

Some exact mathematical solutions for granular stock piles and granular flow in hoppers

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Abstract: The flow of granular materials in the presence of gravity in hoppers and the storage of granular materials as a stock pile occur in many industrial situations. The governing ordinary differential equations for two-dimensional wedges and three-dimensional cones are highly nonlinear and there are no known general solutions, apart from that given by the authors for a special angle of internal friction. Here, we give the overall picture relating to those special cases which give rise to analytical solutions for the two problems of granular flow through hoppers and the stress distributions at the base of stock piles. These equations are fundamental to granular mechanics and previously only some special isolated exact solutions were known. We list here a number of new exact analytical solutions applying for the two special cases of $\beta = \pm 1$, noting that $\beta = \sin \phi$ where ϕ is the angle of internal friction. The case $\beta = -1$ corresponds to a non-physical material, but there are materials such as Silica and Alumina cake which do indeed exhibit large angles of internal friction and the case $\beta = 1$ is by no means unrealistic. However, all the solutions presented are meaningful mathematical solutions of the governing equations and constitute the only known general solutions of these important equations. For certain cases, a full independent numerical solution has been obtained and shown to coincide with the appropriate exact analytical solution.

1 INTRODUCTION

Granular materials occur in many industrial processes throughout the world. These industries frequently store the granulated material in hoppers and more simply as stock piles, from which the material can be subsequently reclaimed. For steady flow from a hopper, Jenike [1, 2, 3] and Johanson [4] examine radial flow solutions for which the equilibrium equations and the Coulomb-Mohr yield condition reduce to give two highly nonlinear coupled ordinary differential equations for the determination of the stress field. These solutions were recently re-examined by Bradley [5] and Spencer and Bradley [6]. For convenience we follow the notation of these latter authors, noting however, that for the stock pile problem gravity is acting in the opposite direction to their problem. We also note that these authors adopt the unusual convention of the x -axis being vertical. In general, these coupled equations can only be solved numerically due to the high nonlinearity of the equations. However, for a two-dimensional converging wedge shaped hopper, an exact parametric solution of these equations is given by Hill and Cox [7] for the special case of the angle of internal friction equal to ninety degrees. This exact parametric solution was later exploited by the same authors in Hill and Cox [8] to determine an exact parametric solution for a two-dimensional wedge shaped stock pile. This exact solution is the first exact solution of the highly nonlinear coupled ordinary differential equations which involves two arbitrary constants.

Previously, only a simple exact solution of these equations with a single arbitrary constant, as determined by Sokolovsky [9], was known (namely (2.14)) along with the corresponding three-dimensional solutions (namely (2.29) - (2.31)). The object of this paper is to present the general picture relating to the more general exact solutions applying to special values of $\beta = \sin \phi$, where ϕ is the angle of internal friction. For completeness we deduce the previously determined solution for $\beta = 1$, as given in [7], and derive an additional exact solution for the two-dimensional wedge for the special case of $\beta = -1$. We also derive exact solutions for three-

Granular material	Measured values of $\beta = \sin \phi$
Coal	0.939 0.958 0.973 0.985
Alumina cake	0.941
Waste rock	0.974
Silica	0.979

Table 1: Measured values of $\beta = \sin \phi$ for certain granular materials, where ϕ is the angle of internal friction.

dimensional cones for the special cases of $\beta = \pm 1$, and apply the solutions to the flow through hoppers and stock piles. While of course the special case $\beta = -1$ corresponds to a non-physical material, there are however many materials which do indeed exhibit large angles of internal friction as shown in Table 1 and the case $\beta = 1$ is physically reasonable as a limiting ideal material. All the special solutions presented are meaningful mathematical solutions of the governing equations and might provide limiting bounds for solutions of physically meaningful materials and benchmarks for numerical schemes.

In the following section, we briefly state the basic equations of the proper continuum mechanical theory of granular materials for quasi-static flow of an ideal cohesionless material which satisfies the Coulomb-Mohr yield condition, for both the hopper and stock pile problems. Some simple exact solutions of the appropriate equations are also stated. In Section 3, we derive the exact solutions for the special cases of $\beta = \pm 1$ for the two-dimensional governing equations and we apply the solution for $\beta = 1$ to both the stock pile problem and the wedge-flow problem, but we find that the solution for $\beta = -1$ is unable to accommodate the boundary conditions of either problem. Also in Section 3 we note some results for the special case of $\beta = 0$ which is also of interest as a limiting case for small angle of internal friction, but we are unable to solve the equation in this case. In Section 4, we derive the exact solutions for the special cases of $\beta = \pm 1$ for the three-dimensional governing equations, where we adopt the Haar-von Karman hypothesis as stated in Cox *et*

al [10], that is we assume the hoop stress to be equal to either the maximum or minimum principal stress. The solution obtained for $\beta = 1$ is applied to both the stock pile and hopper problems, but again we find that the solution for $\beta = -1$ is unable to accommodate the boundary conditions of either problem. Also in Section 4 we note some results for the special case of $\beta = 0$. Numerical solutions confirming the exact solution for the three-dimensional stock pile and hopper problems for $\beta = 1$ are presented in Section 5, noting that as with the exact parametric solutions for $\beta = -1$, it was not possible to determine a numerical solution for this special case which satisfies the boundary conditions of either problem.

2 BASIC EQUATIONS

In the following two subsections we state briefly the basic equations for the proper continuum mechanical theory of granular materials for the two problems of gravity flow through hoppers and stock piles for two-dimensional wedges and three-dimensional cones.

2.1 *Two-Dimensional Basic Equations*

In terms of cylindrical polar coordinates (r, θ, z) as defined in Figure 1, the stress components for quasi-static plane flow in hoppers and stock piles satisfy the equilibrium equations

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = -\epsilon \rho g \cos \theta, \quad (2.1)$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{2\sigma_{r\theta}}{r} = \epsilon \rho g \sin \theta,$$

where ρ is the bulk density, g is acceleration due to gravity, ϵ is a parameter whose value is 1 for the stock pile problem and -1 for the hopper problem and σ_{rr} , $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$ denote the usual in-plane physical Cauchy stress components, which are assumed to be positive in tension. Namely, the usual convention in continuum mechanics is adopted that positive forces are assumed to produce positive extensions. Following

Spencer and Bradley [6] these components can be expressed in the standard form

$$\sigma_{rr} = -p + q \cos 2\psi, \quad \sigma_{\theta\theta} = -p - q \cos 2\psi, \quad \sigma_{r\theta} = q \sin 2\psi, \quad (2.2)$$

where p and q are defined as

$$p = -\frac{1}{2}(\sigma_{rr} + \sigma_{\theta\theta}), \quad q = \frac{1}{2}\left\{(\sigma_{rr} - \sigma_{\theta\theta})^2 + 4\sigma_{r\theta}^2\right\}^{1/2}, \quad (2.3)$$

while ψ is given by

$$\tan 2\psi = \frac{2\sigma_{r\theta}}{(\sigma_{rr} - \sigma_{\theta\theta})}, \quad (2.4)$$

and ψ is the angle between the maximum principal stress axis and the x direction, in the direction of increasing θ . For a cohesionless material, the Coulomb-Mohr yield condition takes the form

$$q = \beta p, \quad (2.5)$$

where $\beta = \sin \phi$ and ϕ is a material constant referred to as the angle of internal friction.

Following Jenike [2] and Spencer and Bradley [6] we look for a wedge field solution of the form

$$\psi = \psi(\theta), \quad q = -\epsilon \rho g r F(\theta), \quad (2.6)$$

recalling that $\epsilon = 1$ for the stock pile problem and $\epsilon = -1$ for the hopper problem. We note from (2.6)₂ that since q is defined as a positive quantity, then F must be negative for the stock pile problem but positive for the hopper problem. From the above equations we may deduce the following governing nonlinear equations

$$\begin{aligned} \frac{dF}{d\theta} &= \frac{F \sin 2\psi + \beta \sin(\theta + 2\psi)}{\beta + \cos 2\psi}, \\ \frac{d\psi}{d\theta} + 1 &= \frac{F(\beta^{-1} - \beta) + \cos \theta + \beta \cos(\theta + 2\psi)}{2F(\beta + \cos 2\psi)}, \end{aligned} \quad (2.7)$$

noting that the same equations arise for both the stock pile problem and the hopper problem. We also note that if we eliminate F from (2.7) then we may deduce the

following single second order ordinary differential equation for $\psi(\theta)$,

$$\begin{aligned}
 & (\beta + \cos 2\psi)[\cos \theta + \beta \cos(\theta + 2\psi)]\psi'' \\
 & = 2(\psi' + 1) \{ \sin 2\psi[\cos \theta + \beta \cos(\theta + 2\psi)]\psi' \\
 & \quad - 2\beta(\beta + \cos 2\psi) \sin(\theta + 2\psi)\psi' - (3\beta^2 + 2\beta \cos 2\psi - 1) \sin(\theta + 2\psi) \},
 \end{aligned} \tag{2.8}$$

where throughout the paper primes denote differentiation with respect to θ . This formulation forms the basis for the exact analysis for $\beta = \pm 1$ given in Section 3 for both the stock pile and hopper problems and it is also the basis for an independent numerical scheme for the special case of $\beta = 1$. In order to determine the appropriate boundary conditions, we must consider each problem individually.

2.1.1 BOUNDARY CONDITIONS FOR THE STOCK PILE PROBLEM

Here, we assume that the slope of the stock pile is stress free which means that $\sigma_{r\theta} = \sigma_{\theta\theta} = 0$ at $\theta = \alpha$, and generally this gives rise to the condition

$$F(\alpha) = 0, \tag{2.9}$$

where α denotes the semi-vertex angle. However, for the special case of $\beta = 1$ a possible alternate boundary condition to (2.9) is $\psi(\alpha) = \pm\pi/2$, which is sufficient in the special case to guarantee zero stress on the surface, and also indicates possible non-uniqueness in the decomposition (2.2). Secondly, we observe from the equilibrium equations (2.1) with $\epsilon = 1$ that σ_{rr} and $\sigma_{\theta\theta}$ must be even functions of θ and that $\sigma_{r\theta}$ must be an odd function or skew-symmetric, and therefore, $\sigma_{r\theta}$ must vanish at the origin giving rise to the second boundary condition

$$\psi(0) = 0. \tag{2.10}$$

Thus, we need to solve (2.1) subject to the two-point boundary conditions (2.9) and (2.10).

