Non-dilatant double-shearing theory applied to dynamical granular chute flow

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Summary. The non-dilatant double-shearing theory for granular flow is applied to the problem of dynamical chute flow down an inclined slope. Four distinct families of solutions are examined and a number of simple exact solutions are determined which apply to the special case of a granular material with an angle of internal friction equal to ninety degrees. Some of the solutions obtained are illustrated graphically.

1 Introduction

The dynamics of a granular mass flowing down an inclined slope has particular relevance in the study of avalanches and debris flow, for which following the huge devastation which occurs it is apparent that the granular mass may experience huge forces and velocities. In recent papers (Hill [1, 2] and Hill and Williams [3]) it is shown that the hypoplastic model for granular materials admits "hot-spot" solutions, by which we mean that the stresses and velocities may become unbounded in finite time. The purpose of this paper is to examine dynamical granular chute flow down an inclined slope for an alternative granular model, namely the non-dilatant double-shearing theory proposed by Spencer [4, 5], with a view to whether or not simple exact solutions for this model also exhibit similar hot-spot behaviour.

Specifically, we extend the simple horizontal shear-flow model proposed by Hill and Spencer [6], but formulated here to include flow down an inclined

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Granular material	Measured values of $\beta = \sin\phi$			
Coal	0.939	0.958	0.973	0.985
Alumina cake		0.941		
Waste rock		0.974		
Silica		0.979		

Table 1: Measured values of $\beta = \sin\phi$ for certain granular materials, where ϕ is the angle of internal friction.

slope, and we examine the corresponding three families of exact solutions to those studied in [6]. The cohesionless double-shearing theory is characterized by a single physical parameter ϕ termed the angle of internal friction, and each of the families of solutions assumes a particularly simple form when the angle of internal friction takes on the value 90°. There exists in reality many materials with large angles of internal friction such as those shown in Table 1, and this idealization is meaningful both for these highly frictional materials and for the determination of limiting bounds. The non-dilatant double-shearing granular model has been successfully applied to a variety of problems (see for example Spencer [7, 8, 9], Spencer and Bradley [10] and Spencer and Hill [11]).

We comment here that there exists a huge literature on the modelling of "steady" granular chute flow and we refer the reader to Zheng and Hill [12] and Hill and Zheng [13] for appropriate references. The reader may also consult, for example, Hutter [14], Savage and Hutter [15, 16] and Hutter and Koch [17] for recent papers relating specifically to the modelling of granular avalanches.

In the following section we state the basic plane strain dynamical equations for the non-dilatant double-shearing model of granular flow. In the subsequent section we specialize these equations to describe the dynamics of a granular layer accelerating down an inclined plane. Special solutions for this model are examined in Sections 4, 5 and 6. In Section 7 we examine a separable solution for the special case of the angle of internal friction equal to ninety degrees, for which in general ($\phi \neq 90^\circ$) there is no corresponding solution. The four major solutions obtained are illustrated graphically in Section 8.

2 Basic equations for non-dilatant double-shearing theory

In this section we state briefly the basic equations for the continuum mechanical theory of granular materials for the problem of two-dimensional plane strain dynamical flow down an inclined plane or chute. In terms of the Cartesian coordinates (x, y, z) defined by Figure 1, the stress components for dynamical plane flow down an inclined plane or chute satisfy the equilibrium equations with inertia terms, namely

$$\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} &= \rho \left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - g\sin\alpha \right], \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} &= \rho \left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + g\cos\alpha \right], \end{aligned} \tag{1}$$

where ρ denotes the bulk density which here is assumed to be constant, g is acceleration due to gravity, and α is the angle between the slope of the inclined plane and the horizontal, as shown in Figure 1. The in-plane physical Cauchy stress components are denoted by σ_{xx}, σ_{xy} and σ_{yy}, which are assumed to be positive in tension, and u and v denote the velocities in the x and y directions respectively. We note that both stress and velocity components are assumed to be independent of the depth z.

Upon introducing the usual positive quantities p and q defined by

$$p = -\frac{1}{2}(\sigma_{xx} + \sigma_{yy}), \quad q = \frac{1}{2}\left[(\sigma_{xx} - \sigma_{yy})^2 + 4\sigma_{xy}^2\right]^{1/2}, \tag{2}$$

and the stress angle ψ given by

$$\tan 2\psi = \frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}}, \tag{3}$$

where ψ is the angle between the maximum principal stress axis and the x axis, in the direction of increasing θ, then the stresses can be expressed in terms of the usual decomposition

$$\sigma_{xx} = -p + q\cos 2\psi, \quad \sigma_{yy} = -p - q\cos 2\psi, \quad \sigma_{xy} = q\sin 2\psi. \tag{4}$$

This stress decomposition, combined with the usual cohesionless Coulomb-Mohr yield condition

$$q = \beta p, \tag{5}$$

enables the three stresses to be expressed in terms of just two unknowns, where $\beta = \sin\phi$ and ϕ is the angle of internal friction, assumed constant.

The above equations are generally accepted as a reasonable basis for the determination of the plane strain stress profile for free flowing (cohesionless) granular material. However, to prescribe equations for the determination of the associated velocity profile is far more controversial. Here we assume the non-dilatant double-shearing theory as enunciated by Spencer [4, 5]. This theory assumes that the non-zero components of velocity $u(x, y, t)$ and $v(x, y, t)$ satisfy the following equations

$$\begin{gathered} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \\ \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\sin 2\psi - \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\cos 2\psi = \beta\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} + 2\Omega\right), \end{gathered} \tag{6}$$

where the quantity Ω is given by

$$\Omega = \frac{\partial \psi}{\partial t} + u\frac{\partial \psi}{\partial x} + v\frac{\partial \psi}{\partial y}. \tag{7}$$

Physically speaking, equation $(6)_1$ corresponds to the assumption that the flow is isochoric, while $(6)_2$ expresses the condition that the flow arises due to simultaneous shearing on the two families of surfaces on which the critical shear stress is mobilized. Together with appropriate boundary conditions for the stresses and velocities, the above equations represent a complete description for the dynamical behaviour of non-dilatant cohesionless granular flow down a smooth inclined plane or chute.

3 Model equations for dynamical chute flow

In this section, we exploit the previously stated equations for dynamical flow of material down an inclined slope or chute to determine a simple model which gives rise to analytical solutions that exhibit hot-spots. In order to achieve this, we assume that the material undergoes a shearing deformation in the x direction, so that the components of the displacement, velocity and acceleration in the y direction are zero. Thus, we assume that the velocity components take the form

$$u = u(y,t), \quad v = 0, \tag{8}$$

and we note that this deformation is consistent with the material being non-dilatant or incompressible. From (8), we see that the equilibrium equations with inertia terms (1) becomes

$$\frac{\partial \sigma_{xy}}{\partial y} = \rho\frac{\partial u}{\partial t} - \rho g \sin\alpha, \quad \frac{\partial \sigma_{yy}}{\partial y} = \rho g \cos\alpha, \tag{9}$$

and clearly $(9)_2$ can be integrated directly to obtain

$$\sigma_{yy} = \rho g(y-h)\cos\alpha, \tag{10}$$

where h is at most an arbitrary function of time t, but here is taken to be constant. We note that if the surface $y=0$ is stress free, then $h=0$. Now, from $(4)_2$, (5) and (10) we find

$$q = \frac{\rho g\beta(h-y)\cos\alpha}{1+\beta\cos 2\psi}, \tag{11}$$

which combined with $(4)_3$ and the equilibrium equation $(9)_1$, gives

$$\frac{\partial u}{\partial t} - \frac{2g\beta(h-y)(\beta+\cos 2\psi)\cos\alpha}{(1+\beta\cos 2\psi)^2}\frac{\partial \psi}{\partial y} = \frac{g[\sin\alpha + \beta\sin(\alpha-2\psi)]}{1+\beta\cos 2\psi}. \tag{12}$$

Also from (8), we find that the non-dilatant double-shearing velocity equation $(6)_2$ becomes simply

$$(\beta+\cos 2\psi)\frac{\partial u}{\partial y} + 2\beta\frac{\partial \psi}{\partial t} = 0. \tag{13}$$

Therefore, equations (12) and (13) are the governing partial differential equations for dynamical flow of granular material down an inclined slope or chute. In the following sections we examine these governing equations for the existence of hot-spot solutions.

