

FLOW EQUIVALENCE OF GRAPH ALGEBRAS

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ABSTRACT. This paper explores the effect of various graphical constructions upon the associated graph C^* -algebras. The graphical constructions in question arise naturally in the study of flow equivalence for topological Markov chains. We prove that out-splittings give rise to isomorphic graph algebras, and in-splittings give rise to strongly Morita equivalent C^* -algebras. We generalise the notion of a delay as defined in [D] to form in-delays and out-delays. We prove that these constructions give rise to Morita equivalent graph C^* -algebras. We provide examples which suggest that our results are the most general possible in the setting of the C^* -algebras of arbitrary directed graphs.

1. INTRODUCTION

The purpose of this paper is to describe various constructions on a directed graph which give rise to equivalences between the associated graph C^* -algebras. The graphical constructions in question all have their roots in the theory of flow equivalence for topological Markov chains. Our results will unify the work of several authors over the last few years who have studied similar constructions for Cuntz-Krieger algebras and, more recently, graph C^* -algebras (see [CK, MRS, Ash, D, DS, Br, B1] amongst others).

The motivation for the graphical constructions we use lies in the theory of subshifts of finite type. A shift space (X, σ) over a finite alphabet \mathcal{A} is a compact subset X of $\mathcal{A}^{\mathbb{Z}}$ invariant under the shift map σ . To a directed graph E with finitely many edges and no sources or sinks one may associate a shift space X_E , called the edge shift of E , whose alphabet is the edge set of E (see [LM, Definition 2.2.5]). Edge shifts are examples of subshifts of finite type. Alternatively, to every square 0-1 matrix A with no zero rows or columns one may associate a subshift of finite type X_A (see [LM, Definition 2.3.7]).

Two important types of equivalence between shift spaces are conjugacy and flow equivalence. Shift spaces (X, σ_X) and (Y, σ_Y) are conjugate ($X \cong Y$) if there is an isomorphism $\phi : X \rightarrow Y$ such that $\sigma_Y \circ \phi = \phi \circ \sigma_X$. The suspension of (X, σ_X) is

$$SX := (X \times \mathbf{R}) / [(x, t + 1) \sim (\sigma_X(x), t)].$$

Shift spaces (X, σ_X) and (Y, σ_Y) are flow equivalent ($X \sim Y$) if there is a homeomorphism between SX and SY preserving the orientation of flow lines.

By [LM, Proposition 2.3.9] every subshift of finite type is conjugate to an edge shift X_E for some directed graph E . Since the edge connectivity matrix B_E of E is a 0-1 matrix such that $X_E \cong X_{B_E}$, every subshift of finite type is conjugate to a shift described by a 0-1 matrix. Conversely, every shift described by a 0-1 matrix A is conjugate to an edge shift: Let E_A be the directed graph with vertex connectivity matrix A , then X_{E_A} is conjugate to X_A (see [LM, Exercise 1.5.6, Proposition 2.3.9]). Hence subshifts of finite type are edge shifts or shifts associated to 0-1 matrices.

Conjugacy and flow equivalence for subshifts of finite type may be expressed in terms of 0-1 matrices: An elementary strong shift equivalence between square 0-1 matrices A, B is a pair (R, S) of 0-1 matrices such that $A = RS$ and $B = SR$. We say A and B are strong shift equivalent if there is a chain of elementary strong shift equivalences from A to B . From [W, Theorem A] (see also [LM, Theorem 7.2.7]) $\mathcal{X}_A \cong \mathcal{X}_B$ if and only if A and B are strong shift equivalent. By [PS] $\mathcal{X}_A \sim \mathcal{X}_B$ if and only if A and B are related via a chain of elementary strong shift equivalences and certain matrix expansions. Both these matrix operations have graphical interpretations: Following [LM, Theorem 2.4.14 and Exercise 2.4.9] an elementary strong shift equivalence corresponds to either an in- or out-splitting of the corresponding graphs. Following [D, §3], the matrix expansions in [PS] correspond to an out-delay of the corresponding graph.

To a 0-1 matrix A with n non-zero rows and columns is associated a C^* -algebra generated by partial isometries $\{S_i\}_{i=1}^n$ with mutually orthogonal ranges satisfying

$$S_j^* S_j = \sum_{i=1}^n A(i, j) S_i S_i^*.$$

If A satisfies condition (I) the Cuntz-Krieger algebra \mathcal{O}_A is unique up to isomorphism. Results about Cuntz-Krieger algebras may be expressed in terms of the directed graph E_A associated to A (see [EW] and [FW], for instance). More recent results are expressed entirely in terms of E_A (see [KPRR, KPR] amongst others).

To a row-finite directed graph E with finitely many edges and no sources or sinks is associated the universal C^* -algebra, $C^*(E)$ generated by partial isometries $\{s_e : e \in E^1\}$ with mutually orthogonal ranges satisfying

$$s_e^* s_e = \sum_{s(f)=r(e)} s_f s_f^*.$$

If A is a square 0-1 matrix which satisfies condition (I), and E_A is the associated directed graph, then $\mathcal{O}_A \cong C^*(E_A)$ (see [MRS, Proposition 4.1]). On the other hand, if E satisfies condition (L), then the associated edge connectivity matrix B_E satisfies condition (I) and $C^*(E) \cong \mathcal{O}_{B_E}$ (see [KPRR, Proposition 4.1]). There are similar equivalences between Cuntz-Krieger algebras associated to infinite 0-1 matrices which are row-finite and certain row-finite directed graphs (see [PR], [BPRSz, Theorem 3.1]).

By [CK, Proposition 2.17, Theorems 3.8 and 4.1] if A and B satisfy condition (I) and $\mathcal{X}_A \cong \mathcal{X}_B$ then $\mathcal{O}_A \cong \mathcal{O}_B$; moreover, if $\mathcal{X}_A \sim \mathcal{X}_B$ then \mathcal{O}_A is stably isomorphic to \mathcal{O}_B . The aim of this paper is to show that the graphical procedures involved in flow equivalence and conjugacy for edge shifts may be applied to arbitrary graphs, and give rise to isomorphisms or Morita equivalences of the corresponding graph C^* -algebras. Initial results in this direction were proved in [D] for graphs with no sinks and finitely many vertices: Out-splittings lead to isomorphisms of the underlying graph C^* -algebras whilst the C^* -algebra of an in-split graph is isomorphic to the C^* -algebra of a certain out-delayed graph. Further partial results may be found in [DS] and [Br].

The paper is organised as follows. Section 2 describes the C^* -algebra of any directed graph and the gauge invariant uniqueness result used to establish our results. Section 3 deals with out-split graphs, section 4 with in- and out-delays, and section 5 with in-split graphs. Finally section 6 relates our results to those in [B1]. Our main results are:

1. If E is a directed graph and F is a proper out-split graph formed from E then $C^*(E) \cong C^*(F)$ (Theorem 3.2).

2. If E is a directed graph and F is an out-delayed graph formed from E then $C^*(E)$ is strongly Morita equivalent to $C^*(F)$ if and only if F arises from a proper out-delay (Theorem 4.2).
3. If E is a directed graph and F is an in-delayed graph formed from E then $C^*(E)$ is strongly Morita equivalent to $C^*(F)$ (Theorem 4.5).
4. If E is a directed graph and F is an in-split graph formed from E then $C^*(E)$ is strongly Morita equivalent to $C^*(F)$ if and only if F arises from a proper in-splitting (Corollary 5.4).

2. THE C^* -ALGEBRA OF A DIRECTED GRAPH

Here we briefly set out some of the basic definitions and terminology which we use throughout this paper. A directed graph E consists of countable sets of vertices and edges E^0 and E^1 , together with maps $r, s : E^1 \rightarrow E^0$ giving direction of each edge. The maps r, s extend naturally to E^* , the collection of all finite paths in E . The graph E is called *row-finite* if every vertex emits a finite number of edges.