2.1.2 BOUNDARY CONDITIONS FOR THE HOPPER PROBLEM

Now for the hopper problem, we find from the equilibrium equations (2.1) with $\epsilon = -1$ that σ_{rr} and $\sigma_{\theta\theta}$ must also be even functions of θ and that $\sigma_{r\theta}$ must again be an odd function or skew-symmetric, and therefore, $\sigma_{r\theta}$ must vanish at the origin again giving rise to the boundary condition (2.10). To determine the second boundary condition, following Spencer and Bradley [6] we assume a Coulomb friction condition at the wall of the hopper at $\theta = \alpha$, such that

$$\sigma_{r\theta} = -\sigma_{\theta\theta} \tan \mu, \quad \text{at } \theta = \alpha, \quad (2.11)$$

where μ is the angle of wall friction. From (2.2) and (2.5) we find that (2.11) gives

$$\sin [2\psi(\alpha) - \mu] = \frac{\sin \mu}{\sin \phi}, \quad (2.12)$$

which is only meaningful provided $\mu \leq \phi$, recalling that $\beta = \sin \phi$. If $\mu \geq \phi$ then the wall is ‘perfectly rough’ and the material slips on itself. In this case we find

$$\psi(\alpha) = \frac{\phi}{2} + \frac{\pi}{4}, \quad (2.13)$$

and we observe that for ψ positive, this value of ψ provides the first singularity of both equations in (2.7) in the sense that $\psi(\alpha)$ defined by (2.13) satisfies $\cos 2\psi = -\beta$. Thus, we need to solve (2.7) subject to (2.10) and either (2.12) or (2.13), depending on the value of μ .

Now, we note for both the stock pile problem and the hopper problem that (2.7) admits the special exact solution given by Sokolovsky [9]

$$\psi(\theta) = -\theta + \psi_0, \quad F(\theta) = -\frac{\beta}{(1 - \beta^2)} [\cos \theta + \beta \cos(2\psi_0 - \theta)], \quad (2.14)$$

where ψ_0 is some constant, but in general, we observe that (2.14) can only satisfy one of the necessary boundary conditions. We also note from Hill and Cox [8] that the vertical and horizontal force distributions acting along a plane $x = \text{constant} = h$ in a two-dimensional, wedge shaped stock pile are

$$\sigma_x = \rho gh \frac{F(\theta)}{\cos \theta} \left\{ \frac{1}{\beta} - \cos 2[\theta + \psi(\theta)] \right\}, \quad \sigma_y = -\rho gh \frac{F(\theta)}{\cos \theta} \sin 2[\theta + \psi(\theta)], \quad (2.15)$$

respectively.

2.2 Three-Dimensional Basic Equations

In terms of spherical polar coordinates (R, Θ, Φ) as defined in Figure 2, the stress components for quasi-static axially symmetric flow in cone shaped hoppers and stock piles satisfy the equilibrium equations

$$\frac{\partial \sigma_{RR}}{\partial R} + \frac{1}{R} \frac{\partial \sigma_{R\Theta}}{\partial \Theta} + \frac{1}{R} (2\sigma_{RR} - \sigma_{\Theta\Theta} - \sigma_{\Phi\Phi} + \sigma_{R\Theta} \cot \Theta) = -\epsilon \rho g \cos \Theta, \quad (2.16)$$

$$\frac{\partial \sigma_{R\Theta}}{\partial R} + \frac{1}{R} \frac{\partial \sigma_{\Theta\Theta}}{\partial \Theta} + \frac{1}{R} (\sigma_{\Theta\Theta} - \sigma_{\Phi\Phi}) \cot \Theta + \frac{3}{R} \sigma_{R\Theta} = \epsilon \rho g \sin \Theta,$$

where ρ is the bulk density, g is acceleration due to gravity, ϵ is a parameter whose value is 1 for the stock pile problem and -1 for the hopper problem and $\sigma_{RR}, \sigma_{\Theta\Theta}, \sigma_{\Phi\Phi}$ and $\sigma_{R\Theta}$ denote the physical components of the Cauchy stress, which are again assumed to be positive in tension. Again, following Spencer and Bradley [6] these components can be expressed in the standard form

$$\sigma_{RR} = -p + q \cos 2\Psi, \quad \sigma_{\Theta\Theta} = -p - q \cos 2\Psi, \quad \sigma_{R\Theta} = q \sin 2\Psi, \quad (2.17)$$

where p and q are defined as

$$p = -\frac{1}{2} (\sigma_{RR} + \sigma_{\Theta\Theta}), \quad q = \frac{1}{2} \left\{ (\sigma_{RR} - \sigma_{\Theta\Theta})^2 + 4\sigma_{R\Theta}^2 \right\}^{1/2}, \quad (2.18)$$

while Ψ is given by

$$\tan 2\Psi = \frac{2\sigma_{R\Theta}}{(\sigma_{RR} - \sigma_{\Theta\Theta})}, \quad (2.19)$$

and Ψ is the angle between the direction of the maximum principal stress axis and the z direction, in the direction of increasing Θ .

Now, we need to make an assumption about the hoop stress in order to determine an expression for the $\sigma_{\Phi\Phi}$ in terms of p, q and Ψ . Cox *et al* [10] state that the plastic regimes which agree with the Haar-von Karman hypothesis are likely to be of the greatest significance in the solution of problems of interest. The heuristic Haar-von Karman principle states, under an axially symmetric condition, that the hoop stress is equal to either the maximum or minimum principal stress. This gives rise to the notion of the Haar-von Karman regimes, and in particular, either

$\sigma_I = \sigma_{\Phi\Phi} = \sigma_{II} > \sigma_{III}$ or $\sigma_I > \sigma_{II} = \sigma_{\Phi\Phi} = \sigma_{III}$, where σ_I, σ_{II} and σ_{III} denote the maximum, intermediate and minimum principal stresses respectively. Thus, we may deduce

$$\sigma_{\Phi\Phi} = -p + \delta q, \tag{2.20}$$

where δ is a parameter whose value is 1 if the hoop stress is the maximum principal stress and -1 if the hoop stress is the minimum principal stress. For a cohesionless material, the Coulomb-Mohr yield condition again becomes (2.5).

Following Jenike [2] and Spencer and Bradley [6] we look for solutions of the form

$$\Psi = \Psi(\Theta), \quad q = -\epsilon\rho g R G(\Theta), \tag{2.21}$$

recalling that $\epsilon = 1$ for the stock pile problem and $\epsilon = -1$ for the hopper problem. Again we note from (2.21)₂ that since q is defined to be a positive quantity, then G must be negative for the stock pile problem and positive for the hopper problem. From the above equations we may deduce the three-dimensional governing equations

$$\frac{dG}{d\Theta} = \frac{G \{ \sin 2\Psi - \beta \operatorname{cosec}\Theta [\cos \Theta + \delta \cos(\Theta + 2\Psi)] \} + \beta \sin(\Theta + 2\Psi)}{\beta + \cos 2\Psi}, \tag{2.22}$$

$$\frac{d\Psi}{d\Theta} + 1 = \frac{G(1 - \delta\beta) \{ \beta^{-1}(1 + 2\delta\beta) - \operatorname{cosec}\Theta \sin(\Theta + 2\Psi) \} + \cos \Theta + \beta \cos(\Theta + 2\Psi)}{2G(\beta + \cos 2\Psi)},$$

noting that the same equations arise for both the stock pile problem and the hopper problem, and recalling that $\delta = 1$ if the hoop stress is the maximum principal stress and $\delta = -1$ if the hoop stress is the minimum principal stress. We also note that if we eliminate G from (2.22) then we may deduce the following single second order

ordinary equation for $\Psi(\theta)$,

$$\begin{aligned}
 & 2(\beta + \cos 2\Psi)[\cos \Theta + \beta \cos(\Theta + 2\Psi)]\Psi'' \\
 & = 2(\Psi' + 1) \left\{ \operatorname{cosec} \Theta \left[\beta(\beta + \cos 2\Psi)[\delta + \cos\{2(\Theta + \Psi)\}] \right. \right. \\
 & \quad - [\cos \Theta + \beta \cos(\Theta + 2\Psi)][2 \sin \Theta \sin 2\Psi + (1 - \delta\beta) \cos(\Theta + 2\Psi)] \\
 & \quad \left. \left. + 2(1 - \delta\beta) \sin(\Theta + 2\Psi)[(1 + 2\delta\beta) \sin \Theta - \beta \sin(\Theta + 2\Psi)] \right] \right. \tag{2.23} \\
 & \quad \left. + 2(\Psi' + 1) \left[\sin 2\Psi[\cos \Theta + \beta \cos(\Theta + 2\Psi)] - 2\beta \sin(\Theta + 2\Psi)(\beta + \cos 2\Psi) \right] \right\} \\
 & \quad + (1 - \delta\beta) \operatorname{cosec}^2 \Theta \left\{ \sin\{2(\Theta + \Psi)\}[\cos \Theta + \beta \cos(\Theta + 2\Psi)] \right. \\
 & \quad \left. - [\delta + \cos\{2(\Theta + \Psi)\}][(1 + 2\delta\beta) \sin \Theta - \beta \sin(\Theta + 2\Psi)] \right\},
 \end{aligned}$$

where throughout the paper primes denote differentiation with respect to Θ . Again we note that this formulation forms the basis for the exact analysis for $\beta = \pm 1$ given in Section 4 for both the stock pile and hopper problems and it is also the basis for an independent numerical scheme for the special case of $\beta = 1$.

2.2.1 BOUNDARY CONDITIONS FOR THE STOCK PILE PROBLEM

Here, we assume that the slope of the stock pile is stress free which means that $\sigma_{R\Theta} = \sigma_{\Theta\Theta} = 0$ at $\Theta = \alpha$, and in general this gives rise to the condition

$$G(\alpha) = 0, \tag{2.24}$$

where α denotes the semi-vertex angle. However, for the special case of $\beta = 1$ a possible alternate boundary condition to (2.24) is $\Psi(\alpha) = \pm\pi/2$, which is sufficient in the special case to guarantee zero stress on the surface. Secondly, we observe from the equilibrium equations (2.16) with $\epsilon = 1$ that $\sigma_{RR}, \sigma_{\Theta\Theta}$ and $\sigma_{\Phi\Phi}$ must be even

functions of Θ while $\sigma_{R\Theta}$ must be an odd function or skew-symmetric, and therefore, $\sigma_{R\Theta}$ must vanish at the origin giving rise to the second boundary condition

$$\Psi(0) = 0. \tag{2.25}$$

Thus, we need to solve (2.16) subject to the two-point boundary conditions (2.24) and (2.25).