We note that for $\beta = 1$ ($\phi = \pi/2$) equations (12) and (13) become respectively

$$\begin{aligned} &\frac{\partial u}{\partial t} - g(h-y)\cos\alpha\frac{\partial T}{\partial y} = g(\sin\alpha - T\cos\alpha), \\ &\frac{\partial u}{\partial y} + \frac{\partial T}{\partial t} = 0, \end{aligned} \tag{14}$$

where $T = \tan\psi$ and from which we may deduce the linear partial differential equations

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} + \frac{\partial}{\partial y}\left\{ g(h-y)\cos\alpha \frac{\partial u}{\partial y} \right\} = 0, \\
&\frac{\partial^2 T}{\partial t^2} + \frac{\partial^2}{\partial y^2}\left\{ g(h-y)T\cos\alpha \right\} = 0.
\end{aligned}
\tag{15}
$$

We observe for $\phi \neq \pi/2$, that while the transformation

$$
T = \frac{\beta \sin 2\psi}{1 + \beta \cos 2\psi}, \tag{16}
$$

transforms (12) into the linear equation $(14)_1$, (16) does not linearize equation (13).

4 Similarity solution for a half-space

In this section, we examine the governing equations for dynamical flow of material down an inclined plane or chute, namely (12) and (13), and we look for hot-spot behaviour arising from a similarity solution. Thus we assume the governing equations are invariant under the stretching group of transformations

$$
(h-y)^* = e^{\epsilon}(h-y), \quad t^* = e^{a\epsilon}t, \quad u^* = e^{b\epsilon}u, \quad \psi^* = \psi, \tag{17}
$$

and we determine the constants a and b such that (12) and (13) remains invariant in the sense that they have the same form in the starred variables as in the non-starred variables. From (12) and (17) we obtain $a = b$, while from (13) and (17) we have $a + b = 1$, so that $a = b = 1/2$. Thus, we find that (12) and (13) admits a similarity solution of the form

$$
u(y,t) = tU(\xi), \quad \psi = \Psi(\xi), \tag{18}
$$

where U and Ψ are functions of ξ, which is defined by

$$\xi = (h - y)/t^2. \tag{19}$$

Therefore, on substituting (18) into (12) and (13) we obtain

$$\begin{gathered}(\beta + \cos 2\Psi)\frac{dU}{d\xi} + 4\beta\xi\frac{d\Psi}{d\xi} = 0, \\ \frac{dU}{d\xi} - \frac{g\beta\cos\alpha(\beta + \cos 2\Psi)}{(1 + \beta\cos 2\Psi)^2}\frac{d\Psi}{d\xi} = \frac{1}{2\xi}\left\{U - \frac{g[\sin\alpha + \beta\sin(\alpha - 2\Psi)]}{1 + \beta\cos 2\Psi}\right\},\end{gathered} \tag{20}$$

which are the governing similarity equations for dynamical granular flow down an inclined plane or chute. In general (20) must be solved numerically, and therefore we solve (20) for $dU/d\xi$ and $d\Psi/d\xi$, to obtain

$$\begin{gathered}\frac{d\Psi}{d\xi} = \frac{\dfrac{(\beta + \cos 2\Psi)(1 + \beta\cos 2\Psi)^2}{2\beta\xi}\left\{\dfrac{g[\sin\alpha + \beta\sin(\alpha - 2\Psi)]}{1 + \beta\cos 2\Psi} - U\right\}}{4\xi(1 + \beta\cos 2\Psi)^2 + g\cos\alpha(\beta + \cos 2\Psi)^2}, \\ \frac{dU}{d\xi} = -\frac{2(1 + \beta\cos 2\Psi)^2\left\{\dfrac{g[\sin\alpha + \beta\sin(\alpha - 2\Psi)]}{1 + \beta\cos 2\Psi} - U\right\}}{4\xi(1 + \beta\cos 2\Psi)^2 + g\cos\alpha(\beta + \cos 2\Psi)^2},\end{gathered} \tag{21}$$

which forms the basis for the determination of a full numerical solution.

However, for the special case of $\beta = 1$ we find that we are able to analytically solve (20). In this case, we see that (20) with $\beta = 1$ becomes

$$\begin{gathered}\frac{dU}{d\xi} = -\frac{2\xi}{\cos^2\Psi}\frac{d\Psi}{d\xi}, \\ \frac{dU}{d\xi} - \frac{g\cos\alpha}{2\cos^2\Psi}\frac{d\Psi}{d\xi} = \frac{1}{2\xi}\left[U - g\sin\alpha + g\cos\alpha\tan\Psi\right],\end{gathered} \tag{22}$$

which upon making the transformation $T = \tan\Psi$, gives

$$\begin{gathered}\frac{dU}{d\xi} = -2\xi\frac{dT}{d\xi}, \\ 2\xi\frac{dU}{d\xi} - g\xi\cos\alpha\frac{dT}{d\xi} = U - g\sin\alpha + gT\cos\alpha.\end{gathered} \tag{23}$$

Now, upon substituting $(23)_1$ into $(23)_2$, we find

$$2\xi\frac{dU}{d\xi}+\frac{g\cos\alpha}{2}\frac{dU}{d\xi}=U-g\sin\alpha+gT\cos\alpha, \tag{24}$$

which upon differentiating gives

$$\left[2\xi+\frac{g\cos\alpha}{2}\right]\frac{dV}{d\xi}=V, \tag{25}$$

where V is defined by

$$V=\xi\frac{dU}{d\xi}. \tag{26}$$

Equation (25) is separable, and hence, can be integrated to give

$$V=c_1\left[2\xi+\frac{g\cos\alpha}{2}\right]^{1/2}, \tag{27}$$

where c_1 is a constant of integration. From (26) and (27) we find

$$U=c_1\int\frac{[2\xi+\gamma]^{1/2}}{\xi}d\xi+c_2, \tag{28}$$

where c_2 is a constant of integration and $\gamma=(g\cos\alpha)/2$. In order to integrate (28) we introduce λ such that $\lambda=2\xi+\gamma$, which enables (28) to be rewritten in the form

$$U=\frac{c_1}{2}\int\left(\frac{1}{\lambda^{1/2}-\gamma^{1/2}}+\frac{1}{\lambda^{1/2}+\gamma^{1/2}}\right)d\lambda+c_2. \tag{29}$$

Next, upon making the substitution $x=\lambda^{1/2}$, then (29) becomes

$$U=c_1\int\left(1+\frac{\gamma^{1/2}}{x-\gamma^{1/2}}\right)dx+c_1\int\left(1-\frac{\gamma^{1/2}}{x+\gamma^{1/2}}\right)dx+c_2, \tag{30}$$

which can be integrated directly to give

$$U=2c_1x+c_1\gamma^{1/2}\ln\left\{\frac{x-\gamma^{1/2}}{x+\gamma^{1/2}}\right\}+c_2. \tag{31}$$

Now, in order to determine the solution for T we consider $(23)_1$ and (28), to find

$$T=-\frac{c_1}{2}\int\frac{[2\xi+\gamma]^{1/2}}{\xi^2}d\xi+c_3, \tag{32}$$

where c_3 is a constant of integration and again, $\gamma = (g\cos\alpha)/2$. To integrate (32) we again introduce $\lambda = 2\xi + \gamma$, to obtain

$$T = -\frac{c_1\lambda^{1/2}}{\gamma - \lambda} + \frac{c_1}{\gamma^{1/2}}\tanh\left\{\left(\frac{\lambda}{\gamma}\right)^{1/2}\right\} + c_3. \tag{33}$$

Therefore, from (31) and (33) we have the similarity solution

$$\begin{aligned} U(\xi) &= 2c_1[2\xi+\gamma]^{1/2} + c_1\gamma^{1/2}\ln\left\{\frac{[2\xi+\gamma]^{1/2} - \gamma^{1/2}}{[2\xi+\gamma]^{1/2} + \gamma^{1/2}}\right\} + c_2, \\ \tan\Psi(\xi) &= \frac{c_1[2\xi+\gamma]^{1/2}}{2\xi} + \frac{c_1}{\gamma^{1/2}}\tanh\left\{\frac{[2\xi+\gamma]^{1/2}}{\gamma^{1/2}}\right\} + c_3, \end{aligned} \tag{34}$$

where ξ is defined by (19). We note that from (24) we require

$$c_2 = g\sin\alpha - c_3 g\cos\alpha, \tag{35}$$

which ensures that (34) is the general solution of (22).