A Cuntz-Krieger E -family consists of a collection $\{s_e : e \in E^1\}$ of partial isometries with orthogonal ranges, and mutually orthogonal projections $\{p_v : v \in E^0\}$ satisfying

- (i) $s_e^* s_e = p_{r(e)}$,
- (ii) $s_e s_e^* \leq p_{s(e)}$,
- (iii) if v emits finitely many edges then $p_v = \sum_{s(e)=v} s_e s_e^*$.

The graph C^* -algebra of E , $C^*(E)$ is the universal C^* -algebra generated by a Cuntz-Krieger E -family. An important property of a graph C^* -algebra is the existence of an action γ of \mathbf{T} , called the gauge action, which is characterised by

$$\gamma_z s_e = z s_e, \text{ and } \gamma_z p_v = p_v$$

where $\{s_e, p_v\} \subseteq C^*(E)$ is the canonical Cuntz-Krieger E -family and $z \in \mathbf{T}$. This gauge action is a key ingredient in the uniqueness theorem which we shall frequently use:

Theorem 2.1. [BHRSz, Theorem 2.1] *Let E be a directed graph, $\{S_e, P_v\}$ be a Cuntz-Krieger E -family and $\pi : C^*(E) \rightarrow C^*(S_e, P_v)$ the homomorphism satisfying $\pi(s_e) = S_e$ and $\pi(p_v) = P_v$. Suppose that each P_v is non-zero, and that there is a strongly continuous action β of \mathbf{T} on $C^*(S_e, P_v)$ such that $\beta_z \circ \pi = \pi \circ \gamma_z$ for all $z \in \mathbf{T}$. Then π is faithful.*

To apply Theorem 2.1, we exhibit a non-trivial Cuntz-Krieger E -family within a C^* -algebra B , which carries a suitable \mathbf{T} -action β .

Some results in this paper require the following result on Morita equivalence of graph algebras. As in [BHRSz, Remark 3.1] we define the saturation $\Sigma H(S)$ of $S \subseteq E^0$ to be the union of the sequence $\Sigma_n(S)$ of subsets of E^0 defined inductively as follows:

$$\begin{aligned} \Sigma_0(S) &:= \{v \in E^0 : v = r(\mu) \text{ for some } \mu \in E^* \text{ with } s(\mu) \in S\} \\ \Sigma_{n+1}(S) &:= \Sigma_n(S) \cup \{w \in E^0 : 0 < |s^{-1}(w)| < \infty \text{ and } s(e) = w \text{ imply } r(e) \in \Sigma_n(S)\} \end{aligned}$$

We note that if E is row-finite then $\Sigma H(S)$ is the saturation of the hereditary set $\Sigma_0(S)$ as defined in [BPRSz].

Lemma 2.2. *Suppose that E is a directed graph, S a subset of E^0 and $\{s_e, p_v\}$ the canonical Cuntz-Krieger E -family. Let $P = \sum_{v \in S} p_v$. Then $P \in \mathcal{M}(C^*(E))$ and the corner $PC^*(E)P$ is full if and only if $\Sigma H(S) = E^0$.*

Proof. By [PR, Lemma 3.3.1] the sum $\sum_{v \in S} p_v$ converges to a projection $P \in \mathcal{M}(C^*(E))$. We claim that $PC^*(E)P \subseteq I_{\Sigma H(S)}$. Let $s_\mu s_\nu^*$ be a nonzero element of $PC^*(E)P$, then $s(\mu) \in S$ and so $p_{s(\mu)} \in I_{\Sigma H(S)}$. Thus $s_\mu s_\nu^* = p_{s(\mu)} s_\mu s_\nu^* \in I_{\Sigma H(S)}$ establishing our claim.

If $PC^*(E)P$ is full then $I_{\Sigma H(S)} = C^*(E)$ and so $\Sigma H(S) = E^0$ by [BHRSz, Section 3]. Conversely, suppose that $\Sigma H(S) = E^0$ and $PC^*(E)P \subseteq I$ for some ideal I in $C^*(E)$. By [BHRSz, Lemma 3.2] $H_I = \{v : p_v \in I\}$ is a saturated hereditary subset of E^0 containing S and hence $\Sigma H(S)$. Thus $C^*(E) = I_{\Sigma H(S)} \subseteq I$ and the result follows. \square

3. OUT-SPLITTINGS

The following definitions are adapted from [LM, Definition 2.4.3]. Let $E = (E^0, E^1, r, s)$ be a directed graph. For each $v \in E^0$ which emits an edge, partition $s^{-1}(v)$ into disjoint nonempty subsets $\mathcal{E}_v^1, \dots, \mathcal{E}_v^{m(v)}$ where $m(v) \geq 1$ (if v is a sink, then we put $m(v) = 0$). Let \mathcal{P} denote the resulting partition of E^1 . We form the *out-split graph* $E_s(\mathcal{P})$ from E using \mathcal{P} as follows: Let

$$\begin{aligned} E_s(\mathcal{P})^0 &= \{v^i : v \in E^0, 1 \leq i \leq m(v)\} \cup \{v : m(v) = 0\}, \\ E_s(\mathcal{P})^1 &= \{e^j : e \in E^1, 1 \leq j \leq m(r(e))\} \cup \{e : m(r(e)) = 0\}, \end{aligned}$$

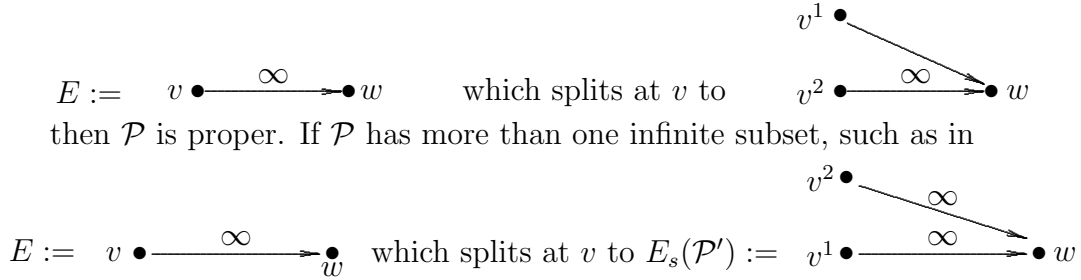
and define $r_{E_s(\mathcal{P})}, s_{E_s(\mathcal{P})} : E_s(\mathcal{P})^1 \rightarrow E_s(\mathcal{P})^0$ for $e \in \mathcal{E}_s^i(e)$ by

$$\begin{aligned} s_{E_s(\mathcal{P})}(e^j) &= s(e)^i \text{ and } s_{E_s(\mathcal{P})}(e) = s(e)^i \\ r_{E_s(\mathcal{P})}(e^j) &= r(e)^j \text{ and } r_{E_s(\mathcal{P})}(e) = r(e). \end{aligned}$$

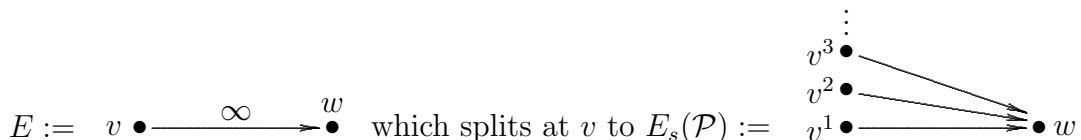
The partition \mathcal{P} is *proper* if for every vertex v with infinite valency we have $m(v) < \infty$ and only one of the partition sets \mathcal{E}_v^i is infinite.

Examples 3.1. (i) The partitions which give rise to the out-splittings described in [LM, Figures 2.4.3–2.4.5] and [D, §4.1] are all examples of proper partitions.