2.2.2 BOUNDARY CONDITIONS FOR THE HOPPER PROBLEM

For the hopper problem, we find from the equilibrium equations (2.16) with $\epsilon = -1$ that $\sigma_{RR}, \sigma_{\Theta\Theta}$ and $\sigma_{\Phi\Phi}$ must also be even functions of Θ while again $\sigma_{R\Theta}$ must be an odd function or skew-symmetric, and therefore, $\sigma_{R\Theta}$ must vanish at the origin again giving rise to the boundary condition (2.25). Again for the second boundary condition, following Spencer and Bradley [6], we assume a Coulomb friction condition at the wall of the hopper at $\Theta = \alpha$, such that

$$\sigma_{R\Theta} = -\sigma_{\Theta\Theta} \tan \mu, \quad \text{at } \Theta = \alpha, \tag{2.26}$$

where μ is the angle of wall friction. Thus, from (2.17) and (2.5) we find that (2.26) gives

$$\sin[2\Psi(\alpha) - \mu] = \frac{\sin \mu}{\sin \phi}, \tag{2.27}$$

which is again only meaningful provided $\mu \leq \phi$, recalling that $\beta = \sin \phi$. If $\mu \geq \phi$ then the wall is ‘perfectly rough’ and the material slips on itself at the wall. In this case we find

$$\Psi(\alpha) = \frac{\phi}{2} + \frac{\pi}{4}, \tag{2.28}$$

and we observe that for positive Ψ , this value of Ψ provides the first singularity of both equations in (2.16) in the sense that $\Psi(\alpha)$ defined by (2.28) satisfies $\cos 2\Psi = -\beta$. Thus, we need to solve (2.16) subject to (2.25) and either (2.27) or (2.28), depending on the value of μ .

Now, we note for both the stock pile problem and the hopper problem that (2.22) admits two special exact solutions, namely

$$\Psi(\Theta) = -\Theta + \lambda_0, \quad G(\Theta) = \frac{\beta \sin \Theta}{[(3\delta\beta + 1)(\delta\beta - 1)]^{1/2}}, \quad (2.29)$$

where λ_0 is defined by

$$\cos 2\lambda_0 = -\frac{(1 + \delta\beta)}{2\beta}, \quad (2.30)$$

and

$$\Psi(\Theta) = -\Theta, \quad G(\Theta) = -\frac{\beta(1 + \beta) \cos \Theta}{(1 - \delta\beta)[1 + (1 + 2\delta)\beta]}, \quad (2.31)$$

noting that (2.29) and (2.31) are distinct solutions when $\lambda_0 = 0$, and recalling that $\delta = 1$ if the hoop stress is the maximum principal stress and $\delta = -1$ if the hoop stress is the minimum principal stress. We also note from Hill and Cox [11] that the vertical and horizontal force distributions acting along a plane $Z = \text{constant} = h$ where $R = h \sec \Theta$ in a three-dimensional cone shaped stock pile to be

$$\sigma_z = \rho gh \frac{G(\Theta)}{\cos \Theta} \left\{ \frac{1}{\beta} - \cos 2[\Theta + \Psi(\Theta)] \right\}, \quad \sigma_r = -\rho gh \frac{G(\Theta)}{\cos \Theta} \sin 2[\Theta + \Psi(\Theta)], \quad (2.32)$$

respectively.

3 EXACT TWO-DIMENSIONAL SOLUTIONS

In this section we derive some analytical results for the two-dimensional governing equations (2.7) for the stock pile and hopper problems for the two special values of $\beta = \pm 1$. We examine the single second order ordinary differential equation (2.8) for ψ and then determine F from (2.7)₂. However, we first note that for $\beta = 0$, (2.8) simplifies to give

$$\psi'' \cos \theta \cos 2\psi = 2(\psi' + 1) \{ \sin(\theta + 2\psi) + \psi' \cos \theta \sin 2\psi \}, \quad (3.1)$$

which we note can be formally obtained from the coupled equations (2.7) in the limit as β tends to zero as follows. Upon rewriting (2.7)₂ in the form

$$F \left\{ 2\beta \cos 2\psi (\psi' + 1) + 2\beta^2 (\psi' + 1) + \beta^2 - 1 \right\} = \beta [\cos \theta + \beta \cos(\theta + 2\psi)], \quad (3.2)$$

we can then approximate F for small β by

$$F \approx -\frac{\beta [\cos \theta + \beta \cos(\theta + 2\psi)]}{[1 - 2\beta \cos 2\psi(\psi' + 1)]}, \quad (3.3)$$

which becomes

$$F \approx -\beta \{ \cos \theta + \beta [\cos(\theta + 2\psi) + 2 \cos \theta \cos 2\psi(\psi' + 1)] \}. \quad (3.4)$$

Finally, on substituting (3.4) into (2.7)₁ and simplifying, equation (3.1) arises. However, we note that although (3.1) appears reasonably simple, we are unable to determine an exact analytical solution for the special case of $\beta = 0$.

3.1 *Exact Solution for $\beta = 1$.*

This solution was first derived in Hill and Cox [7] for the hopper problem and later utilized in Hill and Cox [8] for the stock pile problem. For the sake of completeness we briefly repeat some of these details below. For $\beta = 1$, (2.8) simplifies to give

$$\cos \psi \cos(\theta + \psi) \psi'' = -2(\psi' + 1) \{ (\psi' + 1) \cos \psi \sin(\theta + \psi) + \sin \psi \cos(\theta + \psi) \}, \quad (3.5)$$

which can be rearranged to yield

$$\frac{\psi''}{(1 + \psi')^2} + \frac{2 \tan \psi}{(1 + \psi')} + 2 \tan(\theta + \psi) = 0. \quad (3.6)$$

Thus, if we make the transformations

$$x = \tan \theta, \quad y = \tan(\theta + \psi), \quad (3.7)$$

so that

$$\tan \psi = \frac{y - x}{1 + xy}, \quad (3.8)$$

then (3.6) can be shown to simplify to yield

$$\frac{d^2 y}{dx^2} + \frac{2y}{(1 + xy)} \frac{dy}{dx} = 0. \quad (3.9)$$

Now this equation remains invariant under the stretching group of transformations

$$x_1 = \lambda x, \quad y_1 = \lambda^{-1} y, \quad (3.10)$$

and therefore we introduce the new variable $z = xy$ so that upon making the Euler transformation $s = \log x$, (3.9) becomes

$$\frac{d^2z}{ds^2} - \frac{dz}{ds} - \frac{2}{(1+z)} \left(\frac{dz}{ds} - z \right) = 0. \quad (3.11)$$

We solve (3.10) by introducing ω such that

$$\omega = \frac{dz}{ds} - z, \quad (3.12)$$

so that (3.11) is equivalent to

$$\frac{dz}{ds} = z + \omega, \quad \frac{d\omega}{ds} = \frac{2\omega}{1+z}. \quad (3.13)$$

On eliminating z from these equations and introducing

$$v = \frac{1}{\omega} \frac{d\omega}{ds}, \quad (3.14)$$

we may readily deduce the standard first order differential equation

$$\frac{dv}{d\omega} + \frac{1}{2} \left(1 - \frac{1}{\omega} \right) v = -\frac{1}{\omega}. \quad (3.15)$$

This equation readily integrates to give

$$v = 2 - \omega^{1/2} e^{-\omega/2} I(\omega), \quad (3.16)$$

where $I(\omega)$ is the integral defined by

$$I(\omega) = \int^\omega t^{-1/2} e^{t/2} dt + C_1, \quad (3.17)$$

where C_1 denotes an arbitrary constant of integration. From (3.14) and (3.16) we may perform a second integration to obtain

$$2\omega^{-1/2} e^{\omega/2} - I(\omega) = \frac{C_2}{x}, \quad (3.18)$$

where C_2 denotes an arbitrary constant of integration. Next, from the fact that $v = 2/(1+z)$ we find

$$z = \frac{2}{v} - 1 = \frac{\omega^{1/2}}{2} e^{-\omega/2} I(\omega) \left\{ 1 - \frac{\omega^{1/2}}{2} e^{-\omega/2} I(\omega) \right\}^{-1}, \quad (3.19)$$

and as $z = xy$ then clearly

$$y = \tan(\theta + \psi) = \frac{I(\omega)}{C_2}. \quad (3.20)$$

Therefore, from (3.7), (3.8), (3.18) and (3.20) we find

$$\tan \theta = \frac{C_2}{2\omega^{-1/2}e^{\omega/2} - I(\omega)}, \quad (3.21)$$

$$\tan \psi = \frac{I(\omega)}{C_2} \left\{ 1 - \frac{\omega^{1/2}}{2} e^{-\omega/2} I(\omega) \right\} - \frac{C_2}{2} \omega^{1/2} e^{-\omega/2}.$$

In order to determine a relation for F in terms of the parameter ω , we differentiate (3.20), using (3.21)₁, to find

$$\psi' + 1 = \frac{\omega \cos^2(\theta + \psi)}{\sin^2 \theta} = \frac{\cos(\theta + \psi)}{2F \cos \psi}, \quad (3.22)$$

where the latter equality follows from the differential equation (2.7)₂ with $\beta = 1$.

From (3.22) we obtain

$$F = \frac{\sec \psi \sec(\theta + \psi)}{2\omega \operatorname{cosec}^2 \theta} = \frac{\left\{ 1 + \left(\frac{y-x}{1+xy} \right)^2 \right\}^{1/2} (1+y^2)^{1/2}}{2\omega(1+x^2)} = \frac{x^2(1+y^2)}{2\omega(1+xy)(1+x^2)^{1/2}}, \quad (3.23)$$

where x and y are defined by (3.7). Therefore, on simplifying (3.23) we may deduce the following expression for F in terms of the parameter ω ,

$$F = \frac{1}{4} \frac{\omega^{-1/2} e^{-\omega/2} [C_2^2 + I(\omega)^2]}{\{C_2^2 + [2\omega^{-1/2} e^{\omega/2} - I(\omega)]^2\}^{1/2}}, \quad (3.24)$$

and (3.21) and (3.24) constitutes the exact solution of (2.7) for $\beta = 1$.

In the following two subsections we utilize the exact parametric solution for $\beta = 1$ for determining the stress distribution at the base of stock piles and granular flow through hoppers.

