, Bulgaria, 581-590.

[7] V. V. Sokolovsky, Statics of granular media, Peramin, Oxford (1965).

[8] J. M. Hill and G. M. Cox, An exact parametric solution for granular flow in a converging wedge. *Zeitschrift fur angewandte Mathematik und Physik (ZAMP).* **52**, 657-668.

[9] G. M. Cox and J. M. Hill, Some exact mathematical solutions for granular stock piles and granular flow in hoppers, *Mathematics and Mechanics of Solids*, accepted for publication.

[10] A. J. M. Spencer, A theory of the kinematics of ideal soils under plane strain conditions, *J. Mech. Phy. Solids.* **21** (1964), 337-351.

[11] A. J. M. Spencer, Deformation of ideal granular materials, In: *Mechanics of Solids*, Eds. H. G. hopkins and M. J. Sewell, Oxford, (Pergamon Press, 1982), 607-652.

[12] A. D. Cox, G. Eason and H. G. Hopkins, Axially symmetric plastic deformations in soils, *Phil. Trans. Roy. Soc. Lond.* **A254** (1961), 1-45.

[13] J. M. Hill and G. M. Cox, On the proble of the determination of force distributions in granular heaps using continuum theory. *Q. J. Mech. Appl. Math.* accepted for publication (2002).

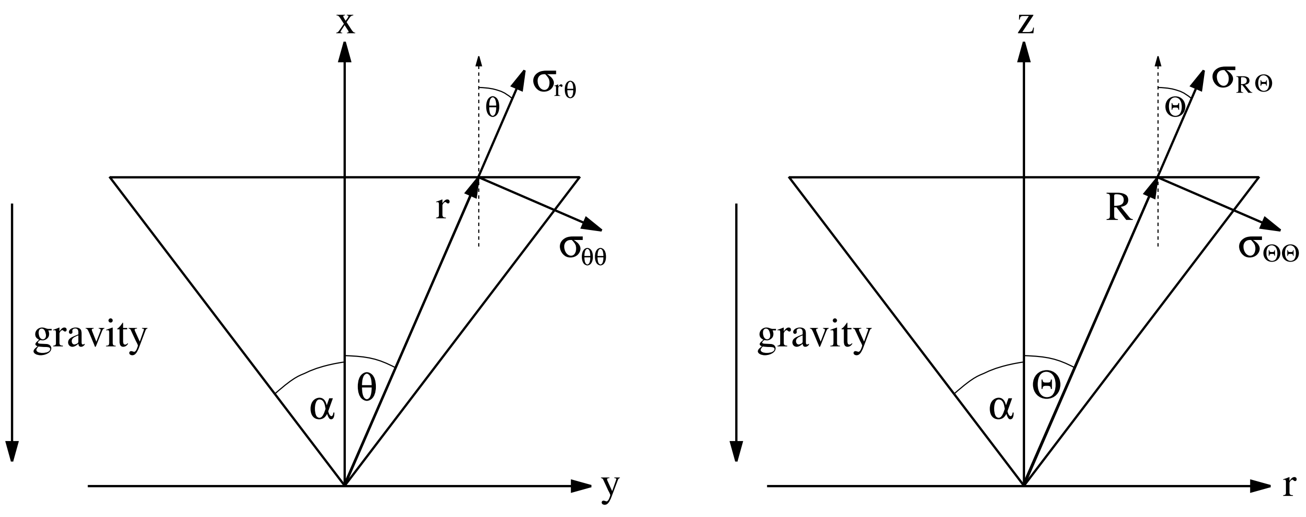
[14] G. M. Cox, J. M. Hill and N. Thamwattana, An analytical solution for a sloping rat-hole in an highly frictional ganular solid. *Proc. Roy. Soc. Lond.* submitted for publication.

Figure Captions

Figure 1. Coordinates for granular flow in a converging hopper ((a) two-dimensional and (b) three-dimensional).

Figure 2. Variation of $\psi(\theta)$ and $u(\theta)$ for two-dimensional hopper flow for three values of the angle of internal friction δ ((a) ψ and (b) u).