(ii) If we out-split at an infinite valence vertex, taking a partition \mathcal{P} which has finitely many subsets, such as in



then \mathcal{P} is proper. Again $C^*(E)$ is not Morita equivalent to $C^*(E_s(\mathcal{P}))$ since the latter has an additional ideal. If \mathcal{P} has infinitely many subsets, such as in



then \mathcal{P} is not proper. In this case, $C^*(E)$ and $C^*(E_s(\mathcal{P}))$ are not Morita equivalent: The former is non-simple and the latter simple.

Theorem 3.2. *Let E be a directed graph, \mathcal{P} a partition of E^1 and $E_s(\mathcal{P})$ the out-split graph formed from E using \mathcal{P} . If \mathcal{P} is proper then $C^*(E) \cong C^*(E_s(\mathcal{P}))$.*

Proof. Let $\{s_f, p_w : f \in E_s(\mathcal{P})^1, w \in E_s(\mathcal{P})^0\}$ be a Cuntz-Krieger $E_s(\mathcal{P})$ -family. For $v \in E^0$ and $e \in E^1$ set $Q_v = p_v$ if $m(v) = 0$, $T_e = s_e$ if $m(r(e)) = 0$,

$$Q_v = \sum_{1 \leq i \leq m(v)} p_{v^i} \text{ if } m(v) \neq 0 \text{ and } T_e = \sum_{1 \leq j \leq m(r(e))} s_{e^j} \text{ if } m(r(e)) \neq 0.$$

Because \mathcal{P} is proper $m(v) < \infty$ for all $v \in E^0$ and all of these sums are finite. We claim that $\{T_e, Q_v : e \in E^1, v \in E^0\}$ is a Cuntz-Krieger E -family in $C^*(E_s(\mathcal{P}))$.

The Q_v 's are non-zero mutually orthogonal projections since they are sums of projections satisfying the same properties. The partial isometries T_e for $e \in E^1$ have mutually orthogonal ranges since they consist of sums of partial isometries with mutually orthogonal ranges. For $e \in E^1$ it is easy to see that $T_e^* T_e = Q_{r(e)}$ and $T_e T_e^* \leq Q_{s(e)}$.

For $e \in E^1$ with $m(r(e)) \neq 0$, then since $r_{E_s}(e^j) \neq r_{E_s}(e^k)$ for $j \neq k$ we have

$$(3.1) \quad T_e T_e^* = \left(\sum_{1 \leq j \leq m(r(e))} s_{e^j} \right) \left(\sum_{1 \leq k \leq m(r(e))} s_{e^k} \right)^* = \sum_{1 \leq j \leq m(r(e))} s_{e^j} s_{e^j}^*.$$

If $m(r(e)) = 0$ then $T_e T_e^* = s_e s_e^*$. For $v \in E^0$ and $1 \leq i \leq m(v)$ put $\mathcal{E}_{1,v}^i = \{e \in \mathcal{E}_v^i : m(r(e)) \geq 1\}$ and $\mathcal{E}_{0,v}^i = \{e \in \mathcal{E}_v^i : m(r(e)) = 0\}$. If $v \in E^0$ has finite valency and is not a sink then $s^{-1}(v) = \bigcup_{i=1}^{m(v)} \mathcal{E}_v^i$ and for $1 \leq i \leq m(v)$ we have

$$s_{E_s(\mathcal{P})}^{-1}(v^i) = \{e^j : e \in \mathcal{E}_{1,v}^i, 1 \leq j \leq m(r(e))\} \cup \mathcal{E}_{0,v}^i$$

Hence using (3.1) we may compute

$$\begin{aligned} Q_v &= \sum_{1 \leq i \leq m(v)} p_{v^i} = \sum_{1 \leq i \leq m(v)} \sum_{e \in \mathcal{E}_{1,v}^i} \sum_{1 \leq j \leq m(r(e))} s_{e^j} s_{e^j}^* + \sum_{1 \leq i \leq m(v)} \sum_{e \in \mathcal{E}_{0,v}^i} s_e s_e^* \\ &= \sum_{1 \leq i \leq m(v)} \sum_{e \in \mathcal{E}_v^i} T_e T_e^* = \sum_{e: s(e)=v} T_e T_e^*, \end{aligned}$$

completing the proof of our claim, since vertices $v \in E^0$ with $m(v) = 0$ are sinks.

Let $\{t_e, q_v\}$ be the canonical generators of $C^*(E)$, then by the universal property of $C^*(E)$ there is a homomorphism $\pi : C^*(E) \rightarrow C^*(E_s(\mathcal{P}))$ taking t_e to T_e and q_v to Q_v . To prove that π is onto we show that the generators of $C^*(E_s(\mathcal{P}))$ lie in $C^*(T_e, Q_v)$. If $w = v^j \in E_s(\mathcal{P})^0$ has finite valency or w is a sink, then $p_w \in C^*(T_e, Q_v)$. If v^j has infinite valency, then without loss of generality we suppose $j = 1$. Since \mathcal{P} is proper it follows that $v^2, \dots, v^{m(v)}$ have finite valency, so $p_{v^2}, \dots, p_{v^{m(v)}} \in C^*(T_e, Q_v)$ and hence

$$p_{v^1} = Q_v - p_{v^2} - \dots - p_{v^{m(v)}} \in C^*(T_e, Q_v).$$

If $e^j \in E_s(\mathcal{P})^1$ then $m(r(e)) \neq 0$. Since $p_{r(e)^j} \in C^*(T_e, Q_v)$ we have $s_{e^j} = T_e p_{r(e)^j} \in C^*(T_e, Q_v)$. If $e \in E_s(\mathcal{P})^1$ then $m(r(e)) = 0$ and so $s_e = T_e \in C^*(T_e, Q_v)$.

Since π commutes with the canonical gauge action on each C^* -algebra and as $Q_v \neq 0$ for all $v \in E^0$ it follows from Theorem 2.1 that π is injective, and the result follows. \square

Remarks 3.3. (i) The *maximal out-splitting* \tilde{E} of E is formed from a partition \mathcal{P} of E^1 which admits no refinements. For a graph $E = (E^0, E^1, r, s)$ without sinks \tilde{E} is isomorphic to the dual graph $\hat{E} = (E^1, E^2, r', s')$ (where $r'(ef) = f$ and $s'(ef) = e$): Since the out-splitting is maximal and there are no sinks, we have

$$\tilde{E}^0 = \{v^e : s(e) = v\}, \text{ and } \tilde{E}^1 = \{e^f : s(f) = r(e)\}.$$

The maps $v^e \mapsto e$ and $e^f \mapsto ef$ induce an isomorphism from \tilde{E} to \hat{E} . We define the dual graph \hat{E} of any directed graph E to be its maximal out-split graph \tilde{E} . Since a maximal out-splitting is proper if and only if E is row-finite we may now use Theorem 3.2 to generalise [BPRSz, Corollary 2.5] to any row-finite graph.