5 Stress angle ψ as a function of time only

In this section, we look for a solution where the stress angle ψ is a function of time t only. With this assumption, namely $\psi = \psi(t)$, we find that the governing equations (12) and (13) become

$$\frac{\partial u}{\partial t} = \frac{g[\sin\alpha + \beta\sin(\alpha - 2\psi)]}{1 + \beta\cos 2\psi}, \quad \frac{\partial u}{\partial y} = -\frac{2\beta}{\beta + \cos 2\psi}\frac{d\psi}{dt}, \tag{36}$$

from which it is clear to see that u must be of the form

$$u(y,t) = u_0 y + U(t), \tag{37}$$

where u_0 is some constant and U is a function of t. Hence, upon substituting (37) into $(36)_2$, we find the separable equation

$$\frac{d\psi}{\beta + \cos 2\psi} = -\frac{u_0}{2\beta}dt, \tag{38}$$

which integrates to give

$$\tan\psi = \left[\frac{\beta+1}{\beta-1}\right]^{1/2} \tan\left\{\left[\beta^2-1\right]^{1/2}(c_1 - u_0 t/2\beta)\right\}, \tag{39}$$

where c_1 is a constant of integration. Now, from (39) we find that $(36)_1$ becomes

$$\frac{dU}{dt} = g\sin\alpha - \frac{g\beta\cos\alpha[\beta-1]^{1/2}\sin\left\{2[\beta^2-1]^{1/2}(c_1-u_0t/2\beta)\right\}}{\beta-(1-\beta^2)\cos\{2[\beta^2-1]^{1/2}(c_1-u_0t/2\beta)\}}, \tag{40}$$

which can be integrated directly to give

$$\begin{aligned} U(t) &= c_2 + gt\sin\alpha \\ &\quad + \frac{g\beta^2\cos\alpha}{u_0(1-\beta^2)}\ln\left\{\beta-(1-\beta^2)\cos\left[2[\beta^2-1]^{1/2}(c_1-u_0t/2\beta)\right]\right\}, \end{aligned} \tag{41}$$

where c_2 is a constant of integration. Thus, from (37), (39) and (41) an exact analytical solution for dynamical flow of material down an inclined plane or chute, where the stress angle ψ is a function of time only, is given by

$$\begin{aligned} \tan\psi(t) &= \left[\frac{1+\beta}{1-\beta}\right]^{1/2}\tanh\left\{[1-\beta^2]^{1/2}(c_1-u_0t/2\beta)\right\}, \\ u(y,t) &= c_2 + u_0 y + gt\sin\alpha \\ &\quad + \frac{g\beta^2\cos\alpha}{u_0(1-\beta^2)}\ln\left\{\beta-(1-\beta^2)\cosh\left[2[1-\beta^2]^{1/2}(c_1-u_0t/2\beta)\right]\right\}, \end{aligned} \tag{42}$$

noting again that $\beta = \sin\phi \leq 1$.

6 Stress angle ψ as a function of velocity only

In this section, we look for a solution where the stress angle ψ is a function of the velocity only. With this assumption, namely $\psi = \psi(u)$, we find that the governing equations (12) and (13) become

$$
\begin{gathered}
\frac{\partial u}{\partial y} = -\frac{2\beta}{\beta+\cos 2\psi}\frac{d\psi}{du}\frac{\partial u}{\partial t}, \\
\frac{\partial u}{\partial t} - \frac{2g\beta\cos\alpha(h-y)(\beta+\cos 2\psi)}{(1+\beta\cos 2\psi)^2}\frac{d\psi}{du}\frac{\partial u}{\partial y} = \frac{g[\sin\alpha+\beta\sin(\alpha-2\psi)]}{1+\beta\cos 2\psi},
\end{gathered}
\tag{43}
$$

which upon solving for $\partial u/\partial t$ and $\partial u/\partial y$ gives

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= f_1 f_2\left[f_2^2+\gamma(h-y)\left(\frac{d\psi}{du}\right)^2\right]^{-1}, \\
\frac{\partial u}{\partial y} &= -2\beta\frac{f_1 f_2}{f_3}\frac{d\psi}{du}\left[f_2^2+\gamma(h-y)\left(\frac{d\psi}{du}\right)^2\right]^{-1},
\end{aligned}
\tag{44}
$$

where f_1, f_2 and f_3 are functions of $\psi(u)$ defined by

$$
\begin{aligned}
f_1(\psi) &= g[\sin\alpha+\beta\sin(\alpha-2\psi)], \\
f_2(\psi) &= 1+\beta\cos 2\psi, \\
f_3(\psi) &= \beta+\cos 2\psi,
\end{aligned}
\tag{45}
$$

and $\gamma = 4g\beta^2\cos\alpha$. Upon checking the consistency of (44), we find

$$
\begin{aligned}
&\left[f_2^2+\gamma(h-y)\left(\frac{d\psi}{du}\right)^2\right]\left[(f_1' f_2+f_1 f_2')\frac{d\psi}{du}\frac{\partial u}{\partial y}+2\beta\frac{f_1 f_2}{f_3}\frac{d^2\psi}{du^2}\frac{\partial u}{\partial t}\right. \\
&\left.+2\beta\left(\frac{f_1' f_2+f_1 f_2'}{f_3}-\frac{f_1 f_2 f_3'}{f_3^2}\right)\left(\frac{d\psi}{du}\frac{\partial u}{\partial t}\right)^2\right] \\
&= f_1 f_2\left[-\gamma\left(\frac{d\psi}{du}\right)^2+2\left(f_2 f_2'+\gamma(h-y)\frac{d^2\psi}{du^2}\right)\frac{d\psi}{du}\frac{\partial u}{\partial y}\right. \\
&\left.+\frac{4\beta}{f_3}\left(f_2 f_2'+\gamma(h-y)\frac{d^2\psi}{du^2}\right)\left(\frac{d\psi}{du}\right)^2\frac{\partial\psi}{\partial t}\right],
\end{aligned}
\tag{46}
$$

where primes denote differentiation with respect to ψ. Now, substituting (44) into (46) gives simply

$$\frac{d^2\psi}{du^2} + \left[\frac{\gamma}{2\beta}\frac{f_3}{f_1 f_2} - \frac{f_3'}{f_3}\right]\left(\frac{d\psi}{du}\right)^2 = 0, \tag{47}$$

which upon dividing throughout by $d\psi/du$ and integrating gives

$$\ln\left(\frac{d\psi}{du}\right) + \frac{\gamma}{2\beta}\int \frac{f_3}{f_1 f_2} d\psi - \ln f_3 = \ln c_1, \tag{48}$$

where c_1 is a constant of integration. Thus, we need to evaluate the integral

$$I_1 = \frac{\gamma}{2\beta}\int \frac{f_3}{f_1 f_2} d\psi, \tag{49}$$

so that from (45) we find

$$I_1 = \int \frac{2\beta(\beta + \cos 2\psi)\cos\alpha}{(1+\beta\cos 2\psi)[\sin\alpha(1+\beta\cos 2\psi) - \beta\cos\alpha\sin 2\psi]} d\psi. \tag{50}$$