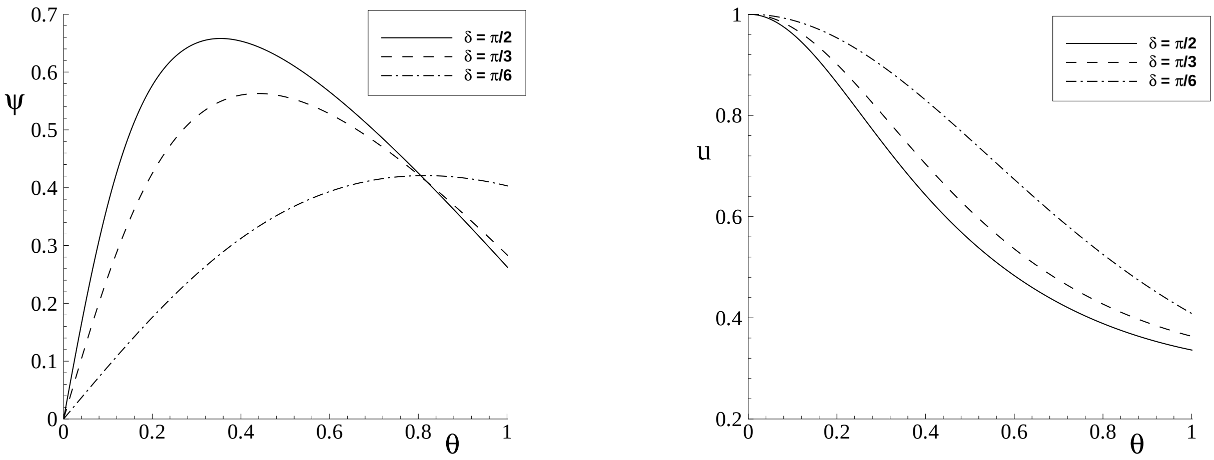
Figure 3. Variation of $\Psi(\Theta)$ and $U(\Theta)$ for three-dimensional hopper flow for three angles of the internal friction δ ((a) Ψ and (b) U).



(a).

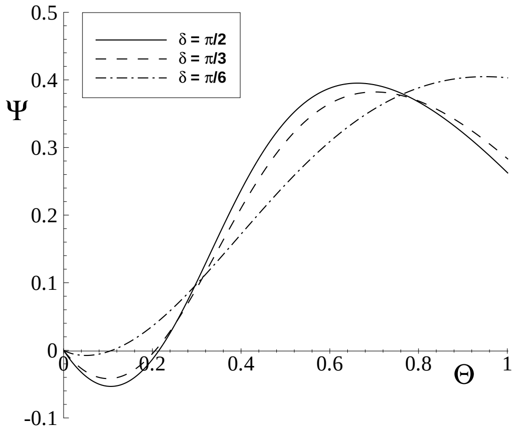
(b).

Figure 1.



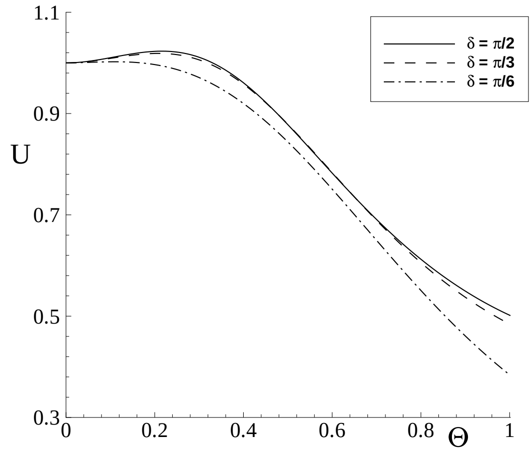
(a). (b).

Figure 2.



0.5
0.4
0.3
0.2
0.1
0
-0.1
Ψ
0
0.2
0.4
0.6
0.8
1
Θ
δ = π/2
δ = π/3
δ = π/6

(a).



1.1
0.9
0.7
0.5
0.3
U
0
0.2
0.4
0.6
0.8
1
Θ
δ = π/2
δ = π/3
δ = π/6

(b).

Figure 3.