C^*-ALGEBRAS OF LABELLED GRAPHS

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ABSTRACT. We describe a class of C^*-algebras which simultaneously generalise the ultragraph algebras of Tomforde and the shift space C^*-algebras of Matsumoto. In doing so we shed some new light on the different C^*-algebras that may be associated to a shift space. Finally, we show how to associate a simple C^*-algebra to an irreducible sofic shift.

KEYWORDS: C^*-algebras, labelled graph, ultragraph, Matsumoto algebra, shift space.

MSC (2000): 46L05, 37B10.

1. INTRODUCTION

The purpose of this paper is to introduce a class of C^*-algebras associated to labelled graphs. Our motivation is to provide a common framework for working with the ultragraph algebras of Tomforde (see [26, 27]) and the C^*-algebras associated to shift spaces studied by Matsumoto and Carlsen (see [14, 17, 6, 8] amongst others). Here a labelled graph $(E, \mathcal{L})$ over an alphabet $\mathcal{A}$ is a directed graph E, together with a map $\mathcal{L} : E^1 \to \mathcal{A}$. An ultragraph $\mathcal{G}$ is a particular example of a labelled graph (see Example 3.3 (ii)), and a shift space Λ has many presentations as a labelled graph (see [13], Example 3.3 (iii)). Hence it is natural to give our common framework in terms of labelled graphs.

To a two-sided shift space Λ over a finite alphabet, Matsumoto associates two C^*-algebras $\mathcal{O}_\Lambda$ and $\mathcal{O}_{\Lambda^*}$ generated by partial isometries (see [8]). Although $\mathcal{O}_\Lambda$ and $\mathcal{O}_{\Lambda^*}$ are generated by elements satisfying the same relations, it turns out that they are not isomorphic in general (see [8, Theorem 4.1]). This fact manifests itself in our realisation in section 6.2 of $\mathcal{O}_\Lambda$ and $\mathcal{O}_{\Lambda^*}$ as the C^*-algebras of the labelled graphs $(E_\Lambda, \mathcal{L}_\Lambda)$ and $(E_{\Lambda^*}, \mathcal{L}_{\Lambda^*})$ respectively, which are not necessarily isomorphic as labelled graphs. Moreover, in Corollary 6.9 we show that using labelled graphs gives us the facility to canonically associate a simple C^*-algebra to an irreducible sofic shift (cf. [8, 6, 7]).

In fact we can associate a number of (possibly different) C^*-algebras to a labelled graph. This leads us to the notion of a labelled space, which we describe in section 3. Briefly, a labelled space $(E, \mathcal{L}, \mathcal{B})$ consists of a labelled graph $(E, \mathcal{L})$ together with a collection $\mathcal{B} \subseteq 2^{E^0}$ which plays the same role as $\mathcal{G}^0$ in [26] and is related to the abelian AF-subalgebra A_Λ (resp. A_{Λ^*}) in $\mathcal{O}_\Lambda$ (resp. $\mathcal{O}_{\Lambda^*}$) generated by the source projections.

In section 4 we define a representation of a labelled space in terms of partial isometries $\{s_a : a \in \mathcal{A}\}$ and projections $\{p_A : A \in \mathcal{B}\}$ subject to certain relations. Our relations generalise those found in [26, 14]. In order to build a nondegenerate C^*-algebra from a representation of $(E, \mathcal{L}, \mathcal{B})$ it is necessary for $\mathcal{B}$ to be weakly left-resolving: a condition which is a generalisation of the left-resolving property for labelled graphs. Hence we may define $C^*(E, \mathcal{L}, \mathcal{B})$ to be the C^*-algebra which is universal for representations of the weakly left-resolving labelled space $(E, \mathcal{L}, \mathcal{B})$. Since any ultragraph has a natural realisation as a left-resolving labelled graph, the class of C^*-algebras of labelled spaces contains the ultragraph algebras (and hence, graph algebras and Exel-Laca algebras).

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In section 5 we give a version of the gauge invariant uniqueness theorem for $C^*(E, \mathcal{L}, \mathcal{B})$ which will ultimately allow us to make the connection with the Matsumoto algebras.

In section 6 we give three applications of our uniqueness theorem: In section 6.1 we show how to construct a dual labelled space, which is the analogue of the higher block presentation of a shift space (cf. [13]). We give an isomorphism theorem for dual labelled spaces which is a generalisation of [2, Corollary 2.5] and forms a starting point for future work (see [4]). In section 6.2 we show that if $\mathcal{O}_\Lambda$ (resp. $\mathcal{O}_{\Lambda^*}$) has a gauge action, then it is isomorphic to the C^*-algebra of a certain labelled space. Then in section 6.3 we give necessary conditions for the C^*-algebra of a labelled space to be isomorphic to the C^*-algebra of the underlying directed graph. We then show how to associate a simple C^*-algebra to an irreducible shift space. By example, we show that in general the C^*-algebra of a labelled space will not be isomorphic to the C^*-algebra of any directed graph; hence labelled graph C^*-algebras form a strictly larger class of C^*-algebras than graph algebras.

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Since we seek to generalise them, we begin by giving a brief description of Ultragraph algebras and Matsumoto algebras.

2. Ultragraph Algebras and Matsumoto Algebras

2.1. **Ultragraph Algebras.** An ultragraph $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ consists of a countable set of vertices G^0, a countable set of edges $\mathcal{G}^1$, and functions $s : \mathcal{G}^1 \to G^0$ and $r : \mathcal{G}^1 \to 2^{G^0}$. Let $\mathcal{G}^0$ be the smallest collection of 2^{G^0} which contains $s(e)$ and $r(e)$ for all $e \in \mathcal{G}^1$ and is closed under finite intersections and unions. The ultragraph algebra $C^*(\mathcal{G})$ is the universal C^*-algebra for Cuntz-Krieger $\mathcal{G}$-families: collections of partial isometries $\{s_e : e \in \mathcal{G}^1\}$ with mutually orthogonal ranges, and projections $\{p_A : A \in \mathcal{G}^0\}$ satisfying the relations

1. $p_\emptyset = 0$, $p_A p_B = p_{A\cap B}$ and $p_{A\cup B} = p_A + p_B - p_{A\cap B}$ for all $A, B \in \mathcal{G}^0$
2. $s_e^* s_e = p_{r(e)}$ and $s_e s_e^* \le p_{s(e)}$ for all $e \in \mathcal{G}^1$
3. $p_v = \sum_{s(e)=v} s_e s_e^*$ whenever $0 < |s^{-1}(v)| < \infty$

(see [26, Definition 2.7]). Recall that $v \in G^0$ is an infinite emitter if $|s^{-1}(v)| = \infty$.

If $\mathcal{G}$ has no infinite emitters, then the underlying graph (see Examples 3.3 (ii)) can still fail to be row-finite. With this in mind we make the following definition (cf. [26, Remark 2.6]):

Definition 2.1. The ultragraph $\mathcal{G}$ is *row-finite* if there are no infinite emitters and $r(e)$ is finite for all $e \in \mathcal{G}^1$.

Ultragraph algebras simultaneously generalise graph C^*-algebras and Exel-Laca algebras (see [26, Sections 3 and 4]). By [27, Corollary 5.5] there is a non row-finite ultragraph whose C^*-algebra is not isomorphic to a graph algebra or an Exel-Laca algebra.

2.2. **Matsumoto Algebras.** For an introduction to shift spaces we refer the reader to the excellent treatment in [13]. Let Λ be a two-sided shift space over a finite alphabet $\mathcal{A}$. Let

$$X_\Lambda = \{(x_i)_{i\ge 1} : (x_i)_{i\in\mathbf{Z}} \in \Lambda\} \tag{1}$$

denote the set of all right-infinite sequences in Λ.

For each $k \ge 1$, let Λ^k be the set of all words with length k appearing in some $x \in \Lambda$. We set $\Lambda_\ell = \bigcup_{k=0}^{\ell} \Lambda^k$ and $\Lambda^* = \bigcup_{k=0}^{\infty} \Lambda^k$ where Λ^0 denotes the empty word $\emptyset$.

Following [8] there are two C^*-algebras associated to Λ. Each C^*-algebra is generated by partial isometries $\{t_a : a \in \mathcal{A}\}$ subject to

$$\sum_{a\in\mathcal{A}} t_a t_a^* = 1, \text{ and } t_\alpha^* t_\alpha t_\beta = t_\beta t_{\alpha\beta}^* t_{\alpha\beta}, \text{ where } \alpha, \beta, \alpha\beta \in \Lambda^*. \tag{2}$$

As in [8] we denote by $\mathcal{O}_\Lambda$ the C^*-algebra defined directly on Hilbert space in [18, 19] and by $\mathcal{O}_{\Lambda^*}$ the C^*-algebra defined using the Fock space construction in [14, 17, 16, 15, 21]. Because of the different ways in which the relations (2) are realised it turns out that $\mathcal{O}_\Lambda$ and $\mathcal{O}_{\Lambda^*}$ are not isomorphic in general (see [8, Section 6]).

There is a uniqueness theorem for $\mathcal{O}_\Lambda$ (resp. $\mathcal{O}_{\Lambda^*}$) when Λ satisfies condition (I) (resp. condition (I^*)) given in [8, Section 4] (resp. [8, Section 3]).

Condition (I)**:** For $x \in X_\Lambda$ and $l \in \mathbf{N}$ put $\Lambda_l(x) = \{\mu \in \Lambda_l : \mu x \in X_\Lambda\}$. Two infinite paths $x, y \in X_\Lambda$ are *l-past equivalent* (written $x \sim_l y$) if $\Lambda_l(x) = \Lambda_l(y)$. The shift space X_Λ satisfies condition (I) if for any $l \in \mathbf{N}$ and $x \in X_\Lambda$ there exists $y \in X_\Lambda$ such that $y \neq x$, $y \sim_l x$.

Condition (I^*)**:** For $\omega \in \Lambda^*$ and $l \in \mathbf{N}$ we set $\Lambda_l(\omega) = \{\mu : |\mu| \leq l,\ \mu\omega \in \Lambda^*\}$. Two words $\mu, \nu \in \Lambda^*$ are said to be *l-past equivalent* (written $\mu \sim_l \nu$) if $\Lambda_l(\mu) = \Lambda_l(\nu)$. The subset $\Lambda_l^* \subseteq \Lambda^*$ is defined by

$$\Lambda_l^* := \{\omega \in \Lambda^* : |\{\mu \in \Lambda^* : \mu \sim_l \omega\}| < \infty\}.$$

The shift space Λ satisfies condition (I^*) if for every $l \in \mathbf{N}$ and $\mu \in \Lambda_l^*$ there exist distinct words $\xi_1, \xi_2 \in \Lambda^*$ with $|\xi_1| = |\xi_2| = m$ such that

$$\mu \sim_l \xi_1\gamma_1 \text{ and } \mu \sim_l \xi_2\gamma_2$$

for some $\gamma_1, \gamma_2 \in \Lambda_{l+m}^*$.