Upon making the substitution (16) we find that (50) becomes simply

$$I_1 = \int \frac{\cos\alpha}{\sin\alpha - T\cos\alpha}\, dT, \tag{51}$$

which clearly can be integrated to give

$$I_1 = -\ln(\sin\alpha - T\cos\alpha). \tag{52}$$

Thus, from (16) and (52) we find that (48) becomes

$$\frac{(1+\beta\cos 2\psi)}{(\beta+\cos 2\psi)[\sin\alpha + \beta\sin(\alpha - 2\psi)]}\frac{d\psi}{du} = c_1, \tag{53}$$

which is a separable equation that can also be integrated to give

$$\int \frac{(1+\beta\cos 2\psi)}{(\beta+\cos 2\psi)[\sin\alpha + \beta\sin(\alpha - 2\psi)]} d\psi = c_1 u + c_2, \tag{54}$$

where c_2 is a further constant of integration. Thus, we now need to evaluate the integral

$$I_2 = \int \frac{(1+\beta\cos 2\psi)}{(\beta+\cos 2\psi)[\sin\alpha + \beta\sin(\alpha - 2\psi)]} d\psi, \tag{55}$$

and the details for this are provided in the Appendix. From the Appendix we find that (54) becomes

$$\frac{(1-\beta^2)^{1/2}\sin\alpha}{(\sin^2\alpha-\beta^2)}\ \mathrm{arctanh}\left\{\left(\frac{1-\beta}{1+\beta}\right)^{1/2}\tan\psi\right\}$$

$$+\frac{\beta\cos\alpha}{2(\sin^2\alpha-\beta^2)}\ln\left\{\frac{\sin\alpha+\beta\sin(\alpha-2\psi)}{\beta+\cos 2\psi}\right\}=c_1u+c_2, \tag{56}$$

which is an equation for u in terms of ψ. Now, from (44), (45) and (53) we find

$$\frac{\partial\psi}{\partial t}=\frac{gc_1(\beta+\cos 2\psi)(1+\beta\cos 2\psi)^2[\sin\alpha+\beta\sin(\alpha-2\psi)]^2}{\{(1+\beta\cos 2\psi)^4+\gamma_1(h-y)(\beta+\cos 2\psi)^2[\sin\alpha+\beta\sin(\alpha-2\psi)]^2\}},$$

$$\frac{\partial\psi}{\partial y}=-\frac{2g\beta c_1^2(\beta+\cos 2\psi)(1+\beta\cos 2\psi)[\sin\alpha+\beta\sin(\alpha-2\psi)]^3}{\{(1+\beta\cos 2\psi)^4+\gamma_1(h-y)(\beta+\cos 2\psi)^2[\sin\alpha+\beta\sin(\alpha-2\psi)]^2\}}, \tag{57}$$

where $\gamma_1=c_1^2\gamma=4g\beta^2c_1^2\cos\alpha$. Now, from $(57)_1$ we find

$$\int\frac{(1+\beta\cos 2\psi)^2}{(\beta+\cos 2\psi)[\sin\alpha+\beta\sin(\alpha-2\psi)]^2}d\psi$$

$$+\gamma_1(h-y)\int\frac{\beta+\cos 2\psi}{(1+\beta\cos 2\psi)^2}d\psi=gc_1t+k_1(y), \tag{58}$$

where $k_1(y)$ is an arbitrary function of integration. Thus, we need to consider the two integrals

$$I_3=\int\frac{\beta+\cos 2\psi}{(1+\beta\cos 2\psi)^2}d\psi,$$

$$I_4=\int\frac{(1+\beta\cos 2\psi)^2}{(\beta+\cos 2\psi)[\sin\alpha+\beta\sin(\alpha-2\psi)]^2}d\psi, \tag{59}$$

which upon making the substitution $T=\tan\psi$, we find that (59) becomes

$$I_3=\int\frac{(1+\beta)-(1-\beta)T^2}{[(1+\beta)+(1-\beta)T^2]^2}dT,$$

$$I_4=\int\frac{[(1+\beta)+(1-\beta)T^2]^2}{[(1+\beta)-(1-\beta)T^2]\left\{[(1+\beta)+(1-\beta)T^2]\sin\alpha-2\beta T\cos\alpha\right\}^2}dT. \tag{60}$$

Now, $(60)_1$ integrates to give

$$I_3 = \frac{T}{(1+\beta)+(1-\beta)T^2}, \tag{61}$$

and using MAPLE we find that $(60)_2$ becomes

$$\begin{aligned} I_4 = {} & \frac{\beta(1-\beta^2)\sin\alpha\cos\alpha}{(\beta^2-\sin^2\alpha)^2}\ln\left[\frac{\{(1+\beta)+(1-\beta)T^2\}\sin\alpha - 2\beta T\cos\alpha}{(1+\beta)-(1-\beta)T^2}\right] \\ & + \frac{(1-\beta^2)^{1/2}[(1-\beta^2)\sin^2\alpha+\beta^2\cos^2\alpha]}{(\beta^2-\sin^2\alpha)^2}\operatorname{arctanh}\left[T\left(\frac{1-\beta}{1+\beta}\right)^{1/2}\right] \\ & + \frac{\beta^2 T\cos^2\alpha}{\sin\alpha(\beta^2-\sin^2\alpha)[\{(1+\beta)+(1-\beta)T^2\}\sin\alpha-2\beta T\cos\alpha]}. \end{aligned} \tag{62}$$

Thus, from (58), (61) and (62) we find that

$$\frac{\gamma_1(h-y)\tan\psi}{[(1+\beta)+(1-\beta)\tan^2\psi]} + I_4 = gc_1 t + k_1(y), \tag{63}$$

which is a transcendental equation for the determination of ψ in terms of t and y, where I_4 is given by (62) and $T=\tan\psi$. We note from $(57)_2$ and (58) that $k_1(y)$ is determined to take the form

$$k_1(y) = -2g\beta c_1^2 y\sin\alpha + c_3, \tag{64}$$

where c_3 is a constant of integration.

Now, we note that (A2) gives rise to a special case when $\beta=\sin\alpha$. Thus, in this case we find from (A1) that

$$I_2 = \frac{1}{\beta}\int\frac{[(1+\beta)+(1-\beta)T^2]}{[(1+\beta)-(1-\beta)T^2][(1+\beta)+(1-\beta)T^2-2(1-\beta^2)^{1/2}T]}dT, \tag{65}$$

which can be rewritten in the form

$$I_2 = \frac{1}{\beta}\int\frac{a^2+b^2T^2}{(a+bT)(a-bT)^3}dT, \tag{66}$$

where $a = (1+\beta)^{1/2}$ and $b = (1-\beta)^{1/2}$. Upon using partial fractions and after integrating, we find that (66) becomes

$$I_2 = \frac{1}{4ab\beta} \ln \left[\frac{a+bT}{a-bT} \right] - \frac{1}{2b\beta} \frac{1}{a-bT} + \frac{a}{2b\beta} \frac{1}{[a-bT]^2} + c_4, \tag{67}$$

which can be simplified to give

$$\begin{aligned} I_2 = \frac{1}{2\beta(1-\beta^2)^{1/2}} \operatorname{arctanh} \left[\left(\frac{1-\beta}{1+\beta} \right)^{1/2} \tan\psi \right] \\ + \frac{\tan\psi}{2\beta[(1+\beta)^{1/2} - (1-\beta)^{1/2}\tan\psi]^2} + c_4, \end{aligned} \tag{68}$$

where c_4 is a constant of integration and recalling that $T = \tan\psi$.

7 Separable solution for special case of $\beta = 1$

In this section, we look for a separable solution for the special case when the angle of internal friction is equal to ninety degrees. Thus, we consider the linear partial differential equations (15) and assume

$$T(y,t) = A(y)B(t), \tag{69}$$

where $A(y)$ and $B(t)$ are functions to be determined. Upon substituting (69) into $(15)_2$ we find

$$\frac{1}{B(t)} \frac{d^2B(t)}{dt^2} = -\frac{1}{A(y)} \frac{d^2}{dy^2} \left\{ g(h-y)A(y)\cos\alpha \right\} = \lambda_1, \tag{70}$$

where λ_1 is some constant. Hence, (70) gives rise to the two ordinary differential equations

$$\begin{aligned} \frac{d^2B(t)}{dt^2} - \lambda_1 B(t) = 0, \\ \frac{d^2}{dy^2} \left\{ g(h-y)A(y)\cos\alpha \right\} + \lambda_1 A(y) = 0. \end{aligned} \tag{71}$$

