CUNTZ–KRIEGER ALGEBRAS ASSOCIATED TO DIRECTED GRAPHS

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1. The universal C^*–algebra of a graph

A *directed graph* E consists of countable sets E^0 of vertices, E^1 of edges, and maps $r, s : E^1 \to E^0$ giving the direction of each edge. The graph E is *row–finite* if for every $v \in E^0$, the set $s^{-1}(v) \subseteq E^1$ is finite; if in addition $r^{-1}(v)$ is finite for all $v \in E^0$, then E is *locally finite*. We write E^* for the finite path space of E, and E^∞ for the infinite path space. The range and source maps r, s extend naturally to E^*.

A path $\alpha \in E^*$ of length $|\alpha| > 0$ is a *loop based at* v, or a *return path for* v if $s(\alpha) = v = r(\alpha)$. A loop is called *simple* if the vertices $\{r(\alpha_i) : 1 \le i \le |\alpha|\}$ are distinct. A vertex $v \in E^0$ which emits no edges is called a *sink*.

If E if a row–finite directed graph, a *Cuntz–Krieger E–family* consists of a set $\{P_v : v \in E^0\}$ of mutually orthogonal projections and a set $\{S_e : e \in E^1\}$ of partial isometries satisfying

$$S_e^* S_e = P_{r(e)} \text{ for } e \in E^1, \text{ and } P_v = \sum_{\{e : s(e) = v\}} S_e S_e^* \text{ for } v \in s(E^1). \qquad (1)$$

The *edge matrix* of E is the $E^1 \times E^1$ matrix defined by

$$A_E(e, f) := \begin{cases} 1 \text{ if } r(e) = s(f), \\ 0 \text{ otherwise.} \end{cases}$$

One may similarly define a positive integer valued *vertex matrix* B_E indexed by E^0. A Cuntz–Krieger E-family $\{P_v, S_e\}$ satisfies

$$S_e^* S_e = \sum_{\{f : s(f) = r(e)\}} S_f S_f^* = \sum_{f \in E} A_E(e, f) S_f S_f^*$$

for every e such that $A_E(e, \cdot)$ has nonzero entries. Thus if E has no sinks, $\{S_e : e \in E^1\}$ is a Cuntz-Krieger A_E-family in the sense of [6]. (Note that here the projections $\{P_v\}$ are the *initial* projections of the partial isometries S_e with $r(e) = v$, and not the range projections as in [6]). The point of our new definition is that the projection P_v can be nonzero even if there are no edges coming out of v.

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Not every $\{0,1\}$-matrix is the edge matrix of a directed graph, but for any $V \times V$ matrix B with entries in $\{0,1\}$, we can construct a graph E with vertex set $E^0 = V$ by joining v to w iff $B(v,w) = 1$, and then there is a natural bijection between Cuntz–Krieger B-families associated to B and those associated to the corresponding edge matrix A_E (see [18,Proposition 4.1] also [11,Theorem 2.3]).

Our first result is to show that for a directed graph E there is a C^*–algebra which is universal for families of projections and partial isometries satisfying the relations (1).

1.1 Theorem. *Let E be a directed graph. Then there is a C^*-algebra B generated by a Cuntz-Krieger E-family $\{s_e, p_v\}$ of non-zero elements such that, for every Cuntz-Krieger E-family $\{S_e, P_v\}$ of partial isometries on $\mathcal{H}$, there is a representation π of B on $\mathcal{H}$ such that $\pi(s_e) = S_e$ and $\pi(p_v) = P_v$ for all $e \in E^1, v \in E^0$.*

Proof. See [2,Theorem 2.1] or [15,Theorem 1.2].