- (ii) Brenken defines a graph algebra $G^*(E)$ which, under certain conditions, is isomorphic to $C^*(E)$. By [Br, Theorem 3.4], $G^*(E)$ and the C^* -algebra of its out-splitting are isomorphic if the graph satisfies a certain properness condition (see [Br, Definition 3.2]). This result only reduces to $C^*(E)$ when E is row-finite.
- (iii) For row-finite graphs with finitely many vertices and no sinks, a proof of Theorem 3.2 may be deduced from [D, §4.1]. In [DS, Section 6] similar results are proved for an ‘‘explosion’’ which is a particular example of an out-splitting operation.
- (iv) The C^* -algebra of an out-split graph is isomorphic to the C^* -algebra of an ultragraph (see [T]). Given a directed graph E and a partition \mathcal{P} define the ultragraph $\mathcal{G}(\mathcal{P}) = (G^0, \mathcal{G}^1, r', s')$ as follows: Put $G^0 = E_s(\mathcal{P})^0$, $\mathcal{G}^1 = E^1$, $s'(e) = s(e)^i$ if $e \in \mathcal{E}_{s(e)}^i$, $r'(e) = r(e)$ if $m(r(e)) = 0$ and

$$r'(e) = \{r(e)^i : 1 \leq i \leq m(r(e))\} \text{ if } m(r(e)) \neq 0.$$

We claim that if \mathcal{P} is proper then $C^*(\mathcal{G}(\mathcal{P})) \cong C^*(E_s(\mathcal{P}))$. When \mathcal{P} is proper \mathcal{G}^0 is the set of finite subsets of $E_s(\mathcal{P})^0$. Let $\{p_A, s_e : A \in \mathcal{G}^0, e \in \mathcal{G}^1\}$ be a Cuntz-Krieger $\mathcal{G}(\mathcal{P})$ -family. For $w \in E_s(\mathcal{P})^0$ set $Q_w = p_w$ and for $f^j \in E_s(\mathcal{P})^1$ put $T_{f^j} = s_{f^j} p_{r(f^j)}$. Then $\{Q_w, T_{f^j}\}$ is a Cuntz-Krieger $E_s(\mathcal{P})$ -family in which each $Q_w \neq 0$. Let $\{t_{f^j}, q_w\}$ be the canonical generators of $C^*(E_s(\mathcal{P}))$, then by the universal property of $C^*(E_s(\mathcal{P}))$ there is a map $\pi : C^*(E_s(\mathcal{P})) \rightarrow C^*(\mathcal{G}(\mathcal{P}))$ sending t_{f^j} to T_{f^j} and q_w to Q_w . As each $A \in \mathcal{G}^0$ is finite, $C^*(T_{f^j}, Q_w)$ contains each generator of $C^*(\mathcal{G}(\mathcal{P}))$, so π is onto. By [T, Section 2] there is an appropriate action of \mathbf{T} on $C^*(\mathcal{G}(\mathcal{P}))$, so π is injective by Theorem 2.1, proving our claim.

- (v) Let Γ act freely on the edges of a row-finite graph E , then the induced Γ -action on the dual graph \hat{E} is free on its vertices. The isomorphism $C^*(E) \cong C^*(\hat{E})$ is Γ -equivariant, so $C^*(E) \times \Gamma \cong C^*(\hat{E}/\Gamma) \times \Gamma$ and by [KQR, Corollary 3.3] we have

$$(3.2) \quad C^*(E) \times \Gamma \cong C^*(\hat{E}/\Gamma) \otimes \mathcal{K}(\ell^2(\mathbf{Z}_2)).$$

For instance, there is a free action of \mathbf{Z}_2 on the edges of graph B_2 which consists of a single vertex and two edges (the ‘‘flip-flop automorphism’’ of $C^*(B_2) \cong \mathcal{O}_2$ described by [Ar]). Equation (3.2) shows that $\mathcal{O}_2 \times \mathbf{Z}_2$ is Morita equivalent \mathcal{O}_2 .

4. DELAYS

Let $E = (E^0, E^1, r, s)$ be a directed graph. A map $d_s : E^0 \cup E^1 \rightarrow \mathbf{N} \cup \{\infty\}$ such that

- (i) if $w \in E^0$ is not a sink then $d_s(w) = \sup\{d_s(e) : s(e) = w\}$,
- (ii) if $d_s(x) = \infty$ for some x then either x is a sink or x emits infinitely many edges

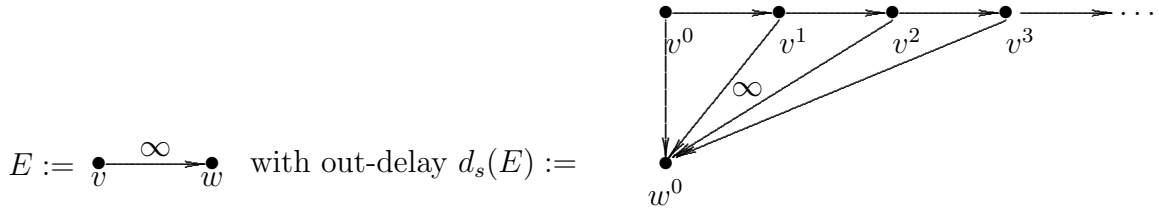
is called a *Drinen source-vector*. Note that only vertices are allowed to have an infinite d_s -value; moreover if $d_s(v) = \infty$ and v is not a sink, then there are edges with source v and arbitrarily large d_s -value. From this data we construct a new graph as follows: Let

$$\begin{aligned} d_s(E)^0 &= \{v^i : v \in E^0, 0 \leq i \leq d_s(v)\} \text{ and} \\ d_s(E)^1 &= E^1 \cup \{f(v)^i : 1 \leq i \leq d_s(v)\}, \end{aligned}$$

and for $e \in E^1$ define $r_{od}(e) = r(e)^0$ and $s_{od}(e) = s(e)^{d_s(e)}$. For $f(v)^i$ define $s_{od}(f(v)^i) = v^{i-1}$ and $r_{od}(f(v)^i) = v^i$. The resulting directed graph $d_s(E)$ is called the *out-delayed graph of E* for the Drinen source-vector d_s .

In the out-delayed graph the original vertices correspond to those vertices with superscript 0; the edge $e \in E^1$ is delayed from leaving $s(e)^0$ and arriving at $r(e)^0$ by a path of length $d_s(e)$. The Drinen source vector d_s is *strictly proper* if, whenever v has infinite valency, there is no v^i with infinite valency unless $i = d_s(v) < \infty$. A Drinen source vector d_s which gives rise to an out-delayed graph $d_s(E)$ which may be constructed using a finite sequence of strictly proper Drinen source vectors is said to be *proper*.

- Examples 4.1.* (i) The notion of an out-delay in the context of graph C^* -algebras was first introduced in [CK, §4] and subsequently generalised in [D]. The graphs shown in [D, §3.1] are all examples of out-delays for some proper Drinen source-vector where all the edges out of a given vertex are delayed by the same amount.
- (ii) The Drinen-Tomforde desingularisation of a graph described in [DT, Definition 2.2] is an example of a out-delay with a proper Drinen source-vector: If v has infinite valency then the edges with source v may be written as $\{e_i : i \in \mathbf{N}\}$; we set $d_s(v) = \infty$ and $d_s(e_i) = i$ for $i \in \mathbf{N}$. If v has finite valency then we set $d_s(v) = 0$ (and so $d_s(e) = 0$ for all $e \in s^{-1}(v)$). If v is a sink then we put $d_s(v) = \infty$. The resulting graph $d_s(E)$ is row-finite with no sinks.
- (iii) Putting $d_s(v) = \infty$ for a sink adds an infinite tail to the sink. If $d_s(v) = \infty$ for all sinks and $d_s(v) = 0$ for all vertices which emit edges then $d_s(E)$ is the graph E with tails added to all sinks (cf. [RS, Lemma 1.4]).
- (iv) Consider the graph E shown below.



The Drinen source-vector for this out-delay is not proper since vertex v_1 has infinite valency. Moreover, the C^* -algebra $C^*(d_s(E))$ is not Morita equivalent to $C^*(E)$ since the former C^* -algebra has 2 proper ideals and the latter only one.