Proposition 2.2. *Let Λ be a two-sided shift space over a finite alphabet which satisfies condition (I). Then there is a strongly continuous action β of $\mathbf{T}$ on $\mathcal{O}_\Lambda$ such that $\beta_z(t_a) = zt_a$ for all $a \in \mathcal{A}$ and $z \in \mathbf{T}$.*

Proof. That each β_z is an automorphism of $\mathcal{O}_\Lambda$ for each $z \in \mathbf{T}$ follows from [8, Proposition 4.2]. A standard $\frac{\epsilon}{3}$ argument shows that β is strongly continuous. □

By [14, p. 363] there is always a gauge action on $\mathcal{O}_{\Lambda^*}$. In [21] Matsumoto defines λ-graph systems $\mathcal{L}_\Lambda$ and $\mathcal{L}_{\Lambda^*}$ associated to a two-sided shift space Λ together with corresponding C^*-algebras $\mathcal{O}_{\mathcal{L}_\Lambda}$ and $\mathcal{O}_{\mathcal{L}_{\Lambda^*}}$. By [8, Theorem 5.6] we see that if Λ satisfies condition (I) then $\mathcal{O}_\Lambda \cong \mathcal{O}_{\mathcal{L}_\Lambda}$ and if Λ satisfies condition (I^*) then $\mathcal{O}_{\Lambda^*} \cong \mathcal{O}_{\mathcal{L}_{\Lambda^*}}$. Hence, for our purposes, it suffices to work with $\mathcal{O}_\Lambda$ and $\mathcal{O}_{\Lambda^*}$.

3. Labelled spaces

A directed graph E consists of a quadruple (E^0, E^1, r, s) where E^0 and E^1 are countable sets of vertices and edges respectively and $r, s : E^1 \to E^0$ are maps giving the direction of each edge. A path $\lambda = e_1 \ldots e_n$ is a sequence of edges $e_i \in E^1$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, \ldots, n-1$. The collection of paths of length n in E is denoted E^n and the collection of all finite paths in E by E^*, so that $E^* = \cup_{n\geq 0} E^n$. The edge shift (X_E, σ_E) associated to a directed graph E with no sinks or sources is defined by:

$$\mathsf{X}_E = \{x \in (E^1)^{\mathbf{Z}} : s(x_{i+1}) = r(x_i) \text{ for all } i \in \mathbf{Z}\} \text{ and } (\sigma_E x)_i = x_{i+1} \text{ for } i \in \mathbf{Z}.$$

The following definition is adapted from [13, Definition 3.1.1]:

Definition 3.1. A *labelled graph* $(E, \mathcal{L})$ over an alphabet $\mathcal{A}$ consists of a directed graph E together with a labelling map $\mathcal{L} : E^1 \to \mathcal{A}$.

Without loss of generality we may assume that the map $\mathcal{L}$ is onto. We say that the labelled graph $(E, \mathcal{L})$ is *row-finite* if the underlying graph E is row-finite.

Given a labelled graph $(E, \mathcal{L})$ such that every vertex in E emits and receives an edge, we may define a subshift $(\mathsf{X}_{(E,\mathcal{L})}, \sigma)$ of $\mathcal{A}^{\mathbf{Z}}$ by

$$\mathsf{X}_{(E,\mathcal{L})} = \{y \in \mathcal{A}^{\mathbf{Z}} : \text{ there exists } x \in \mathsf{X}_E \text{ such that } y_i = \mathcal{L}(x_i) \text{ for all } i \in \mathbf{Z}\},$$

where σ is the shift map. The labelled graph $(E, \mathcal{L})$ is said to be a *presentation* of the shift space $X = \mathsf{X}_{(E,\mathcal{L})}$. As shown in [13, §3.1] a shift space may have many different presentations (see Examples 3.3 (ii), (vi), (vii)).

Let $\mathcal{A}^*$ be the collection of all *words* in the symbols of $\mathcal{A}$ (see [25, §0.2]). The map $\mathcal{L}$ extends naturally to a map $\mathcal{L} : E^n \to \mathcal{A}^*$, where $n \geq 1$: for $\lambda = e_1 \ldots e_n \in E^n$ put $\mathcal{L}(\lambda) = \mathcal{L}(e_1) \ldots \mathcal{L}(e_n)$; in this case the path $\lambda \in E^n$ is said to be a *representative* of the *labelled path* $\mathcal{L}(e_1) \ldots \mathcal{L}(e_n)$. Let $\mathcal{L}(E^n)$ denote the collection of all labelled paths in $(E, \mathcal{L})$ of length n, then $\mathcal{L}^*(E) = \cup_{n \geq 1} \mathcal{L}(E^n)$ denotes the collection of all words in the alphabet $\mathcal{A}$ which may be represented by paths in the labelled graph $(E, \mathcal{L})$. In this way $\mathcal{L}$ induces a map from the language $\cup_{n \geq 1} E^n$ of the subshift of finite type X_E associated to E into $\mathcal{L}^*(E)$, the language of the shift space $\mathsf{X}_{(E,\mathcal{L})}$ presented by $(E, \mathcal{L})$ (see [13, §3]). The usual length function $| \cdot | : E^* \to \mathbf{N}$ transfers naturally over to $\mathcal{L}^*(E)$.

For α in $\mathcal{L}^*(E)$ we put

$$s_{\mathcal{L}}(\alpha) = \{s(\lambda) \in E^0 : \mathcal{L}(\lambda) = \alpha\} \text{ and } r_{\mathcal{L}}(\alpha) = \{r(\lambda) \in E^0 : \mathcal{L}(\lambda) = \alpha\},$$

so that $r_{\mathcal{L}}, s_{\mathcal{L}} : \mathcal{L}^*(E) \to 2^{E^0}$. We shall drop the subscript on $r_{\mathcal{L}}$ and $s_{\mathcal{L}}$ if the context in which it is being used is clear. For $\alpha, \beta \in \mathcal{L}^*(E)$ we have $\alpha\beta \in \mathcal{L}^*(E)$ if and only if $r(\alpha) \cap s(\beta) \neq \emptyset$.

Where possible we shall denote the elements of $\mathcal{A} = \mathcal{L}(E^1)$ as a, b, etc., elements of $\mathcal{L}^*(E)$ as α, β, etc., leaving e, f for elements of E^1 and λ, μ for elements of E^*.

Let $(E, \mathcal{L})$ and $(F, \mathcal{L}')$ be graphs labelled by the same alphabet. A graph isomorphism $\phi : E \to F$ is a *labelled graph isomorphism* if $\mathcal{L}'(\phi(e)) = \mathcal{L}(e)$ for all $e \in E^1$ and we write $\phi : (E, \mathcal{L}) \to (F, \mathcal{L}')$.

Definition 3.2. The labelled graph $(E, \mathcal{L})$ is *left-resolving* if for all $v \in E^0$ the map $\mathcal{L} : r^{-1}(v) \to \mathcal{A}$ is injective.

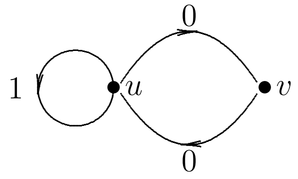
The left-resolving condition ensures that for all $v \in E^0$ the labels $\{\mathcal{L}(e) : r(e) = v\}$ of all incoming edges to v are all different. In particular if $\lambda, \mu \in \cup_{n \geq 1} E^n$ satisfy $\mathcal{L}(\lambda) = \mathcal{L}(\mu)$ and $r(\lambda) = r(\mu)$ then $\lambda = \mu$.

Examples 3.3. (i) Let E be a directed graph. Put $\mathcal{A} = E^1$ and let $\mathcal{L} : E^1 \to E^1$ be the identity map (the *trivial labelling*) then $(E, \mathcal{L})$ is a left-resolving labelled graph.

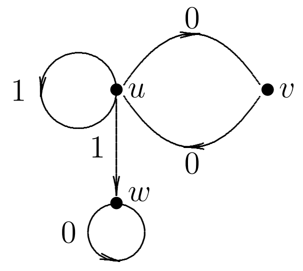
(ii) Let $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ be an ultragraph. Define $E = E_{\mathcal{G}}$ by putting $E^0 = G^0$, $E^1 = \{(e, w) : e \in \mathcal{G}^1, w \in r(e)\}$ and defining $r', s' : E^1 \to E^0$ by $s'(e, w) = s(e)$, $r'(e, w) = w$. Set $\mathcal{A} = \mathcal{G}^1$ and define $\mathcal{L}_{\mathcal{G}} : E^1 \to \mathcal{A}$ by $\mathcal{L}_{\mathcal{G}}(e, w) = e$. The resulting labelled graph $(E_{\mathcal{G}}, \mathcal{L}_{\mathcal{G}})$ is left-resolving since the source map is single-valued. If $\mathcal{G}$ is row-finite in the sense of Definition 2.1 then $E_{\mathcal{G}}$ is row-finite.

Conversely, given a left-resolving labelled graph $(E, \mathcal{L})$ over an alphabet $\mathcal{A}$ where $s_{\mathcal{L}} : \mathcal{L}^*(E) \to 2^{E^0}$ is single-valued, we can form a ultragraph $\mathcal{G}_{(E,\mathcal{L})} = (E^0, \mathcal{A}, r', s')$ with $s' = s_{\mathcal{L}}$ and $r' = r_{\mathcal{L}}$. If $(E, \mathcal{L})$ is row-finite then the ultragraph $\mathcal{G}_{(E,\mathcal{L})}$ is row-finite.

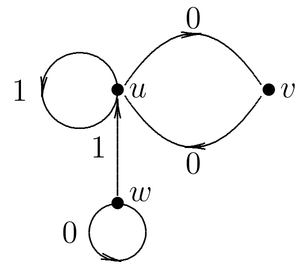
(iii) Following [13, §3] the labelled graphs $(E_1, \mathcal{L}_1) :=$



$(E_2, \mathcal{L}_2) :=$



$(E_3, \mathcal{L}_3) :=$



have the same language as the even shift Y since between any two 1's there must be an even number of 0's. Hence $\mathsf{X}_{(E_i,\mathcal{L}_i)} = Y$ for $i = 1, 2, 3$ by [13, Proposition 1.3.4 (3)]. Only graphs $(E_1, \mathcal{L}_1)$ and $(E_2, \mathcal{L}_2)$ are left-resolving.