The triple (B, p_v, s_e) is unique up to isomorphism, and hence we write $C^*(E)$ for B. The C^*-algebra $C^*(E)$ is unital if and only if E^0 is finite. If E has no sinks, then the projections p_v are redundant, and [16,Theorem 4.2] implies that $C^*(E) \cong C^*(\mathcal{G}_E)$ where $\mathcal{G}_E$ is the path groupoid of E:

For a row finite directed graph E with no sinks we say that $x, y \in E^\infty$ are shift tail equivalent with lag k and write $x \sim_k y$ if there are $k \in \mathbb{Z}$ and $N \in \mathbb{N}$ such that $x_i = y_{i+k}$ for $i \geq N$. The path groupoid $\mathcal{G}_E$ is then the set of triples (x,k,y) such that $x \sim_k y$, with multiplication $(x,k,y)(y,\ell,z) = (x,k+\ell,z)$ and inverse $(x,k,y)^{-1} = (y,-k,x)$. There is a Hausdorff topology on $\mathcal{G}_E$ which makes it an r–discrete, second countable, locally compact groupoid. The topology on $\mathcal{G}_E$ is not the product topology, however the unit space $\mathcal{G}_E^0$ is naturally homeomorphic to E^∞ (see [16,Proposition 2.6]). For a good reference on groupoids and their C^*–algebras see [24].

1.2 Theorem. *Let E be locally finite with no sinks then $\mathcal{G}_E$ is amenable.*

Proof. In [16,§5] we show that $\mathcal{G}_E$ is isomorphic to a reduction of a semidirect product of an AF groupoid by an action of $\mathbb{Z}$.

In [6] Cuntz and Krieger identify a fullness condition (I) on A_E which guarantees the map π in Theorem 1.1 is an isomorphism provided that the S_e are all nonzero. Translating into graphical language E satisfies (I) if every vertex $v \in E^0$ connects to a vertex w which has two distinct return paths. In [15] we introduced an analogue of (L) of (I): E satisfies condition (L) if every loop in E has an exit.

1.3 Lemma. *If E^0 is finite then (L) is equivalent to (I), on the other hand if E^0 is infinite then (L) is weaker than (I).*

In [15,Theorem 3.7] we show the following:

1.4 Theorem. *Let E be a locally finite directed graph which has no sinks and satisfies condition* (L). *Suppose B is a C^*-algebra generated by a Cuntz-Krieger E-family $\{S_e : e \in E^1\}$ with all S_e non-zero. Then there is an isomorphism π of $C^*(E)$ onto B such that $\pi(s_e) = S_e$.*

Indeed, the converse is also true.

2. The ideal structure of $C^*(E)$

We say that the directed graph E *satisfies condition* (K) if there are no vertices $v \in E^0$ which have precisely one simple loop based at v. Condition (K) is the analogue of Cuntz's Condition (II), which is not appropriate for graphs with infinitely many vertices. (In the notation of [5], the set Γ_A of "equivalence classes" could easily be empty.) Condition (K) is logically independent of the analogue (J) of (I) used in [20].

2.1 Lemma. *If E^0 is finite then E satisfies* (K) *if and only if the associated edge matrix A_E satisfies* (II).

We call a subset H of E^0 *hereditary* if $v \in H$ and there is a path from v to w then $w \in H$, and *saturated* if

$$\big[r(e) \in H \text{ for all } e \in E \text{ with } s(e) = v\big] \Longrightarrow v \in H,$$

(cf. [4,§3] and [5,§2]). We may use the groupoid model and [24,II.4.5 and II.4.6] to prove the following generalisation of [5,Theorem 2.5]:

2.2 Theorem. *Let E be a row finite directed graph which has no sinks then for each saturated hereditary $H \subset E^0$, there is an ideal $I(H)$ of $C^*(E)$.*

If E is locally finite and H is a saturated hereditary subset of E^0 then the quotient $C^*(E)/I(H)$ is naturally isomorphic to $C^*(F)$ of the directed graph $F := (E^0 \setminus H, \{e : r(e) \notin H\})$. The ideal $I(H)$ is strongly Morita equivalent to $C^*(K)$ of the directed graph $K := (H, \{e : s(e) \in H\})$.