3.1.1 SOLUTION FOR THE STOCK PILE PROBLEM

In general, to apply the solution for $\beta = 1$ to the problem of determining the stress distribution at the base of two-dimensional stock piles, we must ensure that the

solution satisfies the boundary conditions of (2.9) and (2.10). Thus, from (3.17) and (3.21) we find that (2.10) is satisfied provided we associate the parameter value $\omega \equiv 0$ with $\theta \equiv 0$ and define C_1 such that the integral $I(\omega)$ is given by

$$I(\omega) = \int_0^\omega t^{-1/2} e^{t/2} dt. \quad (3.25)$$

Now, to determine the second constant of integration C_2 , we find that the general boundary condition at $\theta = \alpha$ of (2.9) is inadequate to determine the value of C_2 . Or in other words, applying the boundary condition of (2.9), namely $F(\alpha) = 0$, to (3.24) gives no additional information with which to determine C_2 from. However, recall that for the special case of $\beta = 1$ a possible alternate boundary condition to (2.9) exists, namely

$$\psi(\alpha) = -\frac{\pi}{2}. \quad (3.26)$$

Therefore, upon applying (3.26) to (3.21), noting (3.20), we find

$$\tan \alpha = \frac{C_2}{2\omega_0^{-1/2} e^{\omega_0/2} + C_2 \cot \alpha}, \quad (3.27)$$

where ω_0 denotes the parameter value corresponding to $\theta = \alpha$ and from (3.27) it is clear that $\omega_0 = -\infty$.

Accordingly, we change the parameter from ω to $-\lambda$, make the substitution $t = -s$ in the integral (3.25) so that the exact parametric solution (3.21) and (3.24) now becomes

$$\begin{aligned} \tan \theta &= -\frac{C_3}{2\lambda^{-1/2} e^{-\lambda/2} + J(\lambda)}, \\ \tan \psi &= \frac{J(\lambda)}{C_3} \left\{ 1 + \frac{\lambda^{1/2}}{2} e^{\lambda/2} J(\lambda) \right\} + \frac{C_3}{2} \lambda^{1/2} e^{\lambda/2}, \end{aligned} \quad (3.28)$$

$$F = -\frac{1}{4} \frac{\lambda^{-1/2} e^{\lambda/2} [C_3^2 + J(\lambda)^2]}{\{C_3^2 + [2\lambda^{-1/2} e^{-\lambda/2} + J(\lambda)]^2\}^{1/2}},$$

where $C_2 = iC_3$ and the integral $J(\lambda)$ is defined by

$$J(\lambda) = \int_0^\lambda s^{-1/2} e^{-s/2} ds = 2^{3/2} \int_0^{(\lambda/2)^{1/2}} e^{-x^2} dx = (2\pi)^{1/2} \operatorname{erf}(\lambda/2)^{1/2}, \quad (3.29)$$

where erf denotes the usual error function. We note that the determination of the correct sign of (3.28)₃ is not altogether a straightforward matter. In particular, upon factorizing the minus sign out of the square root on the denominator, the minus sign needs to be expressed in the form $(-1)^{-1}$ in order to cancel out with the $(-1)^{-1/2}$ from the $\omega^{-1/2}$ term on the numerator of (3.24). Alternatively, the minus sign can be seen as arising from the expression (3.23) and $\omega \rightarrow -\lambda$.

Now since $\lambda = \infty$ is the parameter value corresponding to $\theta = \alpha$, we require an estimate of $J(\lambda)$ for λ tending to infinity. Using the standard asymptotic expansion for the complimentary error function (see for example Carslaw and Jaeger [12]) we may deduce

$$J(\lambda) = (2\pi)^{1/2} \left\{ 1 - \operatorname{erfc} \left(\frac{\lambda}{2} \right)^{1/2} \right\} = (2\pi)^{1/2} - \frac{2e^{-\lambda/2}}{\lambda^{1/2}} \left\{ 1 - \frac{1}{\lambda} + O\left(\frac{1}{\lambda^2}\right) \right\}, \quad (3.30)$$

and from this equation and (3.28) we find

$$\begin{aligned} \tan \theta &= -C_3 \left\{ (2\pi)^{1/2} + \frac{2e^{-\lambda/2}}{\lambda^{3/2}} \left[1 + O\left(\frac{1}{\lambda}\right) \right] \right\}^{-1} \\ \tan \psi &= \left(\frac{\pi}{C_3} + \frac{C_3}{2} \right) \lambda^{1/2} e^{\lambda/2} + O(1), \end{aligned} \quad (3.31)$$

$$F = -\frac{e^{\lambda/2}}{4\lambda^{1/2}} \left\{ (C_3^2 + 2\pi)^{1/2} - 4\frac{e^{-\lambda/2}}{\lambda^{1/2}} \left(\frac{2\pi}{C_3^2 + 2\pi} \right)^{1/2} \left[1 + O\left(\frac{1}{\lambda}\right) \right] \right\}.$$

From (3.31)₁ we may deduce

$$C_3 = -(2\pi)^{1/2} \tan \alpha, \quad (3.32)$$

and that

$$\begin{aligned} \tan \psi &= -\left(\frac{\pi\lambda}{2} \right)^{1/2} \frac{e^{\lambda/2}}{\sin \alpha \cos \alpha} + O(1), \\ F &= -\left(\frac{\pi}{2\lambda} \right)^{1/2} \frac{e^{\lambda/2}}{2 \cos \alpha} + O\left(\frac{1}{\lambda}\right), \end{aligned} \quad (3.33)$$

which confirms $\psi(\alpha) = -\pi/2$ as λ tends to infinity, but note that $F(\alpha)$ tends to minus infinity rather than zero. In the case with $\beta = 1$ we have satisfied the alternative boundary conditions of $\psi(0) = 0$ and $\psi(\alpha) = -\pi/2$, which only apply for this special case. From the asymptotic expressions (3.33) and (2.2) we may confirm that $\sigma_{r\theta}$ and $\sigma_{\theta\theta}$ both tend to zero.

Thus, for a two-dimensional stock pile with an angle of internal friction of 90° , an exact parametric solution is given by (3.28) for $0 \leq \lambda \leq \infty$ where $J(\lambda)$ is defined by (3.29) and the constant C_3 is given by equation (3.32).

3.1.2 SOLUTION FOR THE HOPPER PROBLEM

To apply the solution for $\beta = 1$ to the problem of flow under gravity through a wedge hopper, we must ensure that the solution satisfies (2.10) and (2.12) or (2.13), depending on the value of the angle of wall friction μ . Thus, from (3.17) and (3.21) we find that (2.10) is satisfied provided we associate the parameter value $\omega \equiv 0$ with $\theta \equiv 0$ and define C_1 such that the integral $I(\omega)$ is given by

$$I(\omega) = \int_0^\omega t^{-1/2} e^{t/2} dt. \tag{3.34}$$

Now, to determine the second constant of integration C_2 , we first need to determine which boundary condition at the wall of the hopper applies. To do this, we assume $\psi'(\alpha)$ is finite and from Hill and Cox [7] we find which this means that $\mu \leq \phi$, and hence the boundary condition (2.12) applies. Therefore, from (2.12) for $\beta = 1$ we find that (3.21) becomes

$$\tan \alpha = \frac{C_2}{2\omega_0^{-1/2} e^{\omega_0/2} - I(\omega_0)}, \tag{3.35}$$

$$\tan \mu = \frac{I(\omega_0)}{C_2} \left\{ 1 - \frac{\omega_0^{1/2}}{2} e^{-\omega_0/2} I(\omega_0) \right\} - \frac{C_2}{2} \omega_0^{1/2} e^{-\omega_0/2},$$

and from (3.35)₁ we find

$$C_2 = \tan \alpha \left\{ \frac{2e^{\omega_0/2}}{\omega_0^{1/2}} - I(\omega_0) \right\}, \tag{3.36}$$

which upon substituting into (3.35)₂ gives

$$\frac{\sin \alpha}{\cos \mu} \sin(\alpha + \mu) = \frac{\omega_0^{1/2}}{2} e^{-\omega_0/2} I(\omega_0), \quad (3.37)$$

which is a transcendental equation for ω_0 . Thus, C_2 is determined from (3.36) where ω_0 is a solution of (3.37).

Thus, for flow under gravity through a two-dimensional wedge hopper with an angle of internal friction of 90° , an exact parametric solution is given by (3.21) and (3.24) for $0 \leq \omega \leq \omega_0$ where $I(\omega)$ is defined by (3.34), the constant C_2 is defined by (3.36) and ω_0 is determined from the transcendental equation (3.37).

3.2 *Exact Solution for $\beta = -1$.*

For $\beta = -1$, (2.8) simplifies to give

$$\sin \psi \sin(\theta + \psi) \psi'' = 2(\psi' + 1) \{(\psi' + 1) \sin \psi \cos(\theta + \psi) + \cos \psi \sin(\theta + \psi)\}, \quad (3.38)$$

which can be rearranged to yield

$$\frac{\psi''}{(1 + \psi')^2} - \frac{2 \cot \psi}{(1 + \psi')} - 2 \cot(\theta + \psi) = 0. \quad (3.39)$$

Thus, if we make the transformations

$$x = \cot \theta, \quad y = \cot(\theta + \psi), \quad (3.40)$$

so that

$$\cot \psi = \frac{1 + xy}{x - y}, \quad (3.41)$$

then (3.39) can be shown to simplify to yield

$$\frac{d^2 y}{dx^2} + \frac{2}{(x - y)} \frac{dy}{dx} = 0. \quad (3.42)$$

Now, this equation remains invariant under the stretching group of transformations

$$x_1 = \lambda x, \quad y_1 = \lambda y, \quad (3.43)$$

and therefore we introduce the new variable z such that $y = xz$ and so that upon making the Euler transformation $s = \log x$, (3.42) becomes

$$\frac{d^2z}{ds^2} + \frac{dz}{ds} + \frac{2}{(1-z)} \left(\frac{dz}{ds} + z \right) = 0. \quad (3.44)$$

Now, upon introducing ω such that

$$\omega = \frac{dz}{ds} + z, \quad (3.45)$$

then (3.44) is equivalent to

$$\frac{dz}{ds} = \omega - z, \quad \frac{d\omega}{ds} = -\frac{2\omega}{1-z}. \quad (3.46)$$

On eliminating z from these equations and introducing v defined by (3.14) we may readily deduce the standard first order differential equation

$$\frac{dv}{d\omega} + \frac{1}{2} \left(1 - \frac{1}{\omega} \right) v = \frac{1}{\omega}. \quad (3.47)$$

This equation readily integrates to give

$$v = -2 + \omega^{1/2} e^{-\omega/2} I(\omega), \quad (3.48)$$

where $I(\omega)$ is the integral defined by (3.17). From (3.14) and (3.48) we may perform a second integration to obtain

$$2\omega^{-1/2} e^{\omega/2} - I(\omega) = C_2 x, \quad (3.49)$$

where C_2 denotes an arbitrary constant of integration. Next, from the fact that $v = 2/(z - 1)$ we find

$$z = \frac{2}{v} + 1 = -\frac{I(\omega)}{2\omega^{-1/2} e^{\omega/2} - I(\omega)}, \quad (3.50)$$

and as $y = xz$ then clearly

$$y = \cot(\theta + \psi) = -\frac{I(\omega)}{C_2}. \quad (3.51)$$

Therefore, from (3.40), (3.41), (3.49) and (3.51) we find

$$\tan \theta = \frac{C_2}{2\omega^{-1/2}e^{\omega/2} - I(\omega)}, \tag{3.52}$$

$$\cot \psi = -\frac{I(\omega)}{C_2} \left\{ 1 - \frac{\omega^{1/2}}{2} e^{-\omega/2} I(\omega) \right\} + \frac{C_2}{2} \omega^{1/2} e^{-\omega/2}.$$

In order to determine a relation for F in terms of the parameter ω , we differentiate (3.51), using (3.52)₁, to obtain

$$\psi' + 1 = \frac{\omega \operatorname{cosec}^2 \theta}{\operatorname{cosec}^2(\theta + \psi)} = -\frac{\sin(\theta + \psi)}{2F \sin \psi}, \tag{3.53}$$

where the latter equality follows from the differential equation (2.7)₂ with $\beta = -1$.