Theorem 4.2. *Let E be a directed graph and $d_s : E^0 \cup E^1 \rightarrow \mathbf{N} \cup \{\infty\}$ be a Drinen source-vector. Then $C^*(d_s(E))$ is strongly Morita equivalent to $C^*(E)$ if and only if d_s is proper.*

Proof. Without loss of generality we may assume that $d_s : E^0 \cup E^1 \rightarrow \mathbf{N} \cup \{\infty\}$ is an essentially proper Drinen source-vector. Let $\{s_f, p_w : f \in d_s(E)^1, w \in d_s(E)^0\}$ be a Cuntz-Krieger $d_s(E)$ -family. For $e \in E^1$ and $v \in E^0$ define $Q_v = p_{v^0}$ and

$$T_e = s_{f(s(e))^1} \dots s_{f(s(e))^{d_s(e)}} s_e \text{ if } d_s(e) \neq 0 \text{ and } T_e = s_e \text{ otherwise.}$$

We claim that $\{T_e, Q_v : e \in E^1, v \in E^0\}$ is a Cuntz-Krieger E -family. The Q_v 's are non-zero mutually orthogonal projections since the p_v 's are. The T_e 's are partial isometries with mutually orthogonal ranges since they are products of partial isometries with this property. For $e \in E^1$ it is routine to check that $T_e^* T_e = Q_{r(e)}$ and $T_e T_e^* \leq Q_{s(e)}$.

If $v \in E^0$ is neither a sink nor has infinite valence, then $d_s(v) < \infty$. If $d_s(v) = 0$, then we certainly have $Q_v = \sum_{s(e)=v} T_e T_e^*$. Otherwise, for $0 \leq j \leq d_s(v) - 1$ we have

$$(4.1) \quad p_{v^j} = \sum_{s(e)=v, d_s(e)=j} s_e s_e^* + s_{f(v)^{j+1}} p_{v^{j+1}} s_{f(v)^{j+1}}^*,$$

and since we must have some edges with $s(e) = v$ and $d_s(e) = d_s(v)$ we have

$$(4.2) \quad p_{v^{d_s(v)}} = \sum_{s(e)=v, d_s(e)=d_s(v)} s_e s_e^*.$$

Using (4.1) recursively and (4.2) when $j = d_s(v) - 1$ we see that

$$Q_v = p_{v^0} = \sum_{s(e)=v, d_s(e)=0} T_e T_e^* + \dots + \sum_{s(e)=v, d_s(e)=d_s(v)} T_e T_e^* = \sum_{s(e)=v} T_e T_e^*,$$

and this establishes our claim.

Let $\{t_e, q_v\}$ be the canonical generators of $C^*(E)$, then by the universal property of $C^*(E)$ there is a homomorphism $\pi : C^*(E) \rightarrow C^*(d_s(E))$ which takes t_e to T_e and q_v to Q_v . It remains to show that $C^*(T_e, Q_v)$ is a full corner in $C^*(d_s(E))$.

Let α denote the strongly continuous \mathbf{T} -action satisfying, for $z \in \mathbf{T}$,

$$\alpha_z s_e = z s_e, \quad \alpha_z s_{f(v)^i} = s_{f(v)^i} \text{ for } 1 \leq i \leq d_s(v) \text{ and } \alpha_z p_{v^i} = p_{v^i} \text{ for } 0 \leq i \leq d_s(v).$$

It is straightforward to check that $\pi \circ \gamma = \alpha \circ \pi$ where γ is the usual gauge action of \mathbf{T} on $C^*(E)$ and it follows from Theorem 2.1 that π is injective.

By Lemma 2.2 the sum $\sum_{v \in E^0} p_{v^0}$ converges to a projection $P \in \mathcal{M}(C^*(d_s(E)))$. We claim that $C^*(T_e, Q_v)$ is equal to $PC^*(d_s(E))P$. Note that if $P s_\mu s_\nu^* P = s_\mu s_\nu^* \neq 0$ then $s_{od}(\mu) = v^0$ and $s_{od}(\nu) = w^0$ for some $v, w \in E^0$, and $r_{od}(\mu) = r_{od}(\nu)$.

If $\mu = \nu = v^0$ then $s_\mu s_\nu^* = p_{v^0} = Q_v \in C^*(\{T_e, Q_v\})$. If $r_{od}(\mu) = u^0$ for some $u \in E^0$ then there are paths $\alpha, \beta \in E^*$ with $r(\alpha) = r(\beta) = u$ such that $s_\mu s_\nu^* = T_\alpha T_\beta^*$ and so $s_\mu s_\nu^* \in C^*(T_e, Q_v)$. Suppose now that $r_{od}(\mu) \notin E^0$. Then $r_{od}(\mu) = u^q$ for some $u \in E^0$ and $0 < q \leq d_s(u)$ and we can write

$$s_\mu s_\nu^* = T_\alpha s_{f(u)^1} \dots s_{f(u)^q} s_{f(u)^q}^* \dots s_{f(u)^1}^* T_\beta^*.$$

for some $\alpha, \beta \in E^*$. Suppose $q > 1$, then since d_s is proper, $f(u)^{q-1}$ has finite valency. If there are no edges in E with $s(e) = u$ and $d_s(e) = q - 1$ then

$$(4.3) \quad s_\mu s_\nu^* = T_\alpha s_{f(u)^1} \dots s_{f(u)^{q-1}} s_{f(u)^{q-1}}^* \dots s_{f(u)^1}^* T_\beta^*.$$

If there are a finite number of edges $e_1, \dots, e_l \in E^1$ with $s(e_i) = u$ and $d_s(e_i) = q - 1$ for $i = 1, \dots, l$, then

$$(4.4) \quad \begin{aligned} s_\mu s_\nu^* &= T_\alpha s_{f(u)^1} \dots s_{f(u)^{q-1}} (p_{u^{q-1}} - s_{e_1} s_{e_1}^* - \dots - s_{e_l} s_{e_l}^*) s_{f(u)^{q-1}}^* \dots s_{f(u)^1}^* T_\beta^* \\ &= T_\alpha s_{f(u)^1} \dots s_{f(u)^{q-1}} s_{f(u)^{q-1}}^* \dots s_{f(u)^1}^* T_\beta^* - \sum_{i=1}^l T_\alpha T_{e_i} T_{e_i}^* T_\beta^*. \end{aligned}$$

Our new expression for $s_\mu s_\nu^*$ may now be analysed as in (4.3) or (4.4), reducing the value of q until all $s_{f(u)^i}$ terms are removed. Then

$$s_\mu s_\nu^* = T_\alpha T_\beta^* - \sum_{s(e)=u, d_s(e) \leq q-1} T_{\alpha e} T_{\beta e}^* \in C^*(T_e, Q_v),$$

completing the proof of our claim.

Since $\Sigma H(\{v^0 : v \in E^0\}) = d_s(E)^0$, it follows from Lemma 2.2 that $PC^*(d_s(E))P$ is a full corner in $C^*(d_s(E))$ and hence $C^*(d_s(E))$ and $C^*(E)$ are strongly Morita equivalent.

If d_s is not proper, then there are at least two vertices $f(v)^i, f(v)^j$ with $0 \leq i < j \leq d_s(v)$ emitting infinitely many edges. In this case there is an ideal generated by $p_{f(v)^i}$ in $C^*(d_s(E))$ which was not present in $C^*(E)$. \square

We are grateful to Daniel Gow and Tyrone Crisp for pointing out an error in an earlier version of Theorem 4.2 (see also [CG]).