Now, in order to solve (71) we need to make an assumption about the sign of λ_1. Clearly, we have three possible solutions for $B(t)$, namely

$$\begin{aligned} B(t) &= c_1 \cos kt + c_2 \sin kt, && \text{if } \lambda_1 = -k^2, \\ B(t) &= c_1 t + c_2, && \text{if } \lambda_1 = 0, \qquad (72) \\ B(t) &= c_1 e^{kt} + c_2 e^{-kt}, && \text{if } \lambda_1 = k^2, \end{aligned}$$

where c_1 and c_2 are constants of integration and k is some constant. For $A(y)$ we have

$$(h-y)\frac{d^2A}{dy^2} - 2\frac{dA}{dy} + \frac{\lambda_1}{g\cos\alpha}A(y) = 0, \tag{73}$$

which on making the transformation $\zeta(y) = (h-y)^{1/2}A(y)$, becomes

$$(h-y)\frac{d^2\zeta}{dy^2} - \frac{d\zeta}{dy} + \left[\frac{\lambda_1}{g\cos\alpha} - \frac{1}{4(h-y)}\right]\zeta(y) = 0. \tag{74}$$

On making the further transformation $\eta = (h-y)^{1/2}$, we find that (74) gives rise to the equation

$$\eta^2\frac{d^2\zeta}{d\eta^2} + \eta\frac{d\zeta}{d\eta} + \left[\left(\frac{2\lambda_1^{1/2}\eta}{(g\cos\alpha)^{1/2}}\right)^2 - 1\right]\zeta(\eta) = 0, \tag{75}$$

which is Bessel's equation and assuming $\lambda_1 > 0$, has the solution

$$\zeta(\eta) = c_3 J_1\left\{\left(\frac{4\lambda_1}{g\cos\alpha}\right)^{1/2}\eta\right\} + c_4 Y_1\left\{\left(\frac{4\lambda_1}{g\cos\alpha}\right)^{1/2}\eta\right\}, \tag{76}$$

where J_1 and Y_1 are the usual Bessel functions of the first and second kind of order one respectively, and c_3 and c_4 are constants. Thus, we find that (76) becomes

$$A(y) = \frac{c_3}{(h-y)^{1/2}}J_1\left\{\left(\frac{4\lambda_1(h-y)}{g\cos\alpha}\right)^{1/2}\right\} + \frac{c_4}{(h-y)^{1/2}}Y_1\left\{\left(\frac{4\lambda_1(h-y)}{g\cos\alpha}\right)^{1/2}\right\}. \tag{77}$$

Therefore, the solution for $T(y,t)$ is given by (69), (72) and (77), noting that here we assume λ_1 is positive, since we are looking for hot-spot solutions. Thus, altogether we have the solution

$$T(y,t) = \frac{\left[c_1 e^{kt} + c_2 e^{-kt}\right]}{[h-y]^{1/2}} \left\{ c_3 J_1 \left[\gamma (h-y)^{1/2}\right] + c_4 Y_1 \left[\gamma (h-y)^{1/2}\right] \right\}, \tag{78}$$

where γ is defined by

$$\gamma = \left(\frac{4k^2}{g\cos\alpha} \right)^{1/2}. \tag{79}$$

Now, in order to determine the solution for $u(y,t)$, we can either solve $(15)_1$ directly, or we can utilize the above solution for $T(y,t)$ and solve either $(14)_1$ or $(14)_2$. Here, we solve $(14)_1$ which can be rewritten in the form

$$\frac{\partial u}{\partial t} = g\sin\alpha + g\cos\alpha \frac{\partial}{\partial y}\left\{(h-y)T\right\}, \tag{80}$$

and from (78) we find

$$\begin{aligned} &\frac{\partial}{\partial y}\left\{(h-y)T\right\} \\ &= -\frac{k\left[c_1 e^{kt} + c_2 e^{-kt}\right]}{[g\cos\alpha]^{1/2}} \left\{ c_3 J_0 \left[\gamma (h-y)^{1/2}\right] + c_4 Y_0 \left[\gamma (h-y)^{1/2}\right] \right\}, \end{aligned} \tag{81}$$

where J_0 and Y_0 are Bessel functions of the first and second kind of order zero respectively. Thus, from (81) we can integrate (80) with respect to t, to obtain

$$\begin{aligned} &u(y,t) = gt\sin\alpha + c_5 \\ &- [g\cos\alpha]^{1/2}\left[c_1 e^{kt} - c_2 e^{-kt}\right]\left\{ c_3 J_0 \left[\gamma (h-y)^{1/2}\right] + c_4 Y_0 \left[\gamma (h-y)^{1/2}\right] \right\}, \end{aligned} \tag{82}$$

where c_5 is at most a function of y, but here is taken to be constant.

8 Numerical results and conclusions

All the solutions obtained involve particular functional forms, and therefore we are not at liberty to arbitrarily impose initial and boundary conditions. We choose to illustrate these solutions either for an initial or boundary condition involving zero velocity at $y = -d$, where d denotes the layer depth which in the figures is taken to be unity. For the solution presented in Section 4, we assume a perfectly rough inclined plane so that the velocity at $y = -d$ is taken to be zero and we show the variation for the case of an assigned surface velocity which has the form $u(0, t) = U_1 t$. Figure 2 shows the variation of $u(y, t)/t$ and $\psi(y, t)$ for $\xi = -y/t^2$ increasing for the values $\beta = 1/2$ and $U_1 = 0.88$. Figure 3 shows the same variation but for the value $\beta = 1$ and three values of U_1. We note that the exact analytical solution (34) coincides with that obtained from the numerical solution of (21) using MAPLE.

For the solution presented in Section 5, the constant c_2 in (42) is chosen such that $u(-d, 0) = 0$ and Figure 4 shows the variation with time of the velocity of the base layer and the stress angle as determined by (42) with $c_1 = 1$ and $u_0 = -1/3$. In this case we observe that the velocity $u(-1, t)$ becomes infinite at the finite time $t_0 = 4.74$, which generally is determined from

$$t_0 = \frac{\tan\phi}{u_0}\left\{2c_1\cos\phi - \cosh^{-1}\left(\tan\phi\sec\phi\right)\right\}, \tag{83}$$

noting that we require $u_0 < 0$. This particular solution indicates that the non-dilatant double-shearing theory of granular flow admits the possibility of predicting infinite velocities in finite time.

For the solution presented in Section 6, Figure 5 shows the variation with time in the velocity of the bottom layer and the stress angle as determined by (56) and (63) where I_4 is defined by (62) and $k_1(y)$ is given by (64). The constant c_2 is chosen so that initially the base velocity $u(-1, 0)$ is zero, whereas c_1 and c_3 are taken simply to be unity.

For the solution presented in Section 7, Figure 6 shows the variation with time in the velocity of the bottom layer and the stress angle as determined by (78) and (82). The constant c_5 is chosen so that initially the base velocity $u(-1, 0)$ is zero and the constant c_4 is assumed zero to ensure that the stress angle ψ is well-defined at $y = 0$. The constants c_2 and c_3 are taken to be unity, whereas c_1 is assumed to be minus unity to ensure that the velocity remains positive.

We have extended the simple non-dilatant double-shearing dynamic model given in Hill and Spencer [6] for horizontal flow, to the case of chute flow down an inclined plane. We have obtained generalizations of the major solutions presented in [6] and special cases of these are depicted graphically. One solution exhibits "hot-spot" behaviour indicating the possibility of obtaining extremely large velocities in finite time. For the special case of the angle of internal friction equal to 90° the model gives rise to linear partial differential equations for which we have determined a separable solution.

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References

[1] Hill, J. M.: Similarity 'hot-spot' solutions for a hypoplastic granular material. Proc. R. Soc. Lond. A. **456**, 2653 - 2671 (2000).

[2] Hill, J. M.: Dynamical uniaxial and radial flow for hypoplastic granular materials, In: International Workshop on Bifurcation and Localisation in Geomaterials, Perth, Balkema, (2001).

[3] Hill, J. M., Williams, K. A.: Dynamical uniaxial compaction of a hypoplastic granular material, Mechanics of Materials. **32**, 679 - 691 (2000).

[4] Spencer, A. J. M.: A theory of the kinematics of ideal soils under plane strain conditions, J. Mech. Phy. Solids. **12**, 337 - 351 (1964).

[5] Spencer, A. J. M.: Deformation of ideal granular materials, In: Mechanics of Solids, H. G. Hopkins and M. J. Sewell (eds), Oxford, (Pergamon Press, 1982), 607 - 652.

[6] Hill, J. M., Spencer, A. J. M.: Some dynamic shear flow problems for granular materials, Q. Jl. appl. Math. **52**, 253 - 267 (1999).

[7] Spencer, A. J. M.: Dynamic analysis of shear flow of granular materials, In: Advances in Engineering Plasticity and its Applications - AEPA '96, T. Abe and T. Tsuta (eds), Oxford, (Pergamon Press, 1996), 3 - 7.

[8] Spencer, A. J. M.: Remarks on coaxiality in fully developed gravity flows of dry granular materials, In: IUTAM Symposium on Mechanics of Granular and Porous Materials, N. A. Fleck and A. C. F. Cocks (eds), Dordrecht, (Kluwer, 1997), 227 - 238.

[9] Spencer, A. J. M.: A model for granular material mechanics combining double-shearing and critical state concepts, In: Proceedings of the 9th International Symposium on Continuum Models and Discrete Systems, E. Inan and K. Z. Markov (eds), Singapore, (World Scientific, 1998), 803 - 810.

[10] Spencer, A. J. M., Bradley, N. J.: Gravity flow of granular materials in converging wedges and cones, In: Continuum Models and Discrete Systems: Proceedings of the 8th International Symposium, K. Z. Markov (ed.) Singapore, (World Scientific, 1996), 581 - 590.

[11] Spencer, A. J. M., Hill, J. M.: Non-dilatant double-shearing theory applied to granular funnel-flow in hoppers, J. Eng. Math. **41**, 55 - 73 (2001).

[12] Zheng, X. M., Hill, J. M.: Molecular dynamics modelling of granular chute flow: density and velocity profiles, Powder Technology. **86**, 219 - 227 (1996).

[13] Hill, J. M., Zheng, X. M.: Dilatant double shearing theory applied to granular chute flow, Acta Mechanica. **118**, 97 - 108 (1996).

[14] Hutter, K.: Two- and three dimensional evolution of granular avalanche flow - theory and experiments revisited, Acta Mechanica. **[Suppl] 1**, 167 - 181 (1991).

[15] Savage, S. B., Hutter, K.: The motion of a finite mass of granular material down a rough incline, J. Fluid Mech. **199**, 177 - 215 (1989).

[16] Savage, S. B., Hutter, K.: The dynamics of avalanches of granular materials from initiation to runout. Part I: Analysis, Acta Mechanica. **86**, 201 - 223 (1991).

[17] Hutter, K., Koch, T.: Motion of a granular avalanche in an exponentially curved chute: experiments and theoretical predictions, Phil. Trans. R. Soc. Lond. A. **334**, 93 - 138 (1991).

Appendix: Details of the evaluation of integral (55).

In this Appendix we give the details for the evaluation of the integral I_2 defined by (55). Upon making the substitution $T = \tan\psi$, we find that (55) becomes

$$I_2 = \int \frac{[(1+\beta)+(1-\beta)T^2]}{[(1+\beta)-(1-\beta)T^2][\sin\alpha\{(1+\beta)+(1-\beta)T^2\}-2\beta T\cos\alpha]}dT, \tag{A1}$$

and then using partial fractions we find that (A1) can be expressed in the form

$$\begin{aligned} I_2 = {} & \frac{1}{(\sin^2\alpha-\beta^2)}\int \frac{(1-\beta^2)\sin\alpha+\beta(1-\beta)T\cos\alpha}{[(1+\beta)-(1-\beta)T^2]}dT \\ & +\frac{1}{(\sin^2\alpha-\beta^2)}\int \frac{-\beta^2\cos^2\alpha+\beta(1-\beta)T\sin\alpha\cos\alpha}{[\sin\alpha\{(1+\beta)+(1-\beta)T^2\}-2\beta T\cos\alpha]}dT, \end{aligned} \tag{A2}$$

and therefore, we need to consider the three integrals

$$\begin{aligned} I_5 &= \frac{(1-\beta^2)\sin\alpha}{(\sin^2\alpha-\beta^2)}\int \frac{dT}{[(1+\beta)-(1-\beta)T^2]}, \\ I_6 &= \frac{\beta(1-\beta)\cos\alpha}{(\sin^2\alpha-\beta^2)}\int \frac{TdT}{[(1+\beta)-(1-\beta)T^2]}, \\ I_7 &= -\frac{\beta\cos\alpha}{(\sin^2\alpha-\beta^2)}\int \frac{\beta\cos\alpha-(1-\beta)T\sin\alpha}{[\sin\alpha\{(1+\beta)+(1-\beta)T^2\}-2\beta T\cos\alpha]}dT, \end{aligned} \tag{A3}$$

which evaluate respectively to give

$$I_5 = \frac{\sin\alpha(1-\beta^2)^{1/2}}{(\sin^2\alpha-\beta^2)}\text{arctanh}\left\{T\left(\frac{1-\beta}{1+\beta}\right)^{1/2}\right\},$$

$$I_6 = -\frac{\beta\cos\alpha}{2(\sin^2\alpha-\beta^2)}\ln[(1+\beta)-(1-\beta)T^2], \tag{A4}$$

$$I_7 = \frac{\beta\cos\alpha}{2(\sin^2\alpha-\beta^2)}\ln[(1+\beta)\sin\alpha-2\beta T\cos\alpha+(1-\beta)T^2\sin\alpha].$$

Therefore, (A2) becomes

$$\begin{aligned} I_2 = {} & \frac{\sin\alpha(1-\beta^2)^{1/2}}{(\sin^2\alpha-\beta^2)}\text{arctanh}\left\{T\left(\frac{1-\beta}{1+\beta}\right)^{1/2}\right\} \\ & +\frac{\beta\cos\alpha}{2(\sin^2\alpha-\beta^2)}\ln\left\{\frac{(1+\beta)\sin\alpha-2\beta T\cos\alpha+(1-\beta)T^2\sin\alpha}{(1+\beta)-(1-\beta)T^2}\right\}. \end{aligned} \tag{A5}$$

Figure Captions

Figure 1. Coordinates for two-dimensional dynamical granular flow down a plane inclined slope.

Figure 2. Variation of $u(y,t)/t$ and $\psi(y,t)$ as functions of $\xi = -y/t^2$ for the similarity solution (18) as determined by the numerical solution of (21) with $\beta = 1/2$.

Figure 3. Variation of $u(y,t)/t$ and $\psi(y,t)$ as functions of $\xi = -y/t^2$ for the similarity solution (18) as determined by the analytical solution (34) for $\beta = 1$ and three values of $u(0,t)/t$.

Figure 4. Variation of $u(y,t)$ and $\psi(t)$ along the chute base $y = -1$ according to (42), where hot-spot behaviour occurs when $t = 4.74$.

Figure 5. Variation of $u(y,t)$ and $\psi(u)$ along the chute base $y = -1$ according to (56) and (63).

Figure 6. Variation of the separable solutions for $u(y,t)$ and $\psi(y,t)$ as defined by (78) and (82) and valid only for $\beta = 1$.

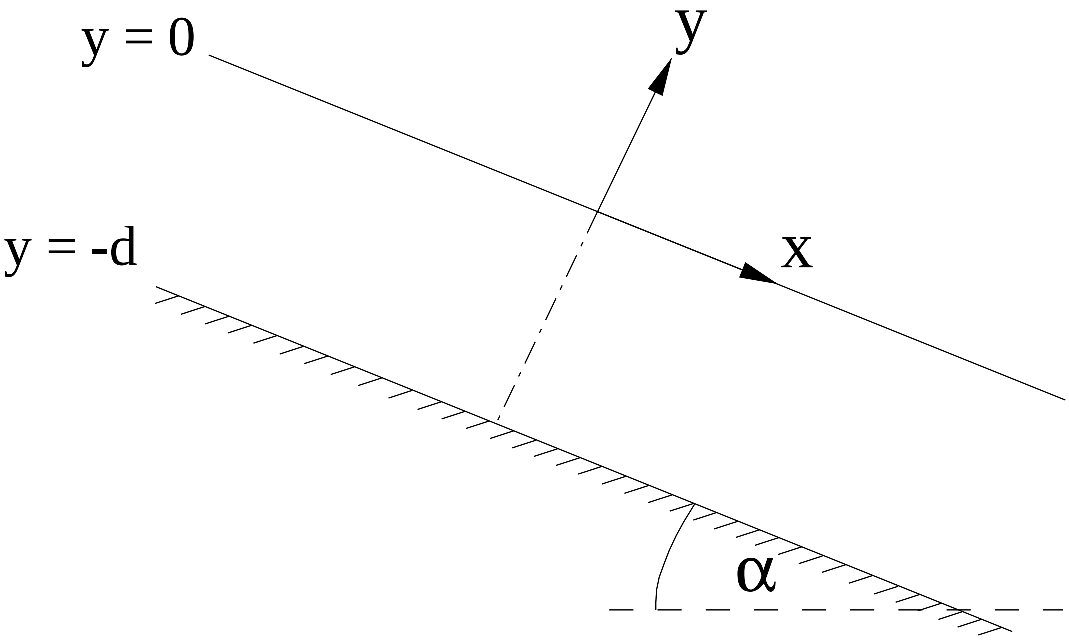


Figure 1.

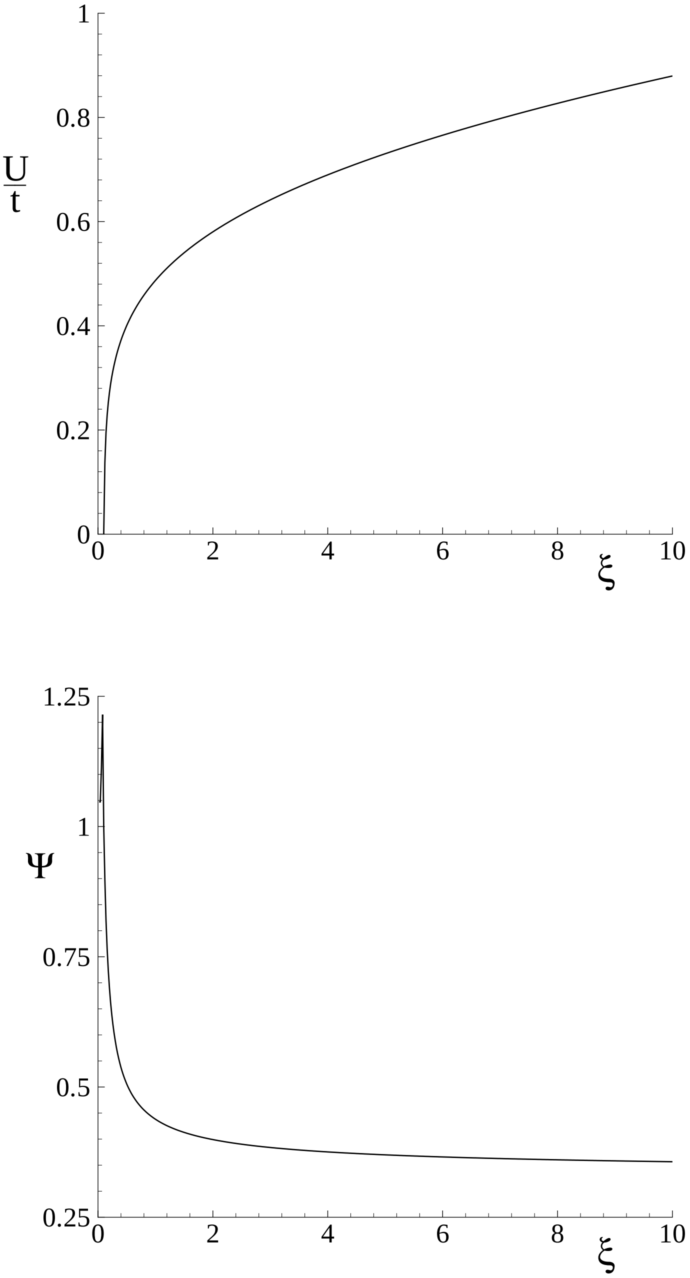
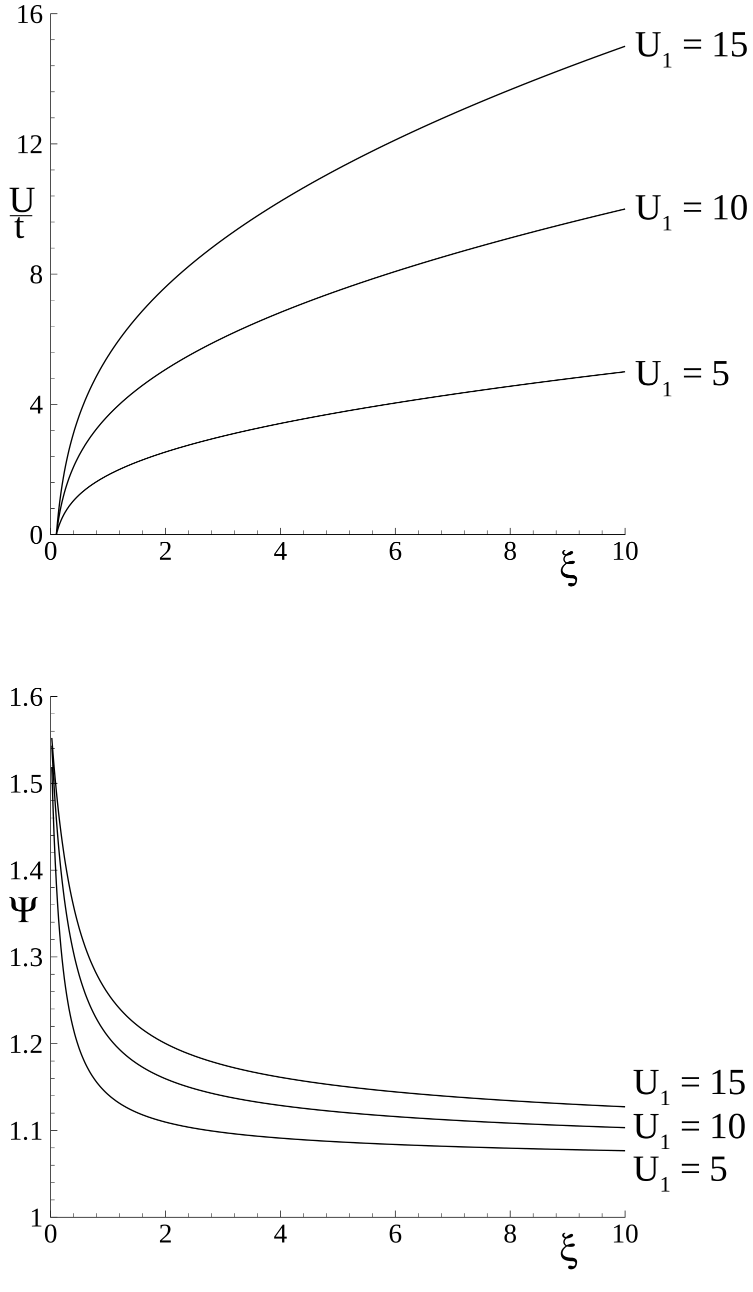


Figure 2.


16
12
8
4
0
$\frac{U}{t}$
0
2
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6
8
10
ξ
$U_1 = 15$
$U_1 = 10$
$U_1 = 5$
1.6
1.5
1.4
1.3
1.2
1.1
1
Ψ
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ξ
$U_1 = 15$
$U_1 = 10$
$U_1 = 5$

Figure 3.

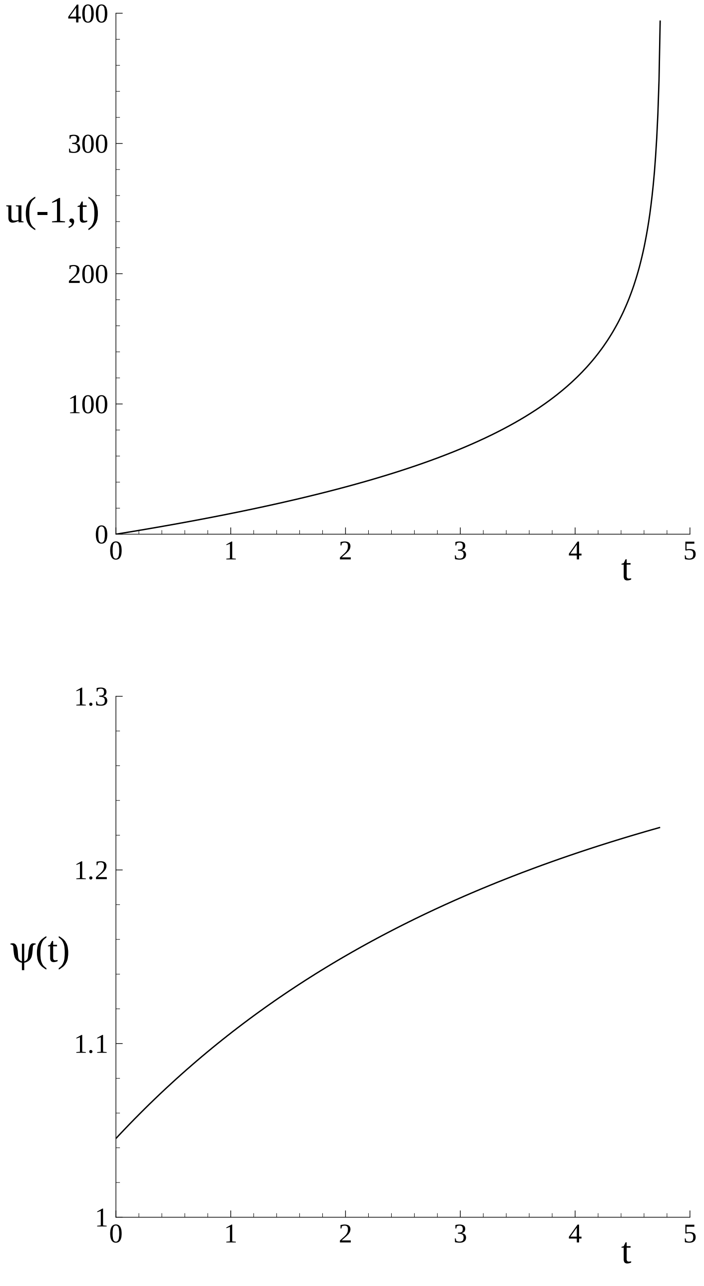
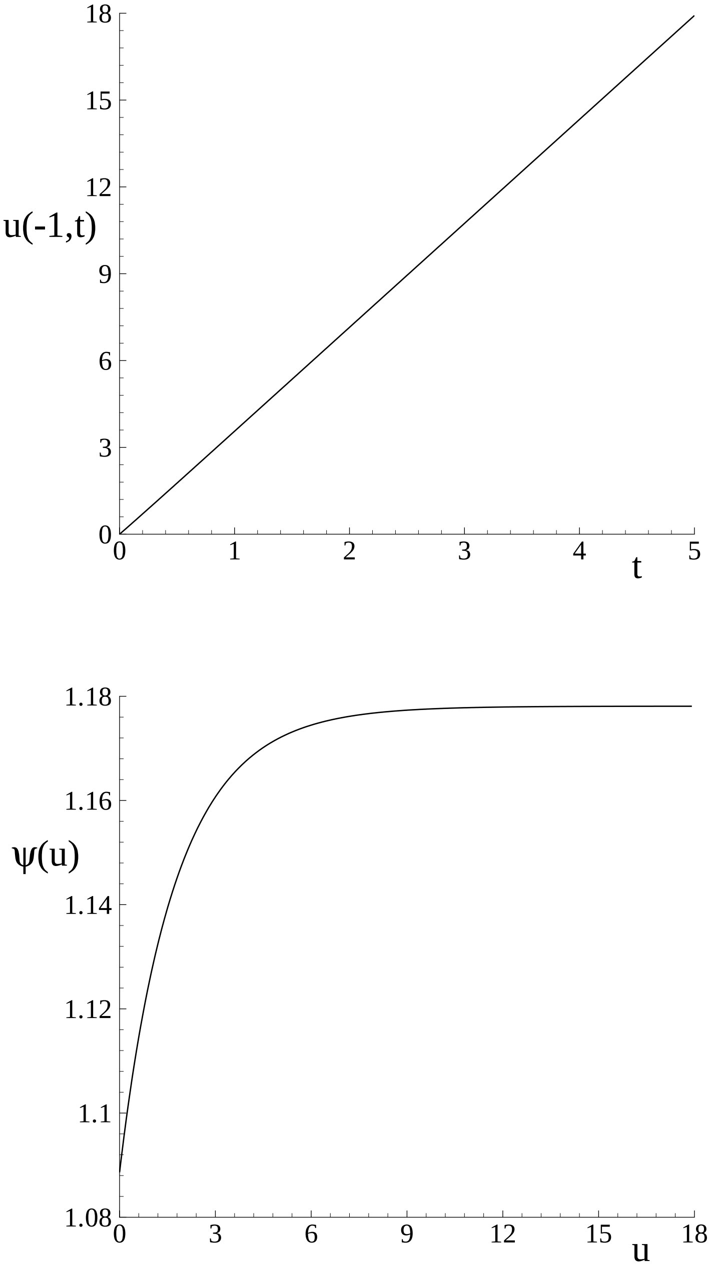


Figure 4.


u(-1,t)
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9
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0
0
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2
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t
ψ(u)
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0
3
6
9
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15
18
u

Figure 5.

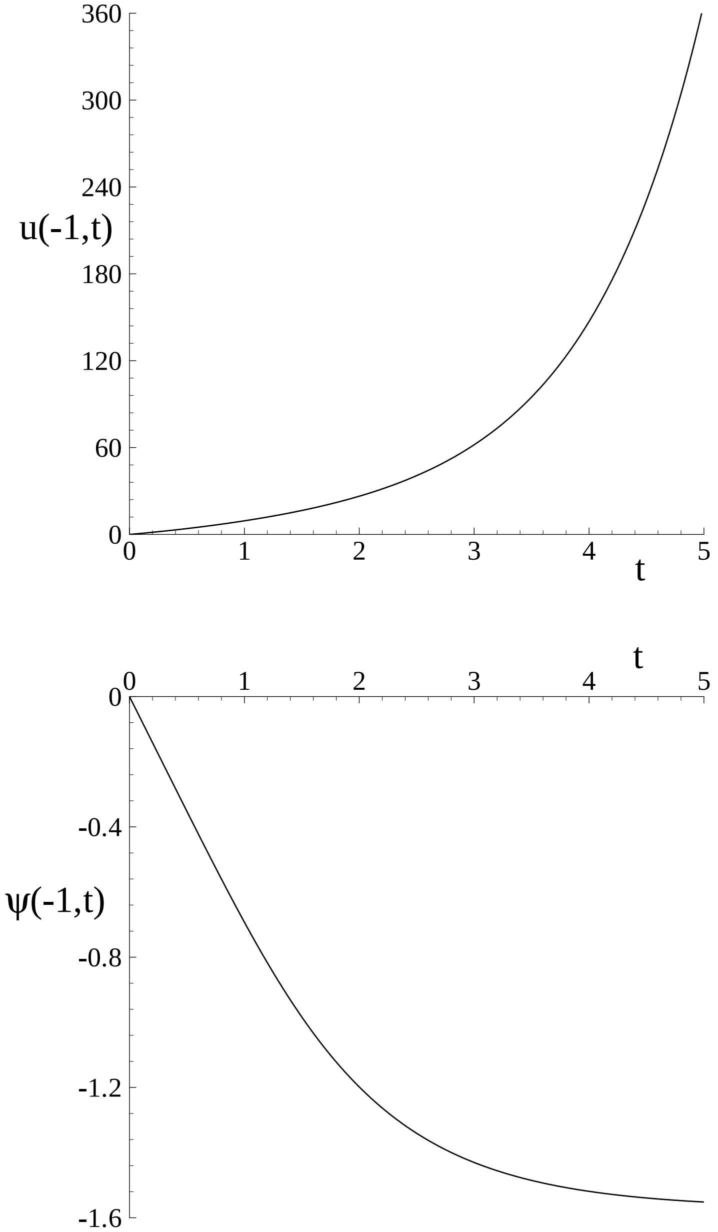

Figure 6.