From (3.53) we obtain

$$F = -\frac{\operatorname{cosec} \psi \operatorname{cosec}(\theta + \psi)}{2\omega \operatorname{cosec}^2 \theta} = -\frac{\left\{ 1 + \left(\frac{1+xy}{x-y} \right)^2 \right\}^{1/2} (1+y^2)^{1/2}}{2\omega(1+x^2)} = -\frac{(1+y^2)}{2\omega(x-y)(1+x^2)^{1/2}}, \tag{3.54}$$

where x and y are defined by (3.40). Therefore, on simplifying (3.54) we may deduce the following expression for F in terms of the parameter ω ,

$$F = -\frac{1}{4} \frac{\omega^{-1/2} e^{-\omega/2} [C_2^2 + I(\omega)^2]}{\{C_2^2 + [2\omega^{-1/2} e^{\omega/2} - I(\omega)]^2\}^{1/2}}. \tag{3.55}$$

Therefore, we have determined an exact parametric solution to the two-dimensional governing equations (2.7) for the special case of $\beta = -1$. However, unlike the exact parametric solution for $\beta = 1$, we are unable to apply the exact parametric solution for $\beta = -1$ to either the stock pile problem or the hopper problem because we are unable to determine a set of values of the constants of integration C_1 and C_2 such that the appropriate boundary conditions are satisfied and the solution remain physical.

4 EXACT THREE-DIMENSIONAL SOLUTIONS

In this section we derive some new analytical results for the three-dimensional governing equations (2.22) for the stock pile and hopper problems for the special values

of $\beta = \pm 1$. However, we first note that for $\beta = 0$, (2.23) simplifies to give

$$2\Psi'' \cos \Theta \cos 2\Psi = 2(\Psi' + 1) \{3 \sin(\Theta + 2\Psi) - \operatorname{cosec} \Theta \cos 2\Psi + 2\Psi' \cos \Theta \sin 2\Psi\} + \operatorname{cosec}^2 \Theta [\sin \Theta + \sin(\Theta + 2\Psi)], \quad (4.1)$$

which can be formally obtained as a limiting process from the coupled equations (2.22) as follows. Upon rewriting (2.22)₂ in the form

$$G \{2\beta(\beta + \cos 2\Psi)(1 + \Psi') - (1 + \beta)(1 - 2\beta) + \beta(1 + \beta)\operatorname{cosec} \Theta \sin(\Theta + 2\Psi)\} = \beta [\cos \Theta + \beta \cos(\Theta + 2\Psi)], \quad (4.2)$$

we can then approximate G for small β by

$$G \approx -\frac{\beta [\cos \Theta + \beta \cos(\Theta + 2\Psi)]}{1 - \beta [1 + \operatorname{cosec} \Theta \sin(\Theta + 2\Psi) + 2 \cos 2\Psi(1 + \Psi')]}, \quad (4.3)$$

which becomes

$$G \approx -\beta \{ \cos \Theta + \beta [\cos(\Theta + 2\Psi) + \cos \Theta + \cot \Theta \sin(\Theta + 2\Psi) + 2 \cos \Theta \cos 2\Psi(\Psi' + 1)] \}. \quad (4.4)$$

Finally, on substituting (4.4) into (2.22)₁ and simplifying, equation (4.1) arises. However, we again note that although (4.1) is much simpler in form, we are unable to determine an analytical solution even for this special case.

4.1 Exact Solution for $\beta = 1$ and $\delta = 1$.

For $\beta = 1$ and $\delta = 1$, (2.22) simplifies to give

$$\frac{\Psi''}{(1 + \Psi')^2} + \frac{[3 \tan \Psi - \cot \Theta]}{(1 + \Psi')} + 2 \tan(\Theta + \Psi) = 0. \quad (4.5)$$

Thus, if we make the transformations

$$x = \tan \Theta, \quad y = \tan(\Theta + \Psi), \quad (4.6)$$

so that

$$\tan \Psi = \frac{y - x}{1 + xy}, \quad (4.7)$$

then (4.5) can be shown to become

$$\frac{d^2y}{dx^2} + \frac{2xy - 1}{x(1 + xy)} \frac{dy}{dx} = 0. \quad (4.8)$$

Now, this equation remains invariant under the stretching group of transformations (3.10) and therefore we again introduce the new variable z such that $z = xy$ so that upon making the Euler transformation $s = \log x$, (4.8) becomes

$$\frac{d^2z}{ds^2} - \frac{dz}{ds} - \frac{3}{(1 + z)} \left(\frac{dz}{ds} - z \right) = 0. \quad (4.9)$$

Now, upon introducing ω given by (3.12) then (4.9) is equivalent to

$$\frac{dz}{ds} = \omega + z, \quad \frac{d\omega}{ds} = \frac{3\omega}{(1 + z)}. \quad (4.10)$$

On eliminating z from these equations and introducing v defined by (3.14) we may readily deduce the standard first order differential equation

$$\frac{dv}{d\omega} + \frac{1}{3} \left(1 - \frac{1}{\omega} \right) = -\frac{1}{\omega}. \quad (4.11)$$

This equation readily integrates to give

$$v = 3 - \omega^{1/3} e^{-\omega/3} I(\omega), \quad (4.12)$$

where $I(\omega)$ is the integral defined by

$$I(\omega) = \int^\omega t^{-1/3} e^{t/3} dt + C_1, \quad (4.13)$$

where C_1 denotes an arbitrary constant of integration. From (3.14) and (4.12) we may perform a second integration to obtain

$$3\omega^{-1/3} e^{\omega/3} - I(\omega) = \frac{C_2}{x}, \quad (4.14)$$

where C_2 denotes an arbitrary constant of integration. Next, from the fact that $v = 3/(1 + z)$ we find

$$z = \frac{3}{v} - 1 = \frac{\omega^{1/3}}{3} e^{-\omega/3} I(\omega) \left\{ 1 - \frac{\omega^{1/3}}{3} e^{-\omega/3} I(\omega) \right\}^{-1}, \quad (4.15)$$

and as $z = xy$ then clearly

$$y = \tan(\Theta + \Psi) = \frac{I(\omega)}{C_2}. \quad (4.16)$$

Therefore, from (4.6), (4.7), (4.14) and (4.16) we find

$$\tan \Theta = \frac{C_2}{3\omega^{-1/3}e^{\omega/3} - I(\omega)}, \quad (4.17)$$

$$\tan \Psi = \frac{I(\omega)}{C_2} \left\{ 1 - \frac{\omega^{1/3}}{3} e^{-\omega/3} I(\omega) \right\} - \frac{C_2}{3} \omega^{1/3} e^{-\omega/3}.$$

In order to determine a relation for G in terms of the parameter ω , we differentiate (4.16), using (4.17)₁, to find

$$\Psi' + 1 = \frac{\omega \cos^2(\Theta + \Psi)}{\sin^2 \Theta} = \frac{\cos(\Theta + \Psi)}{2G \cos \Psi}, \quad (4.18)$$

where the latter equality follows from the differential equation (2.22)₂ with $\beta = 1$ and $\delta = 1$. From (4.18) we obtain

$$G = \frac{\sec \Psi \sec(\Theta + \Psi)}{2\omega \operatorname{cosec}^2 \Theta} = \frac{\left\{ 1 + \left(\frac{y-x}{1+xy} \right)^2 \right\}^{1/2} (1+y^2)^{1/2}}{2\omega(1+x^{-2})} = \frac{x^2(1+y^2)}{2\omega(1+xy)(1+x^2)^{1/2}}, \quad (4.19)$$

where x and y are defined by (4.6). Therefore, on simplify (4.19) we may deduce the following expression for G in terms of the parameter ω ,

$$G = \frac{1}{6} \frac{\omega^{-2/3} e^{-\omega/3} [C_2^2 + I(\omega)^2]}{\{C_2^2 + [3\omega^{-1/3} e^{\omega/3} - I(\omega)]^2\}^{1/2}}. \quad (4.20)$$

Therefore, from (4.17) and (4.20) we know the exact solution for $\beta = 1$ and $\delta = 1$.

In the following two subsection we utilize the exact parametric solution for $\beta = 1$ and $\delta = 1$ for determining the stress distribution at the base of stock piles and granular flow through hoppers.

4.1.1 SOLUTION FOR THE STOCK PILE PROBLEM

In general, to apply the solution for $\beta = 1$ to the problem of determining the stress distribution at the base of three-dimensional stock piles, we must ensure that the

solution satisfies the boundary conditions of (2.24) and (2.25). Thus, from (4.13) and (4.17) we find that (2.25) is satisfied provided we associate the parameter value $\omega \equiv 0$ with $\Theta \equiv 0$ and define C_1 such that the integral $I(\omega)$ is given by

$$I(\omega) = \int_0^\omega t^{-1/3} e^{t/3} dt. \quad (4.21)$$

Now, to determine the second constant of integration C_2 , we find that the general boundary condition at $\Theta = \alpha$ of (2.24) is inadequate to determine the value of C_2 . Or in other words, applying the boundary condition of (2.24), namely $G(\alpha) = 0$, to (4.20) gives no additional information with which to determine C_2 from. However, recall that for the special case of $\beta = 1$ a possible alternate boundary condition to (2.24) exists, namely

$$\Psi(\alpha) = -\frac{\pi}{2}. \quad (4.22)$$

Therefore, upon applying (4.22) to (4.17), noting (4.16), we find

$$\tan \alpha = \frac{C_2}{3\omega_0^{-1/3} e^{\omega_0/3} + C_2 \cot \alpha}, \quad (4.23)$$

where ω_0 denotes the parameter value corresponding to $\Theta = \alpha$ and from (4.23) it is clear that $\omega_0 = -\infty$.