- Remarks 4.3.*
- (i) Theorem 4.2 significantly generalises the results in [D, §3.1]. Drinen shows a limited number of graph groupoid isomorphisms for row-finite graphs with finitely many vertices and no sinks in which each edge is equally delayed.
 - (ii) The desingularisation of a non row-finite graph is an example of an out-delay (see Examples 4.1 (ii)). Moreover, any out-delay of a non row-finite graph using a proper Drinen source vector with $d_s(v) = \infty$ for all vertices of infinite valency provides an example of a row-finite graph $d_s(E)$ whose C^* -algebra is Morita equivalent to $C^*(E)$. It follows by [B2, Corollary 4.6] that if E satisfies condition (K) then $\text{Prim}(C^*(E))$ is the primitive ideal space of some AF-algebra.
 - (iii) If $d_s : E^0 \cup E^1 \rightarrow \mathbf{N} \cup \{\infty\}$ is a Drinen source-vector then E is a deformation retract of $d_s(E)$ (see [St, §3.3]). The construction of an out-delayed graph replaces each vertex v with $d_s(v) \geq 1$ by the tree $\{v^i : 0 \leq i \leq d_s(v), f(v)^i : 1 \leq i \leq d_s(v)\}$ which may be contracted to the root v^0 and identified with v (see also [GT, §1.5.5]). In particular $\pi_1(d_s(E)) \cong \pi_1(E)$ and the universal covering tree T of E is a deformation retract of the universal covering tree T' of $d_s(E)$. It follows that the boundary ∂T of T is homeomorphic to the boundary $\partial T'$ of T' (see [KP, §4]). Hence the Morita equivalence between $C^*(E)$ and $C^*(d_s(E))$ could be obtained for row-finite graphs with no sinks using the Kumjian-Pask description of $C^*(E)$ as a crossed product of $C_0(\partial T)$ by $\pi_1(E)$ (see [KP, Corollary 4.14]).

We now turn our attention to in-delays where edges are delayed from arriving at their range. Let $E = (E^0, E^1, r, s)$ be a graph. A map $d_r : E^0 \cup E^1 \rightarrow \mathbf{N} \cup \{\infty\}$ satisfying

- (i) if w is not a source then $d_r(w) = \sup \{d_r(e) : r(e) = w\}$,
- (ii) if $d_r(x) = \infty$ then x is either a source or receives infinitely many edges

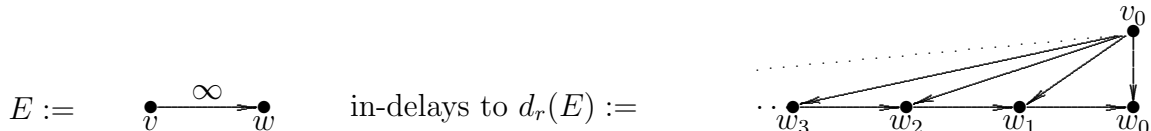
is called a *Drinen range-vector*. We construct a new graph $d_r(E)$ called the *in-delayed graph of E* for the Drinen range-vector d_r as follows:

$$d_r(E)^0 = \{v_i : v \in E^0, 0 \leq i < d_r(v)\} \text{ and}$$

$$d_r(E)^1 = E^1 \cup \{f(v)_i : 1 \leq i < d_r(v)\},$$

and for $e \in E^1$ we define $r_{id}(e) = r(e)_{d_r(e)}$ and $s_{id}(e) = s(e)_0$. For $f(v)_i$ we define $s_{id}(f(v)_i) = v_i$ and $r_{id}(f(v)_i) = v_{i-1}$.

Examples 4.4. (i) Consider the graph E shown below, with edges $\{e_i : i \geq 0\}$ from v to w . If we set $d_r(e_i) = i$, $d_r(v) = 0$ and $d_r(w) = \infty$ then



- (ii) Observe that putting $d_r(v) = \infty$ for a source adds an infinite “head” to the source. If $d_r(v) = \infty$ for all sources and $d_r(v) = 0$ for all vertices which receive edges then $d_r(E)$ is the graph E with heads added to all sources (cf. [RS, Lemma 1.4]).

Theorem 4.5. *If $d_r : E^0 \rightarrow \mathbf{N} \cup \{\infty\}$ is a Drinen range-vector, then $C^*(d_r(E))$ is strongly Morita equivalent to $C^*(E)$.*

Proof. Let $d_r : E^0 \cup E^1 \rightarrow \mathbf{N} \cup \{\infty\}$ be a Drinen range-vector and $\{s_e, p_v : e \in d_r(E)^0, v \in d_r(E)^1\}$ be a Cuntz-Krieger $d_r(E)$ -family. For $v \in E^0$ let $Q_v = p_{v_0}$ and for $e \in E^1$ put

$$T_e = s_e s_{f(r(e))_{d_r(e)}} \cdots s_{f(r(e))_1} \text{ if } d_r(e) \neq 0 \text{ and } T_e = s_e \text{ otherwise.}$$

It is straightforward to check that $\{T_e, Q_v\}$ is a Cuntz-Krieger E -family in $C^*(d_r(E))$ in which all the projections Q_v are non-zero. Let $\{t_e, q_v\}$ be the canonical generators of $C^*(E)$, then by the universal property of $C^*(E)$ there is a homomorphism $\pi : C^*(E) \rightarrow C^*(d_r(E))$ satisfying $\pi(t_e) = T_e$ and $\pi(q_v) = Q_v$. It remains to show that $C^*(T_e, Q_v)$, the image π , is a full corner in $C^*(d_r(E))$.

Let α be the strongly continuous \mathbf{T} -action α on $C^*(d_r(E))$ satisfying, for $z \in \mathbf{T}$,

$$\alpha_z(s_e) = z s_e, \alpha_z(s_{f(v)^i}) = s_{f(v)^i} \text{ for } 1 \leq i \leq d_r(v) \text{ and } \alpha_z(p_{v^i}) = p_{v^i} \text{ for } 0 \leq i \leq d_r(v).$$

It is straightforward to check that $\pi \circ \gamma = \alpha \circ \pi$ where γ is the usual gauge action on $C^*(E)$ and it follows from Theorem 2.1 that π is injective.

By Lemma 2.2, the sum $\sum_{v \in E^0} p_{v_0}$ converges to a projection $P \in \mathcal{M}(C^*(d_r(E)))$. We claim that $C^*(\{T_e, Q_v\})$ is equal to $PC^*(d_r(E))P$. Note that if $P s_\mu s_\nu^* P = s_\mu s_\nu^* \neq 0$ then $s_{id}(\mu) = v_0$ and $s_{id}(\nu) = w_0$ for some $v, w \in E^0$ and $r_{id}(\mu) = r_{id}(\nu)$.

If $r_{id}(\mu) \in E^0$, then $s_\mu s_\nu^* = T_\alpha T_\beta^*$ for some paths $\alpha, \beta \in E^*$ and hence $s_\mu s_\nu^* \in C^*(\{T_e, Q_v\})$. Suppose $r_{id}(\mu) \notin E^0$. Then $r_{id}(\mu) = r(e)_q$ for some $e \in E^1$ with $1 \leq q \leq d_r(e)$ and there are $\alpha, \beta \in E^*$ such that $s_\mu s_\nu^* = T_\alpha s_e p_{f(r(e))_{d_r(e)}} s_e^* T_\beta^*$ if $q = d_r(e)$ and

$$s_\mu s_\nu^* = T_\alpha s_e s_{f(r(e))_{d_r(e)}} \cdots s_{f(r(e))_{q+1}} p_{f(r(e))_q} s_{f(r(e))_{q+1}}^* \cdots s_e^* T_\beta^*,$$

otherwise. Since the vertices $f(r(e))_i$ for $2 \leq i \leq d_r(r(e))$ emit exactly one edge each, we have $p_{f(r(e))_i} = s_{f(r(e))_{i-1}} s_{f(r(e))_{i-1}}^*$ and hence we decrease q in the expression for $s_\mu s_\nu^*$ until we have $s_\mu s_\nu^* = T_\alpha s_e T_\beta^* \in C^*(T_e, Q_v)$ as required.

It remains to check that the corner is full. To see this, we note that $\Sigma H(E^0) = d_r(E)^0$ and apply Lemma 2.2. Our result follows. \square

Remarks 4.6. (i) Using in-delays we can convert row-finite graphs into locally finite graphs (i.e. graphs where every vertex receives and emits finitely many edges): If E is row-finite and $v \in E^0$ receives edges $\{e_i : i \in \mathbf{N}\}$, set $d_r(v) = \infty$ and $d_r(e_i) = i$. If v is a source we put $d_r(v) = \infty$ and if v receives finitely many edges we set $d_r(v) = 0$. Evidently $d_r : E^0 \cup E^1 \rightarrow \mathbf{N}$ is proper. The resulting graph $d_r(E)$ is then locally finite with no sources. Thus, combining Theorem 4.2 and Theorem 4.5 we can show that for any graph E there is a locally finite graph with no sinks and sources F such that $C^*(E)$ is strongly Morita equivalent to $C^*(F)$.

(ii) An in-delay at a vertex v with $d_r(v) \geq 1$ replaces $v \in E^0$ by the tree $\{v_i : 0 \leq i \leq d_r(v), f(v)_i : 1 \leq i \leq d_r(v)\}$ where v is identified with the leaf v_0 . In combination with Remarks 4.3 (iii) it seems that we may get similar Morita equivalence results if we replace vertices with more general trees (i.e. contractible graphs) where the original vertex lies within the tree itself.

(iii) Not every in-delay can be expressed as an out-delay. To see this observe that for the graph E used in Examples 3.1 there can be no out-delay which corresponds to the in-delay described in Example 4.4. It should not be difficult to find examples where the graph contains no sources and sinks.

5. IN-SPLITTINGS

The following is adapted from [LM, Definition 2.4.7]: Let $E = (E^0, E^1, r, s)$ be a directed graph. For each $v \in E^0$ with $r^{-1}(v) \neq \emptyset$ partition the set $r^{-1}(v)$ into disjoint nonempty subsets $\mathcal{E}_1^v, \dots, \mathcal{E}_{m(v)}^v$ where $m(v) \geq 1$ (if v is a source then we put $m(v) = 0$). Let \mathcal{P} denote the resulting partition of E^1 . We form the *in-split graph* $E_r(\mathcal{P})$ from E using the partition \mathcal{P} as follows: Let

$$E_r(\mathcal{P})^0 = \{v_i : v \in E^0, 1 \leq i \leq m(v)\} \cup \{v : m(v) = 0\},$$

$$E_r(\mathcal{P})^1 = \{e_j : e \in E^1, 1 \leq j \leq m(s(e))\} \cup \{e : m(s(e)) = 0\},$$

and define $r_{E_r(\mathcal{P})}, s_{E_r(\mathcal{P})} : E_r(\mathcal{P})^1 \rightarrow E_r(\mathcal{P})^0$ by

$$s_{E_r(\mathcal{P})}(e_j) = s(e)_j \text{ and } s_{E_r(\mathcal{P})}(e) = s(e)$$

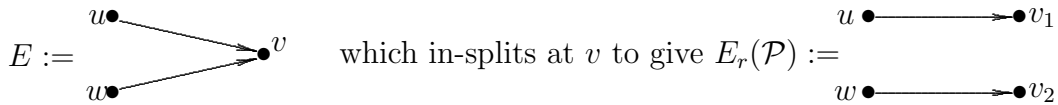
$$r_{E_r(\mathcal{P})}(e_j) = r(e)_i \text{ and } r_{E_r(\mathcal{P})}(e) = r(e)_i \text{ where } e \in \mathcal{E}_i^{r(e)}.$$

Partition \mathcal{P} is *proper* if for every vertex v which is a sink or emits infinitely many edges we have $m(v) = 0, 1$. That is, we cannot in-split at a sink or vertex with infinite valency.

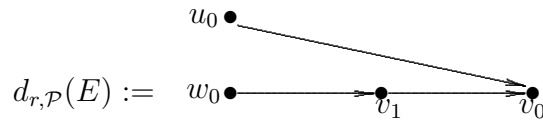
To relate the graph algebras of a graph and its in-splittings we use a variation of the method introduced in [D, §4.2]: If $E_r(\mathcal{P})$ is the in-split graph formed from E using the partition \mathcal{P} then we may define a Drinen range-vector $d_{r,\mathcal{P}} : E^0 \cup E^1 \rightarrow \mathbf{N} \cup \{\infty\}$ by $d_{r,\mathcal{P}}(v) = m(v) - 1$ if $m(v) \geq 1$ and $d_{r,\mathcal{P}}(v) = 0$ otherwise. For $e \in \mathcal{E}_i^{r(e)}$ we put $d_{r,\mathcal{P}}(e) = i - 1$. Hence, if v receives $n \geq 2$ edges then we create an in-delayed graph in which v is given delay of size $m(v) - 1$ and all edges with range v are given a delay one less than their label in the partition of $r^{-1}(v)$. If v is a source or receives only one edge then there is no delay attached to v .

Examples 5.1. (i) Examples of proper in-splittings are found in [LM, Figure 2.4.6] and [D, §4.2].

ii) An in-splitting is not proper if we in-split at a sink, such as for



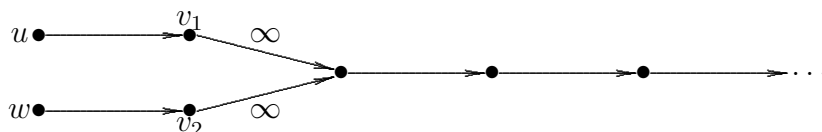
The associated in-delayed graph is



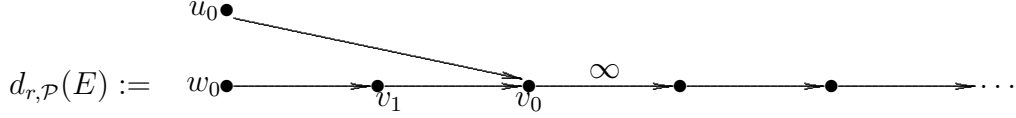
As $C^*(E_r(\mathcal{P}))$ has two ideals and $C^*(d_{r,\mathcal{P}}(E))$ one they are not Morita equivalent.
 (iii) In-splittings at infinite valence vertices are not proper, such as in



which in-splits at v to give $E_r(\mathcal{P}) :=$



The associated in-delayed graph is



In this case $C^*(E_r(\mathcal{P}))$ has two ideals, whereas $C^*(d_r(E))$ only has one. Thus these algebras are not Morita equivalent.

Remark 5.2. If \mathcal{P} is proper then every vertex v which is either a sink or a vertex of infinite valency occurs only as v or v_1 in $E_r(\mathcal{P})$ and only as v_0 in $d_{r,\mathcal{P}}(E)$. In particular, if \mathcal{P} is a proper partition and v is a sink or infinite valence vertex, then there are no edges of the form e_j for $j \geq 2$ with $s(e) = v$ in $E_r(\mathcal{P})$ and no edges of the form $f(v)_i$ in $d_{r,\mathcal{P}}(E)$.

Theorem 5.3. *Let E be a directed graph, \mathcal{P} a partition of E^1 , $E_r(\mathcal{P})$ the in-split graph formed from E using \mathcal{P} and $d_{r,\mathcal{P}} : E^0 \cup E^1 \rightarrow \mathbf{N} \cup \{\infty\}$ the Drinen range-vector defined as above. Then $C^*(E_r(\mathcal{P})) \cong C^*(d_{r,\mathcal{P}}(E))$ if and only if \mathcal{P} is proper.*

Proof. Let $\{s_f, p_w : f \in d_{r,\mathcal{P}}(E)^1, w \in d_{r,\mathcal{P}}(E)^0\}$ be a Cuntz-Krieger $d_{r,\mathcal{P}}(E)$ -family. To simplify our definitions, for $v \in E^0$ we put $s_{f(v)_0} = p_{v_0}$. For $e \in E_r(\mathcal{P})^1$ we define $T_e = s_e$. For $e_j \in E_r(\mathcal{P})^1$ with $1 \leq j \leq m(s(e))$ we define

$$T_{e_j} = s_{f(s(e))_{j-1}} \cdots s_{f(s(e))_1} s_e.$$

For $1 \leq i \leq m(v)$, define $Q_{v_i} = p_{v_{i-1}}$; if $m(v) = 0$, define $Q_v = p_{v_0}$. Then $\{T_g, Q_u : g \in E_r(\mathcal{P})^1, u \in E_r(\mathcal{P})^0\}$ is a Cuntz-Krieger $E_r(\mathcal{P})$ -family with $Q_u \neq 0$ for all u .

Let $\{t_g, q_u\}$ be the canonical generators of $C^*(E_r(\mathcal{P}))$. By the universal property of $C^*(E_r(\mathcal{P}))$ there is a homomorphism $\pi : C^*(E_r(\mathcal{P})) \rightarrow C^*(d_{r,\mathcal{P}}(E))$ such that $\pi(t_g) = T_g$ and $\pi(q_u) = Q_u$. We claim that π is surjective, that is $\{T_g, Q_u\}$ generates $C^*(d_{r,\mathcal{P}}(E))$.

For $w \in d_{r,\mathcal{P}}(E)^0$ we have $p_w \in C^*(\{T_g, Q_u\})$ by definition. For $e \in E_r(\mathcal{P})^1$ we have $s_e = T_e \in C^*(\{T_g, Q_u\})$. Since \mathcal{P} is proper, by Remark 5.2 there are no edges in $d_{r,\mathcal{P}}(E)$ of the form $f(r(e))_j$ with $r(e)$ a sink. In particular, every edge $f(v)_j$ in $d_{r,\mathcal{P}}(E)$ is of the form $f(s(e))_j$ where $s(e)$ has finite valency. For $1 \leq j \leq m(s(e)) - 1 = d_{r,\mathcal{P}}(s(e))$

$$T_{e_{j+1}} T_{e_j}^* = s_{f(s(e))_j} \cdots s_{f(s(e))_1} s_e s_e^* s_{f(s(e))_1} \cdots s_{f(s(e))_{j-1}}$$

and since $v = s(e)$ has finite valency we have

$$\begin{aligned} \sum_{s(e)=v} T_{e_{j+1}} T_{e_j}^* &= s_{f(s(e))_j} \cdots s_{f(s(e))_1} \left(\sum_{s(e)=v_0} s_e s_e^* \right) s_{f(s(e))_1}^* \cdots s_{f(s(e))_{j-1}}^* \\ &= s_{f(s(e))_j} \cdots s_{f(s(e))_1} p_{v_0} s_{f(s(e))_1}^* \cdots s_{f(s(e))_{j-1}}^* = s_{f(s(e))_j}. \end{aligned}$$

Then $s_{f(s(e))_j} \in C^*(T_g, Q_u)$ and our claim follows.

For $z \in \mathbf{T}$ define an action α on $C^*(d_{r,\mathcal{P}}(E))$ by $\alpha_z(p_v) = p_v$ for $v \in d_{r,\mathcal{P}}(E)^0$, $\alpha_z(s_e) = z s_e$ for $e \in E_r(\mathcal{P})^1$, and $\alpha_z(s_{f(v)_i}) = s_{f(v)_i}$ for $1 \leq i \leq d_{r,\mathcal{P}}(v)$. Since $\gamma \circ \pi = \pi \circ \alpha$ where γ is the usual gauge action on $C^*(E_r(\mathcal{P}))$, by Theorem 2.1 $C^*(d_{r,\mathcal{P}}(E)) \cong C^*(E_r(\mathcal{P}))$.

If \mathcal{P} is not proper then there is a non-trivial in-splitting at a sink or a vertex of infinite valency. The graph $E_r(\mathcal{P})$ will have at least one more sink or vertex of infinite valency than $d_{r,\mathcal{P}}(E)$ and hence $C^*(E_r(\mathcal{P}))$ will have more ideals than $C^*(d_{r,\mathcal{P}}(E))$. \square

Applying Theorem 5.3 and Theorem 4.5 we have:

Corollary 5.4. *Let E be a directed graph, \mathcal{P} a partition of E^1 and $E_r(\mathcal{P})$ the in-split graph formed from E using \mathcal{P} , then $C^*(E_r(\mathcal{P}))$ is strongly Morita equivalent to $C^*(E)$ if and only if \mathcal{P} is proper.*

6. CONNECTIONS WITH STRONG SHIFT EQUIVALENCE

In [B1] the following definition (which generalises one given in [Ash]) was given for *elementary strong shift equivalence* of directed graphs which contain no sinks.

Definition 6.1. Let $E_i = (E_i^0, E_i^1, r^i, s^i)$ for $i = 1, 2$ be directed graphs. Suppose there is a directed graph $E_3 = (E_3^0, E_3^1, r_3, s_3)$ such that:

- a) $E_3^0 = E_1^0 \cup E_2^0$ and $E_1^0 \cap E_2^0 = \emptyset$.
- b) $E_3^1 = E_{12}^1 \cup E_{21}^1$ where $E_{ij}^1 := \{e \in E_3^1 : s_3(e) \in E_i^0, r_3(e) \in E_j^0\}$.
- c) For $i = 1, 2$ there are range and source-preserving bijections $\theta_i : E_i^1 \rightarrow E_3^2(E_i^0, E_i^0)$ where for $i \in \{1, 2\}$, $E_3^2(E_i^0, E_i^0) := \{\alpha \in E_3^2 : s_3(\alpha) \in E_i^0, r_3(\alpha) \in E_i^0\}$.

Then we say that E_1 and E_2 are *elementary strong shift equivalent* ($E_1 \sim_{ES} E_2$) via E_3 .

The equivalence relation \sim_S on directed graphs generated by elementary strong shift equivalence is called *strong shift equivalence*. Row-finite graphs which are strong shift equivalent have Morita equivalent C^* -algebras (see [B1, Theorem 5.2]).

Proposition 6.2. *Let E be a directed graph with no sinks and $E_s(\mathcal{P})$ be an out-split graph formed from E using \mathcal{P} . Then $E \sim_{ES} E_s(\mathcal{P})$.*

Proof. One constructs a bipartite graph E_3 in the following manner. Let $E_3^0 = E^0 \cup E_s(\mathcal{P})^0$. For each $v \in E^0$ draw an edge e_v^i to the corresponding split vertices $v^i \in E_s(\mathcal{P})^0$ with $s(e_v^i) = v$ and $r(e_v^i) = v^i$. For each set of edges $\{e^i\}_{i=1}^{m(r(e))} \subseteq E_s(\mathcal{P})^1$ with $s(e^i) = v^i$ and $r(e^i) \in \{w^j\}_{j=1}^{m(w)}$, draw an edge $e_{v,w}^i$ with $s(e_{v,w}^i) = v^i$ and $r(e_{v,w}^i) = w$. The graph E_3 satisfies the conditions of Definition 6.1 and hence $E \sim_{ES} E_s(\mathcal{P})$ via E_3 . \square

In a similar manner we may show:

Proposition 6.3. *Let E be a directed graph with no sinks and $E_r(\mathcal{P})$ be an in-split graph formed from E using \mathcal{P} . Then $E \sim_{ES} E_r(\mathcal{P})$.*

Remark 6.4. Proposition 6.3 and [B1, Theorem 5.2] enable us to give another proof that the C^* -algebras of a row-finite directed graph and its in-splitting are Morita equivalent. We have analogous results for in-amalgamations and out-amalgamations as they are the reverse operations of in-splittings and out-splittings.

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