If E also satisfies condition (K) *then $H \mapsto I(H)$ is an isomorphism of the lattice of saturated hereditary subsets of E^0 onto the lattice of closed ideals in $C^*(E)$.*

If E has a sink at $v \in E^0$, then $C^*(E)$ has an ideal I_v isomorphic to the compact operators on a seperable Hilbert space. The directed graph E is said to be *cofinal* if for every $v \in E^0$ and $x \in E^\infty$, there exist $\alpha \in E^*$ and $n \in \mathbb{N}$ such that $s(\alpha) = v$ and $r(\alpha) = s(x_n)$. If E is cofinal then conditions (K) and (L) are equivalent.

2.3 Corollary. *Suppose E is a locally finite directed graph which satisfies* (L)*. Then $C^*(E)$ is simple if and only if E is cofinal.*

3. The K–theory of $C^*(E)$

In [20,Theorem 4.2.4] we showed that if E is row finite with no sinks then the gauge action on $C^*(E)$ has large spectral subspaces (see [13,19]) and then using [3, §10.6] the computation of the K–theory of $C^*(E)$ is analogous to the classical computation in [5]:

3.1 Theorem. *Let E be a row finite graph with no sinks then if A_E is the edge matrix of E we have*

$$\begin{aligned} K_0(C^*(E)) &\cong \mathbb{Z}^{|E^1|}/(1 - A_E^t)\mathbb{Z}^{|E^1|}, \\ K_1(C^*(E)) &\cong \ker\left\{1 - A_E^t : \mathbb{Z}^{|E^1|} \to \mathbb{Z}^{|E^1|}\right\}. \end{aligned}$$

Moreover, $K_0(C^*(E))$ is generated by the equivalence classes $\{[P_v] : v \in E^0\}$ subject to the relation $[P_v] = \sum_{\{e:s(e)=v\}}[P_{r(e)}]$.

Proof. The first part of the result follows by [16,Corollary 6.12] and the second part is given in [20,Corollary 4.2.5].

The final statement of the above theorem indicates that we may carry out the computation of the K–theory of $C^*(E)$ using the vertex matrix B_E of E:

3.2 Lemma. *Let E be a row finite graph with no sinks, then*

$$\begin{aligned} \mathbb{Z}^{|E^1|}/(1-A_E^t)\mathbb{Z}^{|E^1|} &\cong \mathbb{Z}^{|E^0|}/(1-B_E^t)\mathbb{Z}^{|E^0|}, \\ \ker\left\{1-A_E^t : \mathbb{Z}^{|E^1|} \to \mathbb{Z}^{|E^1|}\right\} &\cong \ker\left\{1-B_E^t : \mathbb{Z}^{|E^0|} \to \mathbb{Z}^{|E^0|}\right\}, \end{aligned}$$

where A_E and B_E are the edge and vertex matrices of E.

Proof. The proof is given in [18,Proposition 4.1] and relies on the fact that A_E and B_E are elementary strong shift equivalent in the sense of [28].

4. Loops and the structure of $C^*(E)$

From Theorem 1.4 and Theorem 2.2 it is clear that the nature of $C^*(E)$ depends on the distribution of loops in E. Indeed, conditions (I), (J), (K), (L) are all statements about the plenitude of loops in E. At the two extremes $C^*(E)$ displays starkly different properties: if there are no loops then $C^*(E)$ is AF, if there are sufficiently many then $C^*(E)$ is purely infinite in the sense that every hereditary subalgebra contains an infinite projection.

4.1 Theorem. *A row finite directed graph E has no loops if and only if $C^*(E)$ is an AF algebra.*

Proof. First we show that if E has finitely many vertices then $C^*(E)$ is isomorphic to a finite direct sum of full matrix algebras. Then, when one looks at the C^*–algebra generated by a finite number of elements of $C^*(E)$ one obtains the C^*–algebra of a certain subgraph of E with finitely many vertices. For more details see [15,Theorem 2.4].

A Bratteli diagram [4,§1] can be thought of as a directed graph with no loops. If B is a Bratteli diagram then $C^*(B)$ is a nonunital AF algebra which is strongly Morita equivalent to a unital AF algebra A which has B for a Bratteli diagram.

4.2 Theorem. *Let E be a locally finite graph with no sinks. Then $C^*(E)$ is purely infinite if and only if E satisfies condition (L) and every vertex connects to a loop.*

Proof. We show that if E satisfies these conditions then $\mathcal{G}_E$ is locally contracting and essentially free. For more details see [15,Theorem 3.9] which relies on work from [1,17].

Indeed we have a dichotomy:

4.3 Corollary. *Let E be a locally finite graph which has no sinks, is cofinal, and satisfies condition (L). Then $C^*(E)$ is simple, and*
(i) if E has no loops, then $C^*(E)$ is AF,
(ii) if E has a loop, then $C^*(E)$ is purely infinite.

5. Doplicher–Roberts algebras

One of our original motivations for looking at Cuntz–Krieger algebras and interpreting them as the C^*–algebra of a directed graph was our interest in understanding the Doplicher–Roberts algebras which occur in their theory of compact group duality (see [9], [10]).

Let K be a compact group and ρ a fixed finite–dimensional unitary representation of K. By decomposing successive tensor powers of ρ we may build a graph E_ρ whose vertices are elements of $\widehat{K}$ (see [18, 21]). In [18,Theorem 2.1] it was shown that the Doplicher–Roberts algebra $\mathcal{O}_\rho$ associated to ρ contains a dense subalgebra which is generated by partial isometries which are indexed by finite paths in E_ρ starting at the class of the trivial representation $[\iota] \in \widehat{K}$.

Given a row finite directed graph E and a finite set of vertices $V \subseteq E^0$ one may form the *pointed graph* (E, V) and construct $C^*(E, V)$ by only considering those paths which begin at a vertex in V.

5.1 Theorem. *Let (E, V) be a pointed row finite directed graph with no sinks such every vertex in E^0 is reachable from V. Then there is a full projection $P_V \in C^*(E)$ such that $C^*(E, V) \cong P_V C^*(E) P_V$.*

Proof. The idea of the proof is that pointing a graph E at $V \subset E^0$ is equivalent to taking a reduction of $\mathcal{G}_E$. For more details see [16,Theorem 3.1].

By adapting the techniques of [20,Lemma 3.3.1] we should be able to extend this result to the case when V is infinite. The relationship between Doplicher–Roberts algebras and pointed graph algebras is then given in [16,Theorem 7.1]:

5.2 Theorem. *Let ρ be a unitary representation of a compact group, and E_ρ the associated directed graph, then $\mathcal{O}_\rho \cong C^*(E_\rho, \{[\iota]\})$.*

Since the vertices of E_ρ are indexed by elements of $\widehat{K}$, combining [16,Corollary 7.3] and [16,Corollary 7.7] we may establish the following:

5.3 Corollary. *Let K be an infinite compact Lie group and ρ a faithful representation of K in $SU(\mathcal{H}_\rho)$ where $1 < \dim(\mathcal{H}_\rho) < \infty$, then $\mathcal{O}_\rho$ is simple. Let $R(K)$ be the representation ring of K and β_ρ be the endomorphism $[\pi] \mapsto [\pi \otimes \rho]$ of $R(K)$, then $K_0(\mathcal{O}_\rho) \cong R(K)/\mathrm{Im}(id - \beta_\rho)$ and $K_1(\mathcal{O}_\rho) \cong 0$.*

In fact one may show that if $\rho : K \to U(\mathcal{H}_\rho)$ is not special unitary then E_ρ has no loops and so $\mathcal{O}_\rho$ is a unital AF algebra.

6. Product graphs and crossed products

Let E be a directed graph, G a countable group and $c : E^1 \to G$ a function. The *skew–product graph* $E(c)$ is defined by setting $E(c)^i = G \times E^i$ for $i = 0, 1$ with source and range maps $s(g, e) = (g, s(e))$ and $r(g, e) = (gc(e), r(e))$. Sometimes these graphs are referred to as "voltage graphs" (see [12,§2]).

6.1 Theorem. *Let E be a row finite directed graph with no sinks and $c : E^1 \to G$ a function where G is a countable group, then there is an action α of G on $C^*(E(c))$ such that $C^*(E(c)) \times_\alpha G \cong C^*(E) \otimes \mathcal{K}(\ell^2(G))$. If G is abelian then $C^*(E(c)) \cong C^*(E) \times_\beta \widehat{G}$ where β is the dual action of $\widehat{G}$ on $C^*(E)$.*

Proof. The key step in the proof is to identify the path groupoid $\mathcal{G}_{E(c)}$ of the skew–product graph with a skew product groupoid $\mathcal{G}_E(\tilde{c})$ where $\tilde{c}$ is a continuous 1–cocycle on $\mathcal{G}_E$ induced by c. For more details see [14].

6.2 Corollary. *Let E be a row finite directed graph with no sinks, then $C^*(E)$ satisfies the hypothesis of the UCT (see* [25,Theorem 1.17]*).*

Proof. If $c : E^1 \to \mathbb{Z}$ is defined by $c(e) = 1$ for all e then $E(c)$ has no loops and so $C^*(E(c))$ is AF. It is then straightforward to show that $C^*(E)$ is stably isomorphic to a crossed product of an AF algebra by a $\mathbb{Z}$–action.

Let E be a directed graph and suppose that a countable group G acts freely on the vertices on E then we may form the quotient graph E/G whose vertices and edges are equivalence classes of vertices and edges of E together with source and range maps given by $s([e]) = [s(e)]$ and $r([e]) = [r(e)]$.

6.3 Theorem. *Let E be a row finite directed graph with no sinks and G a countable group which acts freely on the vertices of E then $C^*(E) \times G \cong C^*(E/G) \otimes \mathcal{K}(\ell^2(G))$. If G is abelian then $C^*(E) \cong C^*(E/G) \times_\beta \widehat{G}$ where β is the dual action of $\widehat{G}$ on $C^*(E/G)$.*

Proof. The key step in the proof is to use [12,Theorem 2.2.2] to give a function $c : (E/G)^1 \to G$ such that $(E/G)(c) \cong E$. For more details see [14].

The duality between Theorem 6.1 and Theorem 6.3 is then captured in the following remarkable result (cf. [27]).

6.4 Theorem. *Let E be a row finite directed graph with no sinks then $C^*(E)$ is strongly Morita equivalent to $C_0(\Omega) \times G$ where Ω is a 0–dimensional space and G is a free group.*

Proof. We define a universal covering graph T of E using the set of reduced walks (undirected paths which do not backtrack) in E starting at a fixed vertex $\star \in E^0$. In fact T is a tree and if we define Ω to be the collection of shift tail equivalence classes of T^∞, then under the quotient topology Ω is a 0–dimensional Hausdorff space. Let G denote the countable group (under concatenation) of reduced walks in E beginning and ending at $\star$, then G acts freely on T and $T/G \cong E$. Since the action of G on T preserves shift tail equivalence it induces an action of G on Ω. As G acts freely on a tree by [26,Theorem 4] it must be a free group. For more details see [14].

7. More generally

Let us briefly mention some significant developments which further abstract the notion of the Cuntz–Krieger algebra of a finite matrix.

In [7,8] Deaconu defines a C^*–algebra $\mathcal{O}_\Phi$ from a certain embedding Φ of circle algebras. Here the vertices of a graph are replaced by circles and the edges are determined by Φ. There is a path groupoid Γ from which one defines $\mathcal{O}_\Phi$ and computes its K–Theory.

In [22,23] Putnam and Spielberg build C^*–algebras which they call *Ruelle algebras* from a Smale space (X, d, ϕ). In this case the zero–dimensional space E^∞ and the equivalence relation which underpins our groupoid $\mathcal{G}_E$ are put into a more general framework. Using the notion of stable and unstable equivalence on X they construct certain groupoid C^*–algebras and their crossed products.

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