Accordingly, we change the parameter from ω to $-\lambda$, make the substitution $t = -s$ in the integral (4.21) so that the exact parametric solution (4.17) and (4.20) now becomes

$$\begin{aligned} \tan \Theta &= \frac{C_3}{3\lambda^{-1/3} e^{-\lambda/3} + J(\lambda)}, \\ \tan \Psi &= -\frac{J(\lambda)}{C_3} \left\{ 1 + \frac{\lambda^{1/3}}{3} e^{\lambda/3} J(\lambda) \right\} - \frac{C_3}{3} \lambda^{1/3} e^{\lambda/3}, \\ G &= -\frac{1}{6} \frac{\lambda^{-2/3} e^{\lambda/3} [C_3^2 + J(\lambda)^2]}{\{C_3^2 + [3\lambda^{-1/3} e^{-\lambda/3} + J(\lambda)]^2\}^{1/2}}, \end{aligned} \quad (4.24)$$

where $C_2 = (-1)^{-1/3} C_3$ and the integral $J(\lambda)$ is defined by

$$J(\lambda) = \int_0^\lambda s^{-1/3} e^{-s/3} ds. \quad (4.25)$$

We again note that the determination of the sign of (4.24)₃ is not a straightforward matter and its validity may be more apparent from the final expression in (4.19) when ω is changed to minus λ . In particular, it is important not to simply express the $(-1)^{-2/3}$ term obtained under the square root on the denominator as 1, as we require the -1 factor obtained upon taking the $(-1)^{-2/3}$ out of the square root. Now since $\lambda = \infty$ is the parameter value corresponding to $\Theta = \alpha$, we require an estimate of $J(\lambda)$ for λ tending to infinity. Thus, from (4.25) we find

$$\lim_{\lambda \rightarrow \infty} J(\lambda) = \int_0^\infty s^{-1/3} e^{-s/3} ds = 3^{2/3} \Gamma(2/3), \quad (4.26)$$

and from this equation and (4.24)₁ we may deduce

$$C_3 = 3^{2/3} \Gamma(2/3) \tan \alpha, \quad (4.27)$$

so that

$$\tan \Theta = \frac{3^{2/3} \Gamma(2/3) \tan \alpha}{3\lambda^{-1/3} e^{-\lambda/3} + J(\lambda)},$$

$$\tan \Psi = -\frac{J(\lambda)}{3^{2/3} \Gamma(2/3) \tan \alpha} \left\{ 1 + \frac{\lambda^{1/3}}{3} e^{\lambda/3} J(\lambda) \right\} - \frac{\Gamma(2/3)}{3^{1/3}} \lambda^{1/3} e^{\lambda/3} \tan \alpha, \quad (4.28)$$

$$G = -\frac{1}{6} \frac{\lambda^{-2/3} e^{\lambda/3} [3^{4/3} \Gamma^2(2/3) \tan^2 \alpha + J^2(\lambda)]}{\{3^{4/3} \Gamma^2(2/3) \tan^2 \alpha + [3\lambda^{-1/3} e^{-\lambda/3} + J(\lambda)]^2\}^{1/2}},$$

which confirms $\Psi(\alpha) = -\pi/2$ as λ tends to infinity, but again note that $G(\alpha)$ tends to minus infinity rather than zero. In the case with $\beta = 1$ we have satisfied the alternative boundary conditions of $\Psi(0) = 0$ and $\Psi(\alpha) = -\pi/2$, which only apply for this special case. From the expressions (4.28) and (2.17) we may confirm that $\sigma_{R\Theta}$ and $\sigma_{\Theta\Theta}$ both tend to zero.

Thus, for a three-dimensional stock pile with an angle of internal friction of 90° , an exact parametric solution is given by (4.24) for $0 \leq \lambda \leq \infty$ where $J(\lambda)$ is defined by (4.25) and the constant C_2 is given by equation (4.27).

4.1.2 SOLUTION FOR THE HOPPER PROBLEM

To apply the solution for $\beta = 1$ to the problem of flow under gravity through a cone shaped hopper, we must ensure that the solution satisfies (2.25) and (2.27) or (2.28), depending on the value of the angle of wall friction μ . Thus, from (4.13) and (4.17) we find that (2.25) is satisfied provided we associate the parameter value $\omega \equiv 0$ with $\Theta \equiv 0$ and define C_1 such that the integral $I(\omega)$ is given by

$$I(\omega) = \int_0^\omega t^{-1/3} e^{t/3} dt. \quad (4.29)$$

Now, to determine the second constant of integration C_2 , we first need to determine which boundary condition at the wall of the hopper applies. To do this, we assume $\Psi'(\alpha)$ is finite and from Hill and Cox [7] we find which this means that $\mu \leq \phi$, and hence the boundary condition (2.27) applies. Therefore, from (2.27) for $\beta = 1$ we find that (4.17) becomes

$$\begin{aligned} \tan \alpha &= \frac{C_2}{3\omega_0^{-1/3} e^{\omega_0/3} - I(\omega_0)}, \\ \tan \mu &= \frac{I(\omega_0)}{C_2} \left\{ 1 - \frac{\omega_0^{1/3}}{3} e^{-\omega_0/3} I(\omega_0) \right\} - \frac{C_2}{3} \omega_0^{1/3} e^{-\omega_0/3}, \end{aligned} \quad (4.30)$$

and from (4.30)₁ we find

$$C_2 = \tan \alpha \left\{ \frac{3e^{\omega_0/3}}{\omega_0^{1/3}} - I(\omega_0) \right\}, \quad (4.31)$$

which upon substituting into (4.30)₂ gives

$$\frac{\sin \alpha}{\cos \mu} \sin(\alpha + \mu) = \frac{\omega_0^{1/3}}{3} e^{-\omega_0/3} I(\omega_0), \quad (4.32)$$

which is a transcendental equation for ω_0 . Thus, C_2 is determined from (4.31) where ω_0 is a solution of (4.32).

Thus, for flow under gravity through a three-dimensional cone shaped hopper with an angle of internal friction of 90° , an exact parametric solution is given by (4.17) and (4.20) for $0 \leq \omega \leq \omega_0$ where $I(\omega)$ is defined by (4.29), the constant C_2 is defined by (4.31) and ω_0 is determined from the transcendental equation (4.32).

4.2 Exact Solution for $\beta = -1$ and $\delta = -1$.

For $\beta = -1$ and $\delta = -1$, (2.22) simplifies to give

$$\frac{\Psi''}{(1 + \Psi')^2} - \frac{[3 \cot \Psi + \cot \Theta]}{(1 + \Psi')} - 2 \cot(\Theta + \Psi) = 0. \quad (4.33)$$

Thus, if we make the transformations

$$x = \cot \Theta, \quad y = \cot(\Theta + \Psi), \quad (4.34)$$

so that

$$\cot \Psi = \frac{1 + xy}{x - y}, \quad (4.35)$$

then (4.33) can be shown to simplify to yield

$$\frac{d^2 y}{dx^2} + \frac{3}{(x - y)} \frac{dy}{dx} = 0. \quad (4.36)$$

Now, this equation remains invariant under the stretching group of transformations (3.43) and therefore we introduce the new variable z such that $y = xz$ and so that upon making the Euler transformation $s = \log x$, (4.36) becomes

$$\frac{d^2 z}{ds^2} + \frac{dz}{ds} + \frac{3}{(1 - z)} \left(\frac{dz}{ds} + z \right) = 0. \quad (4.37)$$

Now, upon introducing ω given by (3.45) then (4.37) is equivalent to

$$\frac{dz}{ds} = \omega - z, \quad \frac{d\omega}{ds} = -\frac{3\omega}{1 - z}. \quad (4.38)$$

On eliminating z from these equations and introducing v defined by (3.14) we may readily deduce the standard first order differential equation

$$\frac{dv}{d\omega} + \frac{1}{3} \left(1 - \frac{1}{\omega} \right) v = \frac{1}{\omega}. \quad (4.39)$$

This equation readily integrates to give

$$v = -3 + \omega^{1/3} e^{-\omega/3} I(\omega), \quad (4.40)$$

where $I(\omega)$ is the integral defined by (4.13). From (3.14) and (4.40) we may perform a second integration to obtain

$$3\omega^{-1/3} e^{\omega/3} - I(\omega) = C_2 x, \quad (4.41)$$

where C_2 denotes an arbitrary constant of integration. Next, from the fact that $v = 3/(z - 1)$ we find

$$z = \frac{3}{v} + 1 = -\frac{I(\omega)}{3\omega^{-1/3}e^{\omega/3} - I(\omega)}, \quad (4.42)$$

and as $y = xz$ then clearly

$$y = \cot(\Theta + \Psi) = -\frac{I(\omega)}{C_2}. \quad (4.43)$$

Therefore, from (4.34), (4.35), (4.41) and (4.43) we find

$$\tan \Theta = \frac{C_2}{3\omega^{-1/3}e^{\omega/3} - I(\omega)}, \quad (4.44)$$

$$\cot \Psi = -\frac{I(\omega)}{C_2} \left\{ 1 - \frac{\omega^{1/3}}{3} e^{-\omega/3} I(\omega) \right\} + \frac{C_2}{3} \omega^{1/3} e^{-\omega/3}.$$

In order to determine a relation for G in terms of the parameter ω , we differentiate (4.43), using (4.44)₁, to obtain

$$\Psi' + 1 = \frac{\omega \operatorname{cosec}^2 \Theta}{\operatorname{cosec}^2(\Theta + \Psi)} = -\frac{\sin(\Theta + \Psi)}{2G \sin \Psi}, \quad (4.45)$$

where the latter equality follows from the differential equation (2.22)₂ with $\beta = -1$.

From (4.45) we obtain

$$G = -\frac{\operatorname{cosec} \Psi \operatorname{cosec}(\Theta + \Psi)}{2\omega \operatorname{cosec}^2 \Theta} = -\frac{\left\{ 1 + \left(\frac{1+xy}{x-y} \right)^2 \right\}^{1/2} (1+y^2)^{1/2}}{2\omega(1+x^2)} = -\frac{(1+y^2)}{2\omega(x-y)(1+x^2)^{1/2}}, \quad (4.46)$$

where x and y are defined by (4.34). Therefore, on simplifying (4.46) we may deduce the following expression for G in terms of the parameter ω ,

$$G = -\frac{1}{6} \frac{\omega^{-2/3} e^{-\omega/3} [C_2^2 + I(\omega)^2]}{\{C_2^2 + [3\omega^{-1/3} e^{\omega/3} - I(\omega)]^2\}^{1/2}}. \quad (4.47)$$

Therefore, we have determined an exact parametric solution to the three-dimensional governing equations (2.22) for the special case of $\beta = -1$. However, we are unable to apply the exact parametric solution for $\beta = -1$ to either the stock pile problem or the hopper problem because we are unable to determine a set of values of the constants of integration C_1 and C_2 such that the appropriate boundary conditions are satisfied and the solution remain physical.