(iv) Let E be a directed graph and Γ a group which acts on (the right of) E. Define $\mathcal{L}_q : E^1 \to E^1/\Gamma$ by $\mathcal{L}_q(e) = q(e)$ where $q : E^1 \to E^1/\Gamma$ is the quotient map. If the action of Γ is free on E^1, then the resulting labelled graph $(E, \mathcal{L}_q)$ is left-resolving. More generally, if $p : F \to E$ is a graph morphism then there is a labelling $\mathcal{L}_p : F^1 \to E^1$ given by $\mathcal{L}_p(f) = p(f)$ for all $f \in E^1$. If p is a covering map then $\mathcal{L}_p$ is left-resolving.

(v) Recall from [3, §3], that an out-splitting of a directed graph E is formed by a partition $\mathcal{P}$ of $s^{-1}(v)$ into $m(v) \geq 1$ non-empty subsets for each $v \in E^0$ (if $s^{-1}(v) = \emptyset$ then $m(v) = 0$). Given such a partition $\mathcal{P}$ one may construct a directed graph $E_s(\mathcal{P})$ where $E_s(\mathcal{P}^1) = \{e^j : e \in E^1, 1 \leq j \leq m(r(e))\} \cup \{e : m(r(e)) = 0\}$. Define $\mathcal{L} : E_s(\mathcal{P})^1 \to E^1$ by $\mathcal{L}(e^j) = e$ for $1 \leq j \leq m(r(e))$ and $\mathcal{L}(e) = e$ if $m(r(e)) = 0$. For an in-splitting (see [3, §5]) of E using a partition $\mathcal{P}$, a similar construction also yields a labelled graph. However the resulting labelling $\mathcal{L}$ of the in-split graph $E_r(\mathcal{P})$ will not be left-resolving in general.

(vi) Let Λ be a two-sided shift space over a finite alphabet $\mathcal{A}$ with X_Λ defined as in (1). Let $X_\Lambda^- = \{(x_i)_{i \leq 0} : (x_i)_{i \in \mathbf{Z}} \in \Lambda\}$ so that any element $x \in \Lambda$ may be written as $x = x^- x^+$. For arbitrary $x^+ \in X_\Lambda$ and $x^- \in X_\Lambda^-$ the bi-infinite sequence $y = x^- x^+$ may not belong to Λ. Define the past set of $t \in X_\Lambda$ as

$$P_\infty(t) = \{x^- \in X_\Lambda^- \; : \; x^- t \in \Lambda\}.$$

A shift is *sofic* if and only if the number of past sets is finite [11, 13].

For $s, t \in X_\Lambda$, we say that s is *past equivalent* to t (denoted $s \sim_\infty t$) if $P_\infty(s) = P_\infty(t)$. Define a labelled graph $(E_\Lambda, \mathcal{L}_\Lambda)$ as follows: let $E_\Lambda^0 = \{[v] : v \in X_\Lambda / \sim_\infty\}$, $E_\Lambda^1 = \{([v], a, [w]) : a \in \mathcal{A}, aw \sim_\infty v\}$ with $s([v], a, [w]) = [v]$ and $r([v], a, [w]) = [w]$. If $([v], a, [w]) \in E_\Lambda^1$ we put $\mathcal{L}_\Lambda([v], a, [w]) = a$. The resulting left-resolving labelled graph is usually referred to as the *left-Krieger cover* of Λ and the construction is evidently independent of the choice of representatives (see [11]).

If Y is the even shift then $(E_Y, \mathcal{L}_Y)$ is labelled graph isomorphic to $(E_2, \mathcal{L}_2)$ in (iii) above. Let Z be shift over the alphabet $\{1, 2, 3, 4\}$ in which the words

$$\{12^k 1, 32^k 12, 32^k 13, 42^k 14 : k \geq 0\}$$

do not occur (see [8, §4]) then $(E_Z, \mathcal{L}_Z)$ has six vertices.

(vii) Let Λ be a two-sided shift over a finite alphabet $\mathcal{A}$. We construct a variant of the predecessor graph $(E_{\Lambda^*}, \mathcal{L}_{\Lambda^*})$ in the following way. For $\mu \in \Lambda^*$ we define

$$P(\mu) := \{\lambda : \lambda\mu \in \Lambda^*\}$$

and define an equivalence relation by $\mu \sim \nu$ if $P(\mu) = P(\nu)$. A shift is sofic if and only if the number of predecessor sets is finite [13].

Let Λ^*_∞ denote those $\mu \in \Lambda^*$ which have an infinite equivalence class. Since $\mathcal{A}$ is finite $\Lambda^*_\infty/\sim$ can be identified with $\Omega_{\Lambda^*} = \lim_{\leftarrow} \Omega^*_l$ as described in [17, Section 2]. We set $E^0_{\Lambda^*} = \Lambda^*_\infty/\sim$, $E^1_{\Lambda^*} = \{([\mu], a, [\nu]) : a \in \mathcal{A}, [\mu] = [a\nu]\}$, $r([\mu], a, [\nu]) = [\nu]$ and $s([\mu], a, [\nu]) = [\mu]$. The labelling map is defined by $\mathcal{L}_{\Lambda^*}([\mu], a, [\nu]) = a$. The resulting labelled graph is evidently left-resolving.

If Y is the even shift then $(E_{Y^*}, \mathcal{L}_{Y^*})$ is labelled graph isomorphic to $(E_2, \mathcal{L}_2)$ in (iii) above (cf. [6, 17]). If Z is the sofic shift described in Example 3.3 (vi) then $(E_{Z^*}, \mathcal{L}_{Z^*})$ has seven vertices and contains $(E_Z, \mathcal{L}_Z)$ as a subgraph.

Definition 3.4. Let $(E, \mathcal{L})$ be a labelled graph. For $A \subseteq E^0$ and $\alpha \in \mathcal{L}^*(E)$ the *relative range of α with respect to A* is defined to be

$$r(A, \alpha) = \{r(\lambda) : \lambda \in E^*, \mathcal{L}(\lambda) = \alpha, s(\lambda) \in A\}.$$

Remark 3.5. For any $A, B \subseteq E^0$ and $\alpha \in \mathcal{L}^*(E)$ we have

$$r(A \cap B, \alpha) \subseteq r(A, \alpha) \cap r(B, \alpha) \text{ and } r(A \cup B, \alpha) = r(A, \alpha) \cup r(B, \alpha).$$

For all $A \subseteq E^0$ and $\alpha \in \mathcal{L}^*(E)$ we have $r(A, \alpha) = r(A \cap s(\alpha), \alpha)$.

A collection $\mathcal{B} \subseteq 2^{E^0}$ of subsets of E^0 is said to be *closed under relative ranges for* $(E, \mathcal{L})$ if for all $A \in \mathcal{B}$ and $\alpha \in \mathcal{L}^*(E)$ we have $r(A, \alpha) \in \mathcal{B}$. If $\mathcal{B}$ is closed under relative ranges for $(E, \mathcal{L})$, contains $r(\alpha)$ for all $\alpha \in \mathcal{L}^*(E)$ and is also closed under finite intersections and unions, then we say that $\mathcal{B}$ is *accommodating* for $(E, \mathcal{L})$.

Definition 3.6. A *labelled space* consists of a triple $(E, \mathcal{L}, \mathcal{B})$, where $(E, \mathcal{L})$ is a labelled graph and $\mathcal{B}$ is accommodating for $(E, \mathcal{L})$.

Definition 3.7. A labelled space $(E, \mathcal{L}, \mathcal{B})$ is *weakly left-resolving* if for every $A, B \in \mathcal{B}$ and every $\alpha \in \mathcal{L}^*(E)$ we have $r(A, \alpha) \cap r(B, \alpha) = r(A \cap B, \alpha)$.

In particular $(E, \mathcal{L}, \mathcal{B})$ is weakly left-resolving if no pair of disjoint sets $A, B \in \mathcal{B}$ can emit paths λ, μ respectively with $\mathcal{L}(\lambda) = \mathcal{L}(\mu)$ and $r(\lambda) = r(\mu)$. If $(E, \mathcal{L})$ is left-resolving then $(E, \mathcal{L}, \mathcal{B})$ is weakly left-resolving for any $\mathcal{B}$. Evidently if $(E, \mathcal{L}, \mathcal{B})$ is weakly left-resolving, then $(E, \mathcal{L}, \mathcal{B}')$ is weakly left-resolving for any $\mathcal{B}' \subseteq \mathcal{B}$.

Consider the following subsets of 2^{E^0}

$$\mathcal{E} = \{\{v\} : v \in E^0 \text{ is a source or a sink }\} \cup \{r(\alpha) : \alpha \in \mathcal{L}^*(E)\} \cup \{s(\alpha) : \alpha \in \mathcal{L}^*(E)\}$$
$$\mathcal{E}^- = \{\{v\} : v \in E^0 \text{ is a sink }\} \cup \{r(\alpha) : \alpha \in \mathcal{L}^*(E)\}.$$

The following definition is analogous to the definition of $\mathcal{G}^0$ in [26].

Definition 3.8. Let $\mathcal{E}^0$ (resp. $\mathcal{E}^{0,-}$) denote the smallest subset of 2^{E^0} containing $\mathcal{E}$ (resp. $\mathcal{E}^-$) which is accommodating for $(E, \mathcal{L})$.

Remark 3.9. If $\alpha, \beta \in \mathcal{L}^*(E)$ are such that $\alpha\beta \in \mathcal{L}^*(E)$ then

$$r(s(\alpha), \alpha\beta) = r(\alpha\beta) \text{ and } r(r(\alpha), \beta) = r(\alpha\beta).$$

For $\alpha, \beta \in \mathcal{L}^*(E)$ with $\alpha\beta \in \mathcal{L}^*(E)$ and $A \subseteq E^0$ we have $r(r(A, \alpha), \beta) = r(A, \alpha\beta)$.