5 Numerical Results

In this section we compare the three-dimensional exact parametric solution for $\beta = 1$ and $\delta = 1$, namely (4.17) and (4.20), with a full numerical solution of the three-dimensional governing equations (2.22) for both the stock pile and hopper problems. We note that the corresponding numerical results for the two-dimensional problems are given by the authors in [7, 8]. We use an iterative scheme to determine successive numerical solutions which converge to the solution which satisfies the appropriate boundary conditions. Specifically, we use a non-linear Finite-Difference method to numerically solve the single second order differential equation for Ψ for the special case of $\beta = 1$ and $\delta = 1$, namely (2.23) with $\beta = 1$ or (4.5), subject to the appropriate boundary conditions. For definiteness, following the three-dimensional experiments of Smid and Novosad [13], we assume a bulk density of $\rho = 1567\text{kg/m}^3$ and $\alpha = (287/900)\pi$, which gives an angle of repose of 32.6° degrees. The experimental results determined by Smid and Novosad [13] are shown in Figure 3 and were obtained at various stages during the pouring of a three-dimensional heap. When the heap was at height h , the horizontal and vertical stresses were measured at the base of the heap. The material used was sand for which the angle of repose was 32.6° and the average bulk density was determined to be $\rho = 1567 \text{ kg/m}^3$.

Figure 4 shows the numerical variation of $\Psi(\Theta)$ and $G(\Theta)$ respectively for the stock pile problem for $\beta = 1$ and $\delta = 1$. We note that the solutions shown in Figure 4 satisfy the stock pile boundary conditions of (2.25) and (4.22). We also note that the numerical solution gives identical results to the exact parametric solution (4.24) with C_3 defined by (4.27) and that both predict the unusual boundary layer behavior shown in the figure. The numerically determined horizontal and vertical stresses at the base of a stock pile for the special case of $\beta = 1$, as given by (2.32), are shown in Figure 5. We note from Figure 5 that the maximum vertical pressure is located directly beneath the vertex of the stock pile, which is in contrast to the experimental M-shaped curves of Smid and Novosad [13], as shown in Figure 3 .

Figure 6 shows the numerical variation of $\Psi(\Theta)$ and $G(\Theta)$ respectively for the hopper problem for $\beta = 1$ and $\delta = 1$, assuming an angle of wall friction of $\mu = \pi/12$. We note that the numerical solutions shown in Figure 6 satisfy the hopper boundary conditions of (2.25) and (2.27). We also note that the numerical solution gives identical results to the exact parametric solution (4.17) and (4.20) where C_2 is defined by (4.31).

Finally, we note that independent numerical schemes for the two-dimensional governing equations for the special case of $\beta = 1$ have been previously determined in Hill and Cox [7] for the hopper problem and Hill and Cox [8] for the stock pile problem, and in both cases, the numerical solutions agree with the corresponding exact parametric solutions for $\beta = 1$.

6 Conclusions

We have considered the two problems of determining the stress distribution at the base of stock piles, and for gravity flow through hoppers. We have given an overview of a previously stated exact solution of the two-dimensional governing equations, and we have determined additional exact solutions to these equations. For the special case of $\beta = 1$, we have shown that the derived exact parametric solutions for both two and three-dimensions, can be applied to both the stock pile and hopper problems. While many materials exhibit large angle of internal friction, such as Coal and Silica with 80 degrees and 78.34 degrees respectively, the exact parametric solution for an angle of internal friction of 90 degrees does not exhibit the experimentally determined profile obtained by Smid and Novosad [13]. For the special case of $\beta = -1$, we have derived new analytical solutions for both the two and three-dimensional equations. Unfortunately however, we are unable to apply these new exact parametric solutions for $\beta = -1$ to either the two or three-dimensional stock pile or hopper problems. Despite this, all the exact mathematical solutions derived here might be utilized for physically meaningful materials either for the determination of limiting bounding solutions or for numerical benchmarking.

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Figure Captions

Figure 1. Coordinates for the two-dimensional stock pile and hopper problems ((a) stock pile and (b) hopper).

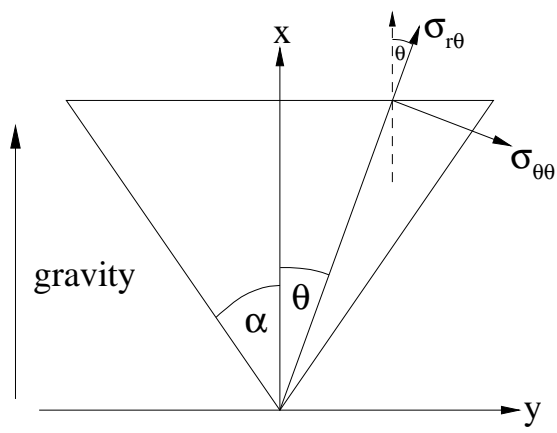
Figure 2. Coordinates for the three-dimensional stock pile and hopper problems ((a) stock pile and (b) hopper).

Figure 3. Horizontal and vertical force distributions as determined experimentally by Smid and Novosad [13] ((a) horizontal and (b) vertical).

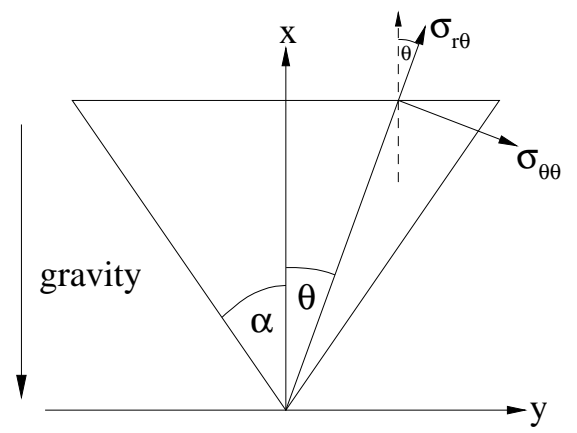
Figure 4. Numerical variation of $\Psi(\Theta)$ and $G(\Theta)$ for the special case of $\beta = 1$ and $\delta = 1$ for the stock pile problem.

Figure 5. Numerical variation of the horizontal and vertical stresses within the stock pile for the special case of $\beta = 1$ and $\delta = 1$.

Figure 6. Numerical variation of $\Psi(\Theta)$ and $G(\Theta)$ for the special case of $\beta = 1$ and $\delta = 1$ for the hopper problem.

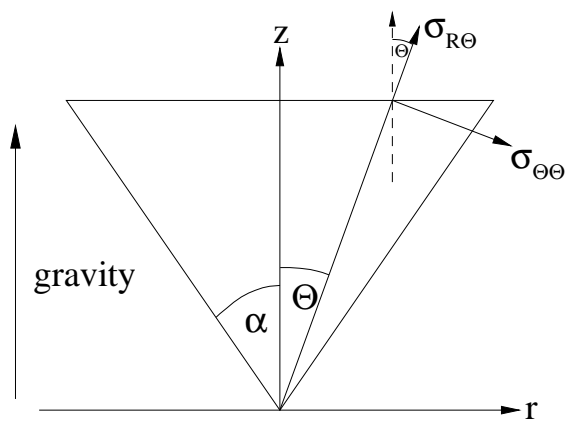


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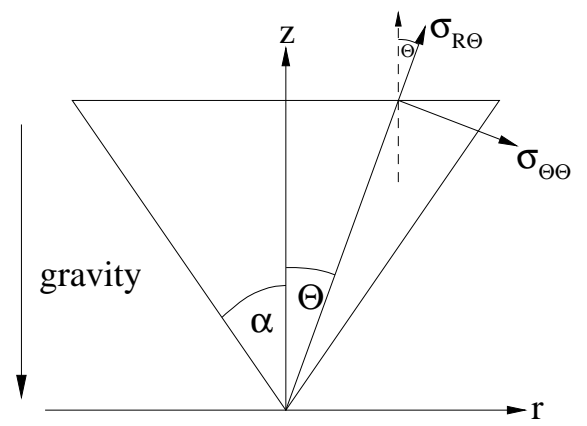


(b).

Figure 1.

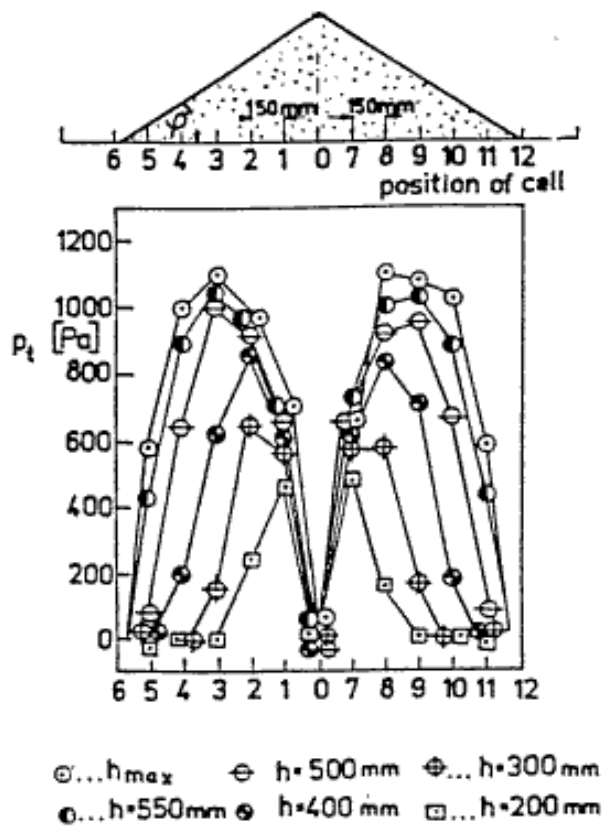


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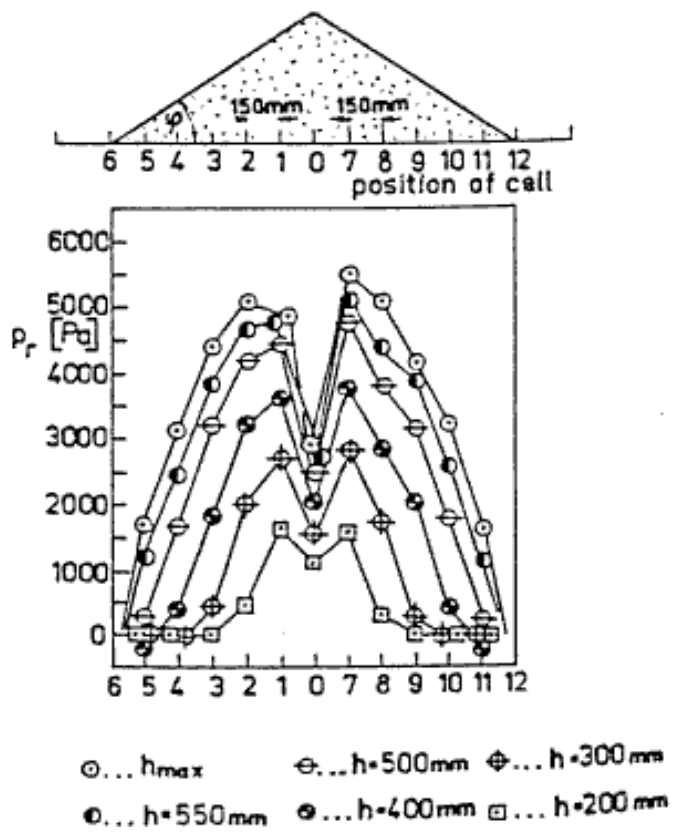


(b).