For labelled spaces $(E,\mathcal{L},\mathcal{E}^0)$ which are weakly left-resolving Remark 3.5 and Remark 3.9 show that to form $\mathcal{E}^0$ it suffices to form

$$\mathcal{E}\cup\{r(A,\alpha) : A\in\mathcal{E},\alpha\in\mathcal{L}^*(E)\}$$

and then close under finite intersections and unions. To form $\mathcal{E}^{0,-}$, by Remark 3.5 it suffices to close $\mathcal{E}^-$ under finite intersections and unions. Evidently, $\mathcal{E}^{0,-}\subseteq\mathcal{E}^0$; the containment can be strict, for instance this occurs when E has sources. One can show that $\mathcal{E}^0=\mathcal{E}^{0,-}$ if and only if for every $\alpha\in\mathcal{L}^*(E)$, $s(\alpha)$ can be written as a finite union of sets of the form $\cap_{i=1}^n r(\beta_i)$. Since E^0, $\mathcal{L}^*(E)$ and $\mathcal{E}$ are countable it follows that $\mathcal{E}^0$ and $\mathcal{E}^{0,-}$ are countable.

For $A\in 2^{E^0}$ and $n\geq 1$ let

$$L_A^n=\{\alpha\in\mathcal{L}(E^n) : A\cap s(\alpha)\neq\emptyset\}$$

denote those labelled paths of length n whose source intersects A nontrivially.

4. C^*-ALGEBRAS OF LABELLED SPACES

Definition 4.1. Let $(E,\mathcal{L},\mathcal{B})$ be a weakly left-resolving labelled space. A representation of $(E,\mathcal{L},\mathcal{B})$ consists of projections $\{p_A : A\in\mathcal{B}\}$ and partial isometries $\{s_a : a\in\mathcal{L}(E^1)\}$ with the properties that

(i) If $A,B\in\mathcal{B}$ then $p_Ap_B=p_{A\cap B}$ and $p_{A\cup B}=p_A+p_B-p_{A\cap B}$, where $p_\emptyset=0$.
(ii) If $a\in\mathcal{L}(E^1)$ and $A\in\mathcal{B}$ then $p_As_a=s_ap_{r(A,a)}$.
(iii) If $a,b\in\mathcal{L}(E^1)$ then $s_a^*s_a=p_{r(a)}$ and $s_a^*s_b=0$ unless $a=b$.
(iv) For $A\in\mathcal{B}$, if L_A^1 is finite and non-empty we have

$$p_A=\sum_{a\in L_A^1}s_ap_{r(A,a)}s_a^*. \tag{3}$$

If $a,b\in\mathcal{L}(E^1)$ are such that $ab\in\mathcal{L}^*(E)$ then we have

$$(s_a^*s_a)(s_bs_b^*)=p_{r(a)}s_bs_b^*=s_bp_{r(r(a),b)}s_b^*=s_bs_b^*p_{r(a)}=(s_bs_b^*)(s_a^*s_a).$$

Hence s_as_b is a partial isometry which is nonzero if and only if s_a and s_b are. Therefore we may define $s_{ab}=s_as_b$ and similarly define s_α for all $\alpha\in\mathcal{L}^*(E)$. One checks that Definition 4.1 (ii) holds for $\alpha\in\mathcal{L}^*(E)$, Definition 4.1 (iii) holds for $\alpha,\beta\in\mathcal{L}(E^n)$ for $n\geq 1$ and Definition 4.1 (iv) holds for $A\in\mathcal{B}$ with finite and nonempty L_A^n for $n\geq 1$. Then (cf. (2)) we have

$$s_\alpha^*s_\alpha s_\beta=p_{r(\alpha)}s_\beta=s_\beta p_{r(r(\alpha),\beta))}=s_\beta p_{r(\alpha\beta)}=s_\beta s_{\alpha\beta}^*s_{\alpha\beta}. \tag{4}$$

To justify the requirement that $(E,\mathcal{L},\mathcal{B})$ is weakly left-resolving in Definitions 4.1, consider the following: Let $\{p_A,s_a\}$ be a representation of $(E,\mathcal{L},\mathcal{B})$ in which $p_A\neq 0$ for all $A\in\mathcal{B}$. By Definition 4.1 (i) we have $(p_A-p_{A\cap B})(p_B-p_{A\cap B})=0$ for all $A,B\in\mathcal{B}$. Suppose, for contradiction, that there is $\alpha\in\mathcal{L}^*(E)$ such that $r(A,\alpha)\cap r(B,\alpha)\neq r(A\cap B,\alpha)$. From Definition 4.1 (iv) we have

$$p_A-p_{A\cap B}\geq s_\alpha\left(p_{r(A,\alpha)}-p_{r(A\cap B,\alpha)}\right)s_\alpha^* \text{ and } p_B-p_{A\cap B}\geq s_\alpha\left(p_{r(B,\alpha)}-p_{r(A\cap B,\alpha)}\right)s_\alpha^*$$

so $(p_A-p_{A\cap B})(p_B-p_{A\cap B})\neq 0$, a contradiction. Thus a representation of $(E,\mathcal{L},\mathcal{B})$ will be degenerate if $(E,\mathcal{L},\mathcal{B})$ is not weakly left-resolving.

Relation (iv) in Definition 4.1 can make sense even if $A\in\mathcal{B}$ emits infinitely many edges in E: If there are only finitely many different labels attached to the edges which A emits then L_A^1 is finite. For directed graphs the analogue of equation (3) holds when a vertex has finite valency; when this is true at every vertex, the graph is called row-finite. With this in mind, we make the following definition:

Definition 4.2. Let $(E,\mathcal{L},\mathcal{B})$ be a labelled space. We say that $A \in \mathcal{B}$ is *singular* if L^1_A is infinite. If no set $A \in \mathcal{B}$ is singular we say that $(E,\mathcal{L},\mathcal{B})$ is *set-finite.*

If $(E,\mathcal{L},\mathcal{B})$ is set-finite, then L^n_A is finite for all $A \in \mathcal{B}$ and all $n \geq 1$. In the examples below, the resulting labelled space will be set-finite whenever the original graph is row-finite.

Examples 4.3.
- (i) Let E be a directed graph with the trivial labelling $\mathcal{L}$. Then $\mathcal{E}^0$ consists of all the finite subsets of E^0. If E is row-finite then $(E,\mathcal{L},\mathcal{E}^0)$ and $(E,\mathcal{L},\mathcal{E}^{0,-})$ are set-finite. One may show that a representation of $(E,\mathcal{L},\mathcal{E}^0)$ is a Cuntz-Krieger E-family and conversely (see [1, 2] for instance). If all sources in E have finite valency, then the $*$-algebra generated by a representation of $(E,\mathcal{L},\mathcal{E}^{0,-})$ contains a representation of $(E,\mathcal{L},\mathcal{E}^0)$. If there is a source $v \in E^0$ with infinite valency then there is no representative of p_v in the $*$-algebra generated by a representation of $(E,\mathcal{L},\mathcal{E}^{0,-})$.
- (ii) Under the identification of an ultragraph $\mathcal{G}$ with a labelled graph $(E_{\mathcal{G}},\mathcal{L}_{\mathcal{G}})$ we have $\mathcal{E}^0_{\mathcal{G}} = \mathcal{G}^0$. Since $\mathcal{A} = \mathcal{G}^1$ a representation of $(E_{\mathcal{G}},\mathcal{L}_{\mathcal{G}},\mathcal{E}^0_{\mathcal{G}})$ is a Cuntz-Krieger $\mathcal{G}$-family (see [26, Definition 2.7]). If $\mathcal{G}$ has sources which are singular then we get similar behaviour to that described in (i) above.
- (iii) In Examples 3.3 (iii) we have $\mathcal{E}^0_i = 2^{E^0_i}$ for $i = 1,2,3$. However $\mathcal{E}^{0,-}_1 = 2^{E^0_1}$, $\mathcal{E}^{0,-}_2 = \{\{w\},\{u,w\},\{v,w\},\{u,v,w\}\}$ and $\mathcal{E}^{0,-}_3 = \{\emptyset,\{u\},\{v\},\{u,v\},\{u,v,w\}\}$. A representation of $(E_2,\mathcal{L}_2,\mathcal{E}^{0,-}_2)$, is generated by partial isometries s_0, s_1 satisfying the relations in [14, Proposition 8.3] and [6, §2] for $\mathcal{O}_Y$, where Y is the even shift.
- (iv) A covering $p : F \to E$ of directed graphs yields a labelling $\mathcal{L}_p : F^1 \to E^1$. We may identify $\mathcal{F}^0$ with the collection of inverse images of the finite subsets of E^0. A representation of $(F,\mathcal{L}_p,\mathcal{F}^0)$ is a Cuntz-Krieger E-family. If F has sources with infinite valency, then we get similar behaviour to that described in (i) above.
- (v) An outsplitting $E_s(\mathcal{P})$ of E gives rise to a labelling $\mathcal{L} : E_s(\mathcal{P})^1 \to E^1$. If $\mathcal{P}$ is proper then we may identify $\mathcal{E}_s(\mathcal{P})^0$ with the collection of finite subsets of E^0, and a representation of $(E_s(\mathcal{P}),\mathcal{L},\mathcal{E}_s(\mathcal{P})^0)$ is a Cuntz-Krieger E-family. If E has sources with infinite valency then, we get similar behaviour to that described in (i) above, even when the outsplitting is proper.
- (vi) An arbitrary shift $\Lambda \subseteq \mathcal{A}^{\mathbf{Z}}$ gives rise to a left-resolving labelled graph $(E_\Lambda,\mathcal{L}_\Lambda)$ with no sources or sinks. If $\mathcal{A}$ is finite then the generators of $\mathcal{O}_\Lambda$ form a representation of $(E_\Lambda,\mathcal{L}_\Lambda,\mathcal{E}^{0,-}_\Lambda)$ (cf. [8, 14]).
- (vii) An arbitrary shift $\Lambda \subseteq \mathcal{A}^{\mathbf{Z}}$ gives rise to a left-resolving labelled graph $(E_{\Lambda^*},\mathcal{L}_{\Lambda^*})$ with no sources or sinks. If $\mathcal{A}$ is finite then the generators of $\mathcal{O}_{\Lambda^*}$ form a representation of $(E_{\Lambda^*},\mathcal{L}_{\Lambda^*},\mathcal{E}^{0,-}_{\Lambda^*})$ (cf. [8, 14]).

Examples 4.3 (i)-(v) show that it is possible for $\mathcal{E}^0$ and $\mathcal{E}^{0,-}$ to be different, but for the $*$-algebras generated by representations of $(E,\mathcal{L},\mathcal{E}^0)$ and $(E,\mathcal{L},\mathcal{E}^{0,-})$ to be the same.