Figure 2.

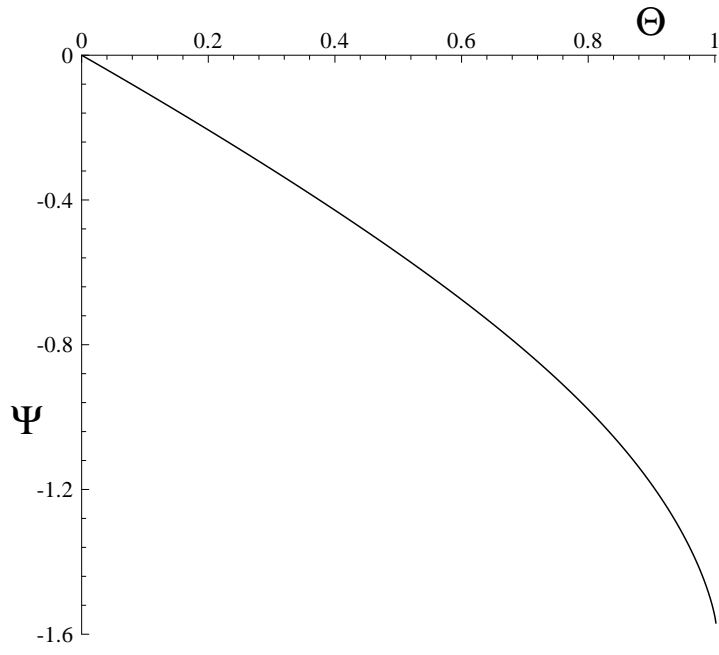


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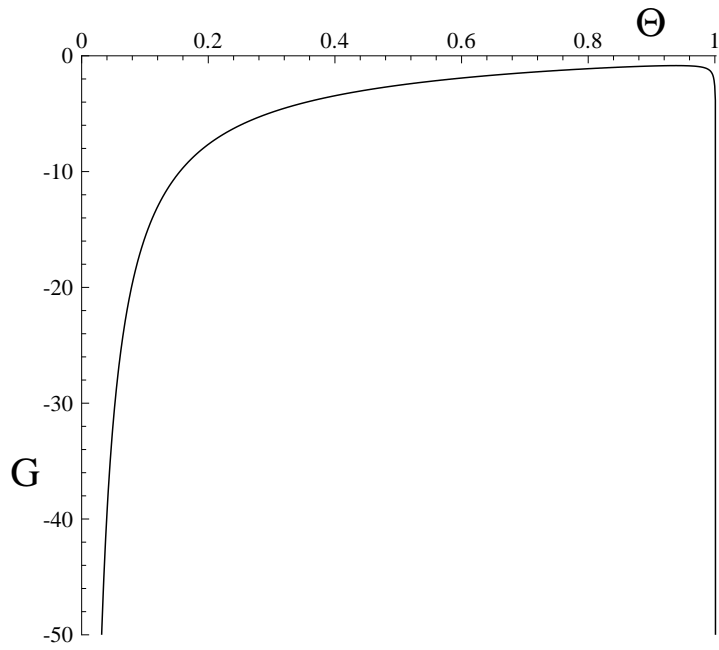


(b).

Figure 3.

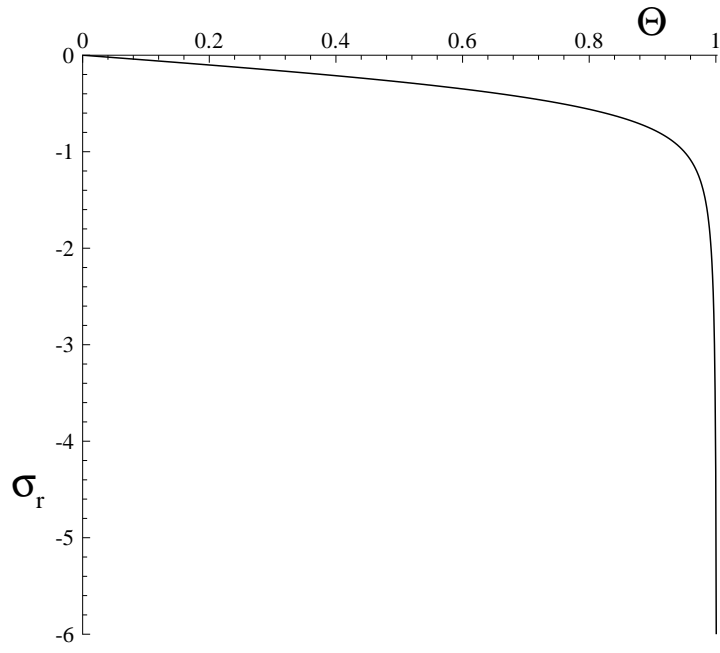


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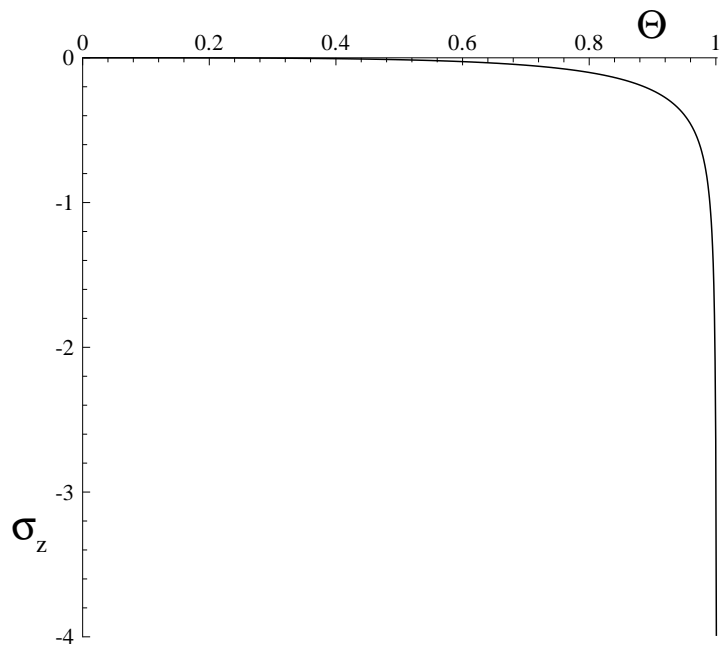


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Figure 4.

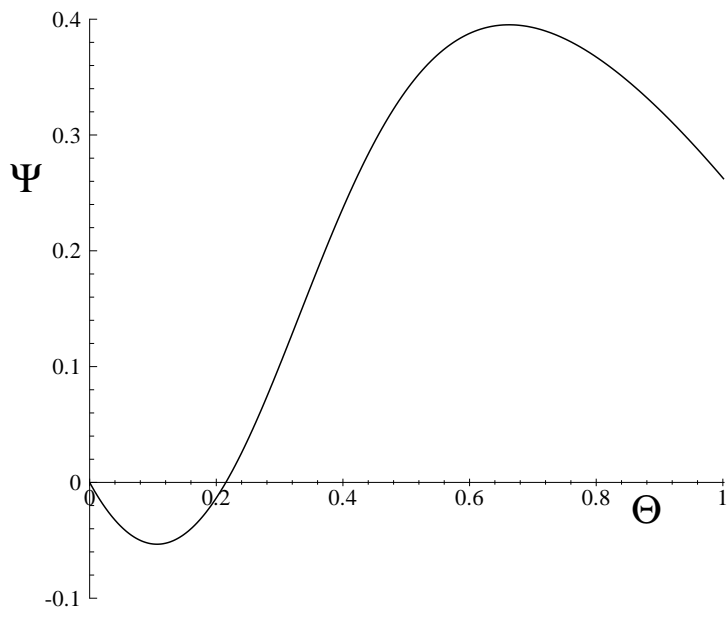


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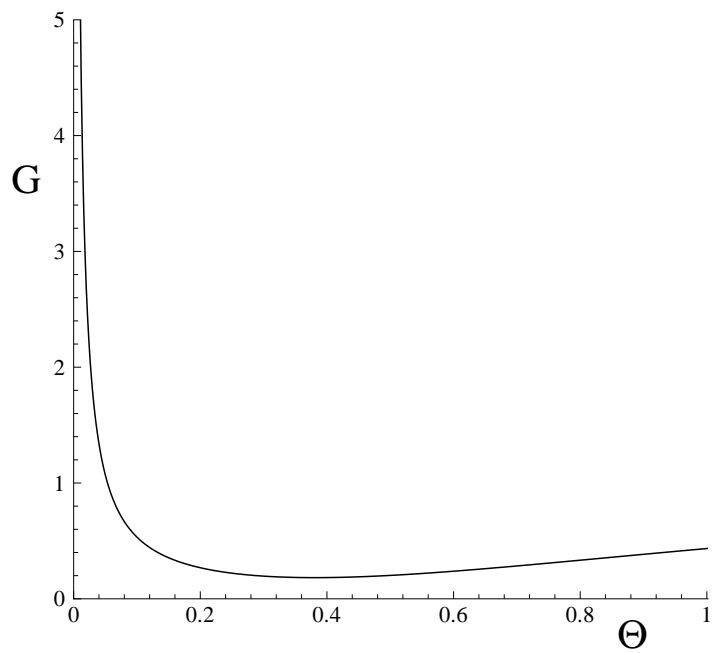


(b).

Figure 5.



(a).



(b).

Figure 6.