Let $(E,\mathcal{L},\mathcal{B})$ be a labelled space. Let $\mathcal{B}^* = \mathcal{L}^*(E) \cup \mathcal{B}$ and extend r, s to $\mathcal{B}^*$ by $r(A) = A$, $s(A) = A$ for all $A \in \mathcal{B}$. For $A \in \mathcal{B}$, put $s_A = p_A$, so s_β is defined for all $\beta \in \mathcal{B}^*$.

Lemma 4.4. *Let $(E,\mathcal{L},\mathcal{B})$ be a weakly left-resolving labelled space and $\{s_a, p_A\}$ a representation of $(E,\mathcal{L},\mathcal{B})$. Then any nonzero product of s_a, p_A and s^*_b can be written as a finite combination of elements of the form $s_\alpha p_A s^*_\beta$ for some $A \in \mathcal{B}$, and $\alpha,\beta \in \mathcal{B}^*$ satisfying $A \subseteq r(\alpha) \cap r(\beta) \neq \emptyset$.*

Proof. Since $s_\alpha p_A s^*_\beta = s_\alpha p_{r(\alpha)\cap A\cap r(\beta)} s^*_\beta$ it follows that $s_\alpha p_A s^*_\beta$ is zero unless $A \cap r(\alpha) \cap r(\beta) \neq \emptyset$ and without loss of generality we may assume that $A \subseteq r(\alpha) \cap r(\beta)$. For

$\alpha, \beta, \gamma, \delta \in \mathcal{L}^*(E)$ and $A, B \in \mathcal{B}$ we have

$$(s_\alpha p_A s_\beta^*)(s_\gamma p_B s_\delta^*) = \begin{cases} s_{\alpha\gamma'} p_{r(A,\gamma')\cap B} s_\delta^* & \text{if } \gamma = \beta\gamma' \\ s_\alpha p_{A\cap r(B,\beta')} s_{\delta\beta'}^* & \text{if } \beta = \gamma\beta' \\ s_\alpha p_{A\cap B} s_\delta^* & \text{if } \beta = \gamma \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

To see this, suppose $\gamma = \beta\gamma'$ then as $A \subseteq r(\beta) \cap r(\alpha)$

$$\begin{aligned} s_\alpha p_A s_\beta^* s_\gamma p_B s_\delta^* &= s_\alpha p_A s_\beta^* s_\beta s_{\gamma'} p_B s_\delta^* = s_\alpha p_A p_{r(\beta)} s_{\gamma'} p_B s_\delta^* \\ &= s_\alpha p_A s_{\gamma'} p_B s_\delta^* = s_{\alpha\gamma'} p_{r(A,\gamma')\cap B} s_\delta^* \end{aligned}$$

A similar calculation gives the desired formulas in the cases $\beta = \gamma\beta'$ and $\beta = \gamma$. If β and γ have no common initial segment, then without loss of generality, assume that $\beta \in \mathcal{L}(E^n)$ and $\gamma \in \mathcal{L}(E^m)$ with $n > m$. Write $\beta = \beta'\beta''$ where $\beta' \in \mathcal{L}(E^m)$, and then by Definition 4.1(iv) we have $s_\beta^* s_\gamma = s_{\beta''}^* s_{\beta'}^* s_\gamma = 0$ since $\beta' \neq \gamma$ and so $s_\alpha p_A s_\beta^* s_\gamma p_A s_\delta^* = 0$. By Definition 4.1 (i) and (ii) we may extend (5) to the case when $\alpha, \beta, \gamma, \delta \in \mathcal{B}^*$. ☐

Theorem 4.5. *Let $(E, \mathcal{L}, \mathcal{B})$ be a weakly left-resolving labelled space. There exists a C^*-algebra B generated by a universal representation of $\{s_a, p_A\}$ of $(E, \mathcal{L}, \mathcal{B})$. Furthermore the s_a's are nonzero and every p_A with $A \neq \emptyset$ is nonzero.*

Proof. Let $S_{(E,\mathcal{L},\mathcal{B})} := \{(\alpha, A, \beta) : \alpha, \beta \in \mathcal{B}^*, A \in \mathcal{B}, A \subseteq r(\alpha) \cap r(\beta)\}$ and let $k_{(E,\mathcal{L},\mathcal{B})}$ be the space of functions of finite support on $S_{(E,\mathcal{L},\mathcal{B})}$. The set of point masses $\{e_\tau : \tau \in S_{(E,\mathcal{L},\mathcal{B})}\}$ forms a basis for $k_{(E,\mathcal{L},\mathcal{B})}$. Set $(\alpha, A, \beta)^* := (\beta, A, \alpha)$; then thinking of $e_{(\alpha,A,\beta)}$ as $s_\alpha p_A s_\beta^*$ and using (5) we can define a multiplication with respect to which $k_{(E,\mathcal{L},\mathcal{B})}$ is a $*$-algebra.

As a $*$-algebra $k_{(E,\mathcal{L},\mathcal{B})}$ is generated by $q_A := e_{(A,A,A)}$ for $A \in \mathcal{B}$ and $t_a := e_{(a,r(a),r(a))}$ for $a \in \mathcal{L}(E^1)$. Our definition of multiplication ensures that properties (ii) and (iii) of Definition 4.1 hold; moreover $q_A q_B = q_{A\cap B}$. We mod out by the ideal J generated by the elements $q_{A\cup B} - q_A - q_B + q_{A\cap B}$ for $A, B \in \mathcal{B}$, and $q_A - \sum_{a\in L_A^1} s_a p_{r(A,a)} s_a^*$ for $A \in \mathcal{B}$ with L_A^1 nonempty and finite. Then the images r_A of q_A and u_a of t_a in $k_{(E,\mathcal{L},\mathcal{B})}/J$ form a representation of $(E, \mathcal{L}, \mathcal{B})$ that generates $k_{(E,\mathcal{L},\mathcal{B})}/J$. The triple $(k_{(E,\mathcal{L},\mathcal{B})}/J, r_A, u_a)$ has the required universal property, but is not a C^*-algebra. Using a standard argument we can convert this triple to a C^*-algebra B satisfying the required properties (see [10, Theorem 2.1] for instance).

Now for each $a \in \mathcal{L}(E^1)$ and $e \in \mathcal{L}^{-1}(a)$, let $\mathcal{H}_{(a,e)}$ be an infinite-dimensional Hilbert space. Also for each $v \in s(a)$ we define $\mathcal{H}_{(a,v)} := \oplus_{\{e:s(e)=v,\mathcal{L}(e)=a\}} \mathcal{H}_{(a,e)}$. If v is a sink let $\mathcal{H}_v$ be an infinite-dimensional Hilbert space. For $A \in \mathcal{B}$ we define $\mathcal{H}_A := \oplus_{b\in L_A^1} \oplus_{v\in s(b)\cap A} \mathcal{H}_{(b,v)}$ and then note that each Hilbert space we have defined is a subspace of

$$\mathcal{H} := \left(\oplus_{a\in\mathcal{L}(E^1)} \oplus_{v\in s(a)} \mathcal{H}_{(a,v)}\right) \oplus_{\{v:s^{-1}(v)=\emptyset\}} \mathcal{H}_v.$$

For each $a \in \mathcal{L}(E^1)$, let S_a be a partial isometry with initial space $\mathcal{H}_{r(a)}$ and final space $\oplus_{v\in s(a)} \mathcal{H}_{(a,v)} \subseteq \mathcal{H}_{s(a)}$. For $A \in \mathcal{B}$, define P_A to be the projection of $\mathcal{H}$ onto $\mathcal{H}_A$, where this is interpreted as the zero projection when $A = \emptyset$.

It is easy to verify that since $(E, \mathcal{L}, \mathcal{B})$ is weakly left-resolving, the operators $\{S_a, P_A\}$ form a representation of $(E, \mathcal{L}, \mathcal{B})$ in which S_a, P_A are nonzero. By the universal property there exists a homomorphism $\pi_{S,P} : B \to C^*(\{S_a, P_A\})$. Since the S_a's and P_A's are nonzero, it follows that the s_a's and p_A's are also nonzero. ☐

Definition 4.6. Let $(E, \mathcal{L}, \mathcal{B})$ be a weakly left-resolving labelled space, then $C^*(E, \mathcal{L}, \mathcal{B})$ is the universal C^*-algebra generated by a representation of $(E, \mathcal{L}, \mathcal{B})$.

Let $(E, \mathcal{L}, \mathcal{B})$ be a weakly left-resolving labelled space and $\{s_a, p_A\}$ be the universal representation of $(E, \mathcal{L}, \mathcal{B})$, then by Lemma 4.4

$$\operatorname{span}\{s_\alpha p_A s_\beta^* : \alpha, \beta \in \mathcal{L}^*(E), A \in \mathcal{B}, A \subseteq r(\alpha) \cap r(\beta)\}$$

is a dense $*$-subalgebra of $C^*(E, \mathcal{L}, \mathcal{B})$. The following result may be proved along the same lines as [26, Lemma 3.2].

Lemma 4.7. *Let $\mathcal{A}$ be finite, E have no sinks, and $(E, \mathcal{L}, \mathcal{B})$ be a weakly left-resolving labelled space. Then $C^*(E, \mathcal{L}, \mathcal{B})$ is unital.*

Proof. Observe that $\sum_{a \in \mathcal{A}} s_a s_a^*$ is a unit for $C^*(E, \mathcal{L}, \mathcal{B})$. □

Lemma 4.8. *If $\phi : (E, \mathcal{L}) \to (F, \mathcal{L}')$ is a labelled graph isomorphism, then for all $\mathcal{B}$ which are accommodating for $(E, \mathcal{L})$ we have $C^*(E, \mathcal{L}, \mathcal{B}) \cong C^*(F, \mathcal{L}', \phi(\mathcal{B}))$.*

Proof. The map ϕ induces a bijection between the generators of $C^*(E, \mathcal{L}, \mathcal{B})$ and $C^*(F, \mathcal{L}', \phi(\mathcal{B}))$ and so by the universal property there are homomorphisms from one C^*-algebra to the other which are also inverses of each other. □

5. Gauge Invariant Uniqueness Theorem

Let $\{s_a, p_A\}$ be the universal representation of $(E, \mathcal{L}, \mathcal{B})$ which generates $C^*(E, \mathcal{L}, \mathcal{B})$. For $z \in \mathbf{T}$, $a \in \mathcal{L}(E^1)$ and $A \in \mathcal{B}$ let

$$t_a := \gamma_z s_a = z s_a \text{ and } q_A := \gamma_z p_A = p_A$$

then the family $\{t_a, q_A\} \in C^*(E, \mathcal{L}, \mathcal{B})$ is also a representation of $(E, \mathcal{L}, \mathcal{B})$. By universality of $C^*(E, \mathcal{L}, \mathcal{B})$ and a routine $\epsilon/3$ argument we see that γ extends to a strongly continuous action

$$\gamma : \mathbf{T} \to \operatorname{Aut}\ C^*(E, \mathcal{L}, \mathcal{B})$$

which we call the *gauge action*.

Proposition 5.1.
- i) *Let E be a directed graph with the trivial labelling $\mathcal{L}$. Then $C^*(E, \mathcal{L}, \mathcal{E}^0) \cong C^*(E)$.*
- ii) *Let $\mathcal{G}$ be an ultragraph. Then $C^*(E_\mathcal{G}, \mathcal{L}_\mathcal{G}, \mathcal{E}^0_\mathcal{G}) \cong C^*(\mathcal{G})$, where $(E_\mathcal{G}, \mathcal{L}_\mathcal{G})$ is the labelled graph associated to $\mathcal{G}$.*
- iii) *Let $p : F \to E$ be a covering map with induced labelling $\mathcal{L}_p : F^1 \to E^1$. Then $C^*(F, \mathcal{L}_p, \mathcal{F}^0) \cong C^*(E)$.*
- iv) *Let E be a directed graph and let $E_s(\mathcal{P})$ be an outsplitting. Let $\mathcal{L}$ be the labelling of $E_s(\mathcal{P})$ induced by the outsplitting. If $\mathcal{P}$ is a proper partition then $C^*(E_s(\mathcal{P}), \mathcal{L}, \mathcal{E}_s(\mathcal{P})^0) \cong C^*(E)$.*

Proof. In each case the left hand side contains a generating set for the C^*-algebra on the right as shown in Examples 4.3. We apply the appropriate gauge-invariant uniqueness theorem for the algebra on the right hand side to obtain the isomorphism. □

To establish connections with the Matsumoto algebras we need a version of the gauge-invariant uniqueness theorem for labelled graph algebras.

Lemma 5.2. *Let $(E, \mathcal{L}, \mathcal{B})$ be a weakly left-resolving labelled space, $\{s_a, p_A\}$ a representation of $(E, \mathcal{L}, \mathcal{B})$, and $Y = \{s_{\alpha_i} p_{A_i} s_{\beta_i}^* : i = 1, \ldots, N\}$ be a set of partial isometries in $C^*(E, \mathcal{L}, \mathcal{B})$ which is closed under multiplication and taking adjoints. If q is a minimal projection in $C^*(Y)$ then either*

(i) *$q = s_{\alpha_i} p_{A_i} s_{\alpha_i}^*$ for some $1 \le i \le N$*

(ii) *$q = s_{\alpha_i} p_{A_i} s^*_{\alpha_i} - q'$ where $q' = \sum_{l=1}^m s_{\alpha_{k(l)}} p_{A_{k(l)}} s^*_{\alpha_{k(l)}}$ and $1 \leq i \leq N$; moreover there is a nonzero $r = s_{\alpha_i \beta} p_{r(A_i,\beta)} s^*_{\alpha_i \beta} \in C^*(E, \mathcal{L}, \mathcal{B})$ such that $q'r = 0$ and $q \geq r$.*

Proof. By equation (5) any projection in $C^*(Y)$ may be written as

$$\sum_{j=1}^{n} s_{\alpha_{i(j)}} p_{A_{i(j)}} s^*_{\alpha_{i(j)}} - \sum_{l=1}^{m} s_{\alpha_{k(l)}} p_{A_{k(l)}} s^*_{\alpha_{k(l)}}$$

where the projections in each sum are mutually orthogonal and for each l there is a unique j such that $s_{\alpha_{i(j)}} p_{A_{i(j)}} s^*_{\alpha_{i(j)}} \geq s_{\alpha_{k(l)}} p_{A_{k(l)}} s^*_{\alpha_{k(l)}}$.

If $q = \sum_{j=1}^n s_{\alpha_{i(j)}} p_{A_{i(j)}} s^*_{\alpha_{i(j)}} - \sum_{l=1}^m s_{\alpha_{k(l)}} p_{A_{k(l)}} s^*_{\alpha_{k(l)}}$ is a minimal projection in $C^*(Y)$ then we must have $n = 1$. If $m = 0$ then $q = s_{\alpha_i} p_{A_i} s^*_{\alpha_i}$ for some $1 \leq i \leq N$. If $m \neq 0$ then $q = s_{\alpha_i} p_{A_i} s^*_{\alpha_i} - q'$ where $q' = \sum_{l=1}^m s_{\alpha_{k(l)}} p_{A_{k(l)}} s^*_{\alpha_{k(l)}}$ and $1 \leq k \leq N$. Since q' is the sum of finitely many projections and $q \neq 0$ it follows by repeated use of Definition 4.1 (iv) that there is a nonzero $r = s_{\alpha_i \beta} p_{r(A_i,\beta)} s^*_{\alpha_i \beta}$ in $C^*(E, \mathcal{L}, \mathcal{B})$ such that $rq' = 0$ and $q \geq r$. □

Theorem 5.3. *Let $(E, \mathcal{L}, \mathcal{B})$ be a weakly left-resolving labelled space and let $\{S_a, P_A\}$ be a representation of $(E, \mathcal{L}, \mathcal{B})$ on Hilbert space. Take $\pi_{S,P}$ to be the representation of $C^*(E, \mathcal{L}, \mathcal{B})$ satisfying $\pi_{S,P}(s_a) = S_a$ and $\pi_{S,P}(p_A) = P_A$. Suppose that each P_A is nonzero whenever $A \neq \emptyset$, and that there is a strongly continuous action β of $\mathbf{T}$ on $C^*(S_\alpha, P_A)$ such that for all $z \in \mathbf{T}$, $\beta_z \circ \pi_{S,P} = \pi_{S,P} \circ \gamma_z$. Then $\pi_{S,P}$ is faithful.*

Proof. A straightforward argument along the lines of [22, Lemma 2.2.3] shows that

$$C^*(E, \mathcal{L}, \mathcal{B})^\gamma = \overline{\text{span}}\{s_\alpha p_A s^*_\beta : \alpha, \beta \in \mathcal{L}(E^n) \text{ for some } n \text{ and } A \subseteq r(\alpha) \cap r(\beta)\}$$

where $C^*(E, \mathcal{L}, \mathcal{B})^\gamma$ is the fixed point algebra of $C^*(E, \mathcal{L}, \mathcal{B})$ under the gauge action γ. We claim that $C^*(E, \mathcal{L}, \mathcal{B})^\gamma$ is AF. Let Y be a finite subset of $C^*(E, \mathcal{L}, \mathcal{B})^\gamma$. Since $y \in Y$ may be approximated by a finite linear combination of elements of the form $s_\alpha p_A s^*_\beta$ where $|\alpha| = |\beta|$ we may assume that $Y = \{s_{\alpha_i} p_{A_i} s^*_{\beta_i} : |\alpha_i| = |\beta_i|, i = 1 \dots N\}$.

Let M be the length of the longest word in $\{\alpha_1, \dots, \alpha_N\}$. Let W denote the collection of all words in $\mathcal{L}^*(E)$ of length at most M that can be formed from composing subwords of $\alpha_1, \dots, \alpha_N, \beta_1, \dots, \beta_N$. Let $\mathcal{C}$ be the collection all finite intersections of $\{A_i\}_{i=1}^n$ and $\{r(A_i, \gamma) : 1 \leq i \leq N, \gamma \in W\}$. By equation (5) a non-zero product of elements of Y is of the form $s_\gamma p_A s^*_\delta$ where $\gamma, \delta \in W$ and $A \in \mathcal{C}$. Since W and $\mathcal{C}$ are finite it follows that $Y' = \{s_\gamma p_A s^*_\delta : \gamma, \delta \in W, A \in \mathcal{C}\}$ is finite, closed under adjoints and $C^*(Y) = C^*(Y')$. Hence we may assume that Y is closed under multiplication and taking adjoints. Thus $C^*(Y) = \overline{\text{span}}(Y)$ is finite dimensional and so $C^*(E, \mathcal{L}, \mathcal{B})^\gamma$ is AF by [5, Theorem 2.2], establishing our claim.

To show that the canonical map $\pi_{S,P} : C^*(E, \mathcal{L}, \mathcal{B}) \to C^*(S_a, P_A)$ is injective on $C^*(E, \mathcal{L}, \mathcal{B})^\gamma$ we write $C^*(E, \mathcal{L}, \mathcal{B})^\gamma$ as $\overline{\cup C^*(Y_n)}$ where $\{Y_n : n \geq 1\}$ is an increasing family of finite sets which are closed under multiplication and taking adjoints. Suppose, for contradiction, that $\pi_{S,P}$ is not faithful on $C^*(Y_n)$ for some n. Then its kernel is an ideal and so must contain a nonzero minimal projection q. If $Y_n = \{s_{\alpha_i} p_{A_i} s^*_{\beta_i} : i = 1 \dots, N(n)\}$ then by Lemma 5.2 either $q = s_{\alpha_i} p_{A_i} s^*_{\alpha_i}$ for some $1 \leq i \leq N(n)$ or $q = s_{\alpha_i} p_{A_i} s^*_{\alpha_i} - q'$ where $q' = \sum_{k=1}^m s_{\alpha_{i(k)}} p_{A_{i(k)}} s^*_{\alpha_{i(k)}}$ and $1 \leq i \leq N(n)$. In the first case $\pi_{S,P}(s_{\alpha_i} p_{A_i}) = S_{\alpha_i} P_{A_i}$ is a partial isometry with initial projection P_{A_i} and final projection $S_{\alpha_i} P_{A_i} S^*_{\alpha_i}$. But $P_{A_i} = \pi_{S,P}(p_{A_i}) \neq 0$ by hypothesis and so $\pi_{S,P}(q) = \pi_{S,P}(s_{\alpha_i} p_{A_i} s^*_{\alpha_i}) = S_{\alpha_i} P_{A_i} S^*_{\alpha_i} \neq 0$ which is a contradiction. In the second case by Lemma 5.2 (ii) there is $r = s_{\alpha_i \beta} p_{r(A_i,\beta)} s^*_{\alpha_i \beta}$ such that $q \geq r$ and $q'r = 0$. We may apply the above argument to show that $\pi_{S,P}(r) \neq 0$ and hence $\pi_{S,P}(q) \geq \pi_{S,P}(r) \neq 0$ which is also a contradiction. Hence $\pi_{S,P}$ is injective on $C^*(Y_n)$ and the result follows by arguments similar to those in [2, Theorem 2.1]. □

6. Applications

6.1. **Dual Labelled Graphs.** Let E have no sinks and $(E,\mathcal{L})$ be a labelled graph over alphabet $\mathcal{A}$. From this data we may form the *dual labelled graph* $(\widehat{E},\widehat{\mathcal{L}})$ over alphabet $\widehat{\mathcal{A}} := \mathcal{L}(E^2)$ as follows: Let $\widehat{E}^0 = E^1$, $\widehat{E}^1 = E^2$ and the maps $r', s' : \widehat{E}^1 \to \widehat{E}^0$ be given by $r'(ef) = f$ and $s'(ef) = e$. The labelling $\widehat{\mathcal{L}} : \widehat{E}^1 \to \widehat{\mathcal{A}}$ is induced by the original labelling, so that $\widehat{\mathcal{L}}(ef) = \mathcal{L}(e)\mathcal{L}(f)$. For $ab \in \widehat{\mathcal{L}}(\widehat{E}^1) = \mathcal{L}(E^2)$ we have

$$r_{\widehat{\mathcal{L}}}(ab) = \{f : \widehat{\mathcal{L}}(ef) = ab\}, \text{ and } s_{\widehat{\mathcal{L}}}(ab) = \{e : \widehat{\mathcal{L}}(ef) = ab\}$$

and for $B \in 2^{E^1}$

$$r_{\widehat{\mathcal{L}}}(B, ab) = \{f : \widehat{\mathcal{L}}(ef) = ab, e \in B\}.$$

These maps extend naturally to $\widehat{\mathcal{L}}^*(\widehat{E}) = \cup_{n\geq 1}\widehat{\mathcal{L}}(\widehat{E}^n)$ where for $n \geq 1$, $\widehat{\mathcal{L}}(\widehat{E}^n)$ is identified with $\mathcal{L}(E^{n+1})$. Consider the following subsets of 2^{E^1}

$$\begin{aligned}\widehat{\mathcal{E}} &= \{\{e\} : s(e) \text{ is a source}\} \cup \{r_{\widehat{\mathcal{L}}}(\alpha) : \alpha \in \widehat{\mathcal{L}}^*(\widehat{E})\} \cup \{s_{\widehat{\mathcal{L}}}(\alpha) : \alpha \in \widehat{\mathcal{L}}^*(\widehat{E})\}\\ \widehat{\mathcal{E}}^- &= \{r_{\widehat{\mathcal{L}}}(\alpha) : \alpha \in \widehat{\mathcal{L}}^*(\widehat{E})\}.\end{aligned}$$

Let $\widehat{\mathcal{E}}^0$ (resp. $\widehat{\mathcal{E}}^{0,-}$) be the smallest collection of subsets of 2^{E^1} containing $\widehat{\mathcal{E}}$ (resp. $\widehat{\mathcal{E}}^-$) which is accommodating for $(\widehat{E},\widehat{\mathcal{L}})$. One checks easily that if $(E,\mathcal{L},\mathcal{B})$ is left-resolving, then $(\widehat{E},\widehat{\mathcal{L}},\widehat{\mathcal{B}})$ is weakly left-resolving for $\mathcal{B} = \mathcal{E}^0, \mathcal{E}^{0,-}$.
For $B \in \widehat{\mathcal{E}}^0$ (resp. $B \in \widehat{\mathcal{E}}^{0,-}$) we set

$$\widehat{L}^1_B = \{ab \in \widehat{\mathcal{L}}(\widehat{E}^1) : s_{\widehat{\mathcal{L}}}(ab) \cap B \neq \emptyset\}.$$

If E has no sources and sinks, the shift $\mathsf{X}_{(\widehat{E},\widehat{\mathcal{L}})}$ determined by the dual labelled graph $(\widehat{E},\widehat{\mathcal{L}})$ of $(E,\mathcal{L})$ is the second higher block shift $\mathsf{X}^{[2]}_{(E,\mathcal{L})}$ formed from $\mathsf{X}_{(E,\mathcal{L})}$ (cf. [13, §1.4]).

Remarks 6.1. Suppose that $ab \in \mathcal{L}(E^2)$ then $c \in L^1_{r(ab)}$ if and only if $bc \in \widehat{L}^1_{r_{\widehat{\mathcal{L}}}(ab)}$; moreover $r(r(ab), c) = r(s(r_{\widehat{\mathcal{L}}}(ab)), bc)$. Suppose that $A \in \mathcal{E}^0$ (resp. $A \in \mathcal{E}^{0,-}$) then $a \in L^1_A$ and $ab \in \mathcal{L}(E^2)$ if and only if $ab \in \widehat{L}^1_{s^{-1}(A)}$.

Theorem 6.2. *Let $(E,\mathcal{L})$ be a set-finite, left-resolving labelled graph with no sinks then $C^*(E,\mathcal{L},\mathcal{E}^0) \cong C^*(\widehat{E},\widehat{\mathcal{L}},\widehat{\mathcal{E}}^0)$, moreover $C^*(E,\mathcal{L},\mathcal{E}^{0,-}) \cong C^*(\widehat{E},\widehat{\mathcal{L}},\widehat{\mathcal{E}}^{0,-})$.*

Proof. Let $\{s_a, p_A\}$ be a representation of $(E,\mathcal{L},\mathcal{E}^0)$ and $\{t_{ab}, q_B\}$ be a representation of $(\widehat{E},\widehat{\mathcal{L}},\widehat{\mathcal{E}}^0)$. For $ab \in \widehat{\mathcal{L}}(\widehat{E}^1)$ and $B \in \widehat{\mathcal{E}}^0$ let $T_{ab} = s_a s_b s_b^*$ and

$$Q_B := \sum_{ab\in\widehat{L}^1_B} s_{ab} p_{r(s(B),ab)} s^*_{ab}.$$

Since $(E,\mathcal{L},\mathcal{E}^0)$ is set-finite $(\widehat{E},\widehat{\mathcal{L}},\widehat{\mathcal{E}}^0)$ is set-finite by Remarks 6.1 and so the above sum is finite. One checks that $\{T_{ab}, Q_B\}$ is a representation of $(\widehat{E},\widehat{\mathcal{L}},\widehat{\mathcal{E}}^0)$.

By the universal property there is a homomorphism $\pi_{T,Q} : C^*(\widehat{E},\widehat{\mathcal{L}},\widehat{\mathcal{E}}^0) \to C^*(E,\mathcal{L},\mathcal{E}^0)$ with $\pi_{T,Q}(t_{ab}) = T_{ab}$ and $\pi_{T,Q}(q_B) = Q_B$. Since $\pi_{T,Q}$ intertwines the respective gauge actions and $Q_B \neq 0$ it follows from Theorem 5.3 that $\pi_{T,Q}$ is faithful. We claim that $\pi_{T,Q}$

is surjective. For $a \in \mathcal{L}(E^1)$ we have

$$\begin{aligned} s_a = s_a p_{r(a)} = s_a \sum_{b\in L^1_{r(a)}} s_b p_{r(r(a),b)} s_b^* &= \sum_{b\in L^1_{r(a)}} s_a s_b s_b^* s_b p_{r(ab)} s_b^* \\ &= \sum_{b\in L^1_{r(a)}} s_a s_b s_b^* \sum_{c\in L^1_{r(ab)}} s_{bc} p_{r(r(ab),c)} s_{bc}^* \\ &= \sum_{b\in L^1_{r(a)}} T_{ab} \sum_{bc\in \widehat{L}^1_{r_{\widehat{\mathcal{L}}}(ab)}} s_{bc} p_{r(s(r_{\widehat{\mathcal{L}}}(ab)),bc)} s_{bc}^* \text{ by Remarks 6.1} \\ &= \sum_{b\in L^1_{r(a)}} T_{ab} Q_{r_{\widehat{\mathcal{L}}}(ab)} \end{aligned}$$

and so $s_a \in C^*(T_{ab}, Q_B)$. For $A \in \mathcal{E}^0$, by Remarks 6.1 we have

$$\begin{aligned} p_A &= \sum_{a\in L^1_A} s_a p_{r(A,a)} s_a^* = \sum_{a\in L^1_A} s_a \sum_{b\in L^1_{r(A,a)}} s_b p_{r(r(A,a),b)} s_b^* s_a^* \\ &= \sum_{ab\in \widehat{L}^1_{s^{-1}(A)}} s_{ab} p_{r(A,ab)} s_{ab}^* = Q_{s^{-1}(A)} \end{aligned}$$

which establishes our claim. The second isomorphism is proved along similar lines. □

6.2. **Matsumoto Algebras.**

Theorem 6.3. *Let Λ be a shift space over a finite alphabet $\mathcal{A}$ which satisfies condition (I) and has left-Krieger cover $(E_\Lambda, \mathcal{L}_\Lambda)$ then $\mathcal{O}_\Lambda \cong C^*(E_\Lambda, \mathcal{L}_\Lambda, \mathcal{E}^{0,-}_\Lambda)$. Moreover, if Λ has predecessor graph $(E_{\Lambda^*}, \mathcal{L}_{\Lambda^*})$ then $\mathcal{O}_{\Lambda^*} \cong C^*(E_{\Lambda^*}, \mathcal{L}_{\Lambda^*}, \mathcal{E}^{0,-}_{\Lambda^*})$.*

Proof. By definition every $A \in \mathcal{E}^{0,-}_\Lambda$ can be written as a union of sets of the form $A_j = \cap_{i=1}^{m(j)} r(\mu_i^j)$ for $j = 1, \ldots, n$. For $\mu \in \Lambda^*$ let $q_{r(\mu)} = t_\mu^* t_\mu$, then since the projections $\{t_\mu^* t_\mu : \mu \in \Lambda^*\}$ are mutually commutative (see [17, p.686]) we may define $q_{r(\mu)\cap r(\nu)} = q_{r(\mu)} q_{r(\nu)}$, and hence define q_{A_j} for $1 \leq j \leq n$. By the inclusion-exclusion principle one may further define

$$q_A = \sum_{j=1}^n q_{A_j} - \sum_{j\neq k} q_{A_j} q_{A_k} + \ldots + (-1)^{n+1} q_{A_1} \cdots q_{A_n}.$$

Using calculations along the lines of those in [14, §3] one checks that $\{t_a, q_A\}$ is a representation of $(E_\Lambda, \mathcal{L}_\Lambda, \mathcal{E}^{0,-}_\Lambda)$.

Let $\{s_a, p_A\}$ be a representation of $(E_\Lambda, \mathcal{L}_\Lambda, \mathcal{E}^{0,-}_\Lambda)$. By the universal property for $C^*(E_\Lambda, \mathcal{L}_\Lambda, \mathcal{E}^{0,-}_\Lambda)$ there is a map $\pi_{t,q} : C^*(E_\Lambda, \mathcal{L}_\Lambda, \mathcal{E}^{0,-}_\Lambda) \to \mathcal{O}_\Lambda$ such that $\pi_{t,q}(s_a) = t_a$ and $\pi_{t,q}(p_A) = q_A$, in particular $\pi_{t,q}$ is surjective.

Since Λ satisfies condition (I) it follows by Proposition 2.2 that $\mathcal{O}_\Lambda$ carries a strongly continuous action β of $\mathbf{T}$. Since $\beta_z \circ \pi_{t,q} = \pi_{t,q} \circ \gamma_z$ for all $z \in \mathbf{T}$ and $\pi_{t,q}(p_A) = q_A \neq 0$ it follows from Theorem 5.3 that $\pi_{t,q}$ is injective, which completes the proof of the first statement.

The second statement is proved similarly. □

Remarks 6.4. (i) In [8, §5] a condition (*) is given under which for shift spaces Λ satisfying (*) conditions (I) and (I*) are equivalent and $\mathcal{O}_\Lambda \cong \mathcal{O}_{\Lambda^*}$. This suggests that if Λ satisfies (*) then $(E_\Lambda, \mathcal{L}_\Lambda)$ is labelled graph isomorphic to $(E_{\Lambda^*}, \mathcal{L}_{\Lambda^*})$ and the isomorphism of $\mathcal{O}_\Lambda$ and $\mathcal{O}_{\Lambda^*}$ can be deduced from Theorem 4.8. However [8, Theorem 6.1] shows that, in general, $\mathcal{O}_\Lambda$ and $\mathcal{O}_{\Lambda^*}$ are not isomorphic. In particular, $(E_\Lambda, \mathcal{L}_\Lambda)$ and $(E_{\Lambda^*}, \mathcal{L}_{\Lambda^*})$ are not labelled graph isomorphic in general.

(ii) The isomorphism of $C^*(E_\Lambda, \mathcal{L}_\Lambda, \mathcal{E}^{0,-}_\Lambda)$ and $\mathcal{O}_\Lambda$ identifies $C^*(p_A : A \in \mathcal{E}^{0,-}_\Lambda)$ with $A_\Lambda \subset \mathcal{O}_\Lambda$. Recall from [17, Corollary 4.7] that $A_\Lambda \cong C(\Omega_\Lambda)$, hence we may think of the elements of $\mathcal{E}^{0,-}_\Lambda$ as indexing closed sets in Ω_Λ.

(iii) In [7] Carlsen constructs a C^*-algebra which has $\mathcal{O}_\Lambda$ as a quotient, that is isomorphic to $\mathcal{O}_\Lambda$ if Λ satisfies condition (I), and always carries a gauge action. A proof along the lines of Theorem 6.3 shows that this new algebra is isomorphic to $C^*(E_\Lambda, \mathcal{L}_\Lambda, \mathcal{E}^{0,-}_\Lambda)$ for all Λ.

6.3. **Finiteness Conditions.**

Definition 6.5. A labelled graph $(E, \mathcal{L})$ is *label-finite* if $|\mathcal{L}^{-1}(a)| < \infty$ for all $a \in \mathcal{L}(E^1)$.

If $(E, \mathcal{L})$ is label-finite then $\mathcal{L}^{-1}(\alpha)$ is finite for all $\alpha \in \mathcal{L}^*(E)$ and so all sets in $\mathcal{E}^0$ are finite (and conversely). If $(E, \mathcal{L})$ is label-finite then $(\widehat{E}, \widehat{\mathcal{L}})$ is label-finite. If E is row-finite and $(E, \mathcal{L})$ is label-finite then $(E, \mathcal{L}, \mathcal{E}^0)$ is set-finite.

The following result generalises [2, Corollary 2.5] (see also [3, Remark 3.3 (i)]).

Theorem 6.6. *Let $(E, \mathcal{L})$ be a row-finite left-resolving labelled graph which is label-finite and satisfies $\{v\} \in \mathcal{E}^0$ for all $v \in E^0$. Then $C^*(E, \mathcal{L}, \mathcal{E}^0) \cong C^*(E)$; moreover if $\{v\} \in \mathcal{E}^{0,-}$ for all $v \in E^0$ then $C^*(E, \mathcal{L}, \mathcal{E}^{0,-}) \cong C^*(E)$.*

Proof. Let $\{s_e, p_v\}$ be the canonical Cuntz-Krieger E-family and $\{t_a, q_A\}$ be the canonical generators of $C^*(E, \mathcal{L}, \mathcal{E}^0)$. For $a \in \mathcal{L}(E^1)$ and $A \in \mathcal{E}^0$ let

$$T_a = \sum_{e \in E^1 : \mathcal{L}(e) = a} s_e, \text{ and } Q_A = \sum_{v \in A} p_v.$$

The above sums make sense since $(E, \mathcal{L})$ is label-finite. Since E is row-finite one may easily check that these operators define a representation of $(E, \mathcal{L}, \mathcal{E}^0)$. By the universal property of $C^*(E, \mathcal{L}, \mathcal{E}^0)$ there is a homomorphism $\pi_{T,Q} : C^*(E, \mathcal{L}, \mathcal{E}^0) \to C^*(E)$ given by $\pi_{T,Q}(t_a) = T_a$ and $\pi_{T,Q}(q_A) = Q_A$ for all $a \in \mathcal{L}(E^1)$ and $A \in \mathcal{E}^0$.

Since $\{v\} \in \mathcal{E}^0$ for all $v \in E^0$, we have $p_v = Q_v \in C^*(T_a, Q_A)$ for all $v \in E^0$. Since our labelled graph is left-resolving we have $s_e = T_{\mathcal{L}(e)} Q_{r(e)} \in C^*(T_a, Q_A)$ for all $e \in E^1$, and so $\pi_{T,Q}$ is surjective. The canonical gauge actions on $C^*(E)$ and $C^*(E, \mathcal{L}, \mathcal{E}^0)$ satisfy the required properties and $\pi_{T,Q}(q_A) = Q_A \neq 0$ for all $A \in \mathcal{E}^0$, so $\pi_{T,Q}$ is an isomorphism by Theorem 5.3.

The proof of the second isomorphism is essentially the same. □

Corollary 6.7. *Let $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ be a row-finite ultragraph then $C^*(\mathcal{G}) \cong C^*(E_\mathcal{G})$ where $E_\mathcal{G}$ is the underlying directed graph of $\mathcal{G}$.*

Proof. From Examples 3.3 (ii) a row-finite ultragraph $\mathcal{G}$ may be realised as a row-finite left-resolving labelled graph $(E_\mathcal{G}, \mathcal{L}_\mathcal{G})$. As $E_\mathcal{G}$ is row-finite it follows that $(E_\mathcal{G}, \mathcal{L}_\mathcal{G})$ is label-finite. Since the source map is single-valued it follows that $v \in \mathcal{E}^0_\mathcal{G}$ for all $v \in G^0 = E^0_\mathcal{G}$ and hence the result follows from Theorem 6.6. □

The following result was first observed in [6, Theorem 3.5] (see also [24, Corollary 3.4.5]).

Corollary 6.8. *Let Λ be a sofic shift over a finite alphabet then*

$$\mathcal{O}_\Lambda \cong C^*(E_\Lambda)$$

where $(E_\Lambda, \mathcal{L}_\Lambda)$ is the left-Krieger cover of Λ.

Proof. As E^0_Λ is finite and each $v \in E^0_\Lambda$ has a different past there are $\alpha_v \in \mathcal{L}^*(E_\Lambda)$ with $r_{\mathcal{L}_\Lambda}(\alpha_v) = \{v\}$. Hence $\{v\} \in \mathcal{E}^{0,-}_\Lambda$ for all $v \in E^0_\Lambda$. The result follows by Theorem 6.6. □

From [13, Theorem 3.3.18] any two minimal left-resolving representations $(E, \mathcal{L})$, $(F, \mathcal{L}')$ of an irreducible sofic shift are labelled graph isomorphic and so $C^*(E, \mathcal{L}, \mathcal{E}^0_-) \cong C^*(F, \mathcal{L}', \mathcal{F}^0_-)$ by Lemma 4.8. Moreover, one may use the minimality of the representation to show that the underlying graph E is irreducible (cf. [13, Lemma 3.3.10]). Hence we have:

Corollary 6.9. *Let $(E, \mathcal{L})$ be a minimal left-resolving presentation of an irreducible sofic shift over a finite alphabet, then $C^*(E, \mathcal{L}, \mathcal{E}^{0,-}) \cong C^*(E, \mathcal{L}, \mathcal{E}^0)$ is simple.*

Remark 6.10. Recall that the graph $(E_2, \mathcal{L}_2)$ in Examples 3.3 (ii) is the left-Krieger cover of the even shift Y. Although Y is irreducible, $(E_2, \mathcal{L}_2)$ is not a minimal left-resolving presentation of Y and $\mathcal{O}_Y \cong C^*(E_2)$ is not simple. However the graph $(E_1, \mathcal{L}_1)$ Examples 3.3 (ii) is a minimal left-resolving cover of Y and so

$$C^*(E_1, \mathcal{L}_1, \mathcal{E}^{0,-}_1) \cong C^*(E_1, \mathcal{L}_1, \mathcal{E}^0_1) \cong C^*(E_1)$$

is simple. Similarly $C^*(E_Z, \mathcal{L}_Z, \mathcal{E}^{0,-}_Z) \cong C^*(E_Z)$ is simple where Z is the irreducible shift introduced in Examples 3.3 (vi).

Thus, if one wishes to associate a simple C^*-algebra to an irreducible sofic shift Λ, then one should use the minimal left-resolving presentation of Λ. (cf. [6, 7]).

For a general shift space Λ, either $(E_\Lambda, \mathcal{L}_\Lambda)$ will not be row-finite or there will be $v \in E^0_\Lambda$ with $v \notin \mathcal{E}^{0,-}_\Lambda$. This indicates that the C^*-algebras corresponding to presentations of such shift spaces will not be Morita equivalent to graph algebras. The shift associated to a certain Shannon graph (see [20, Theorem 7.7]) provides such an example.

References

[1] T. Bates, J-H. Hong, I. Raeburn and W. Szymański. *The ideal structure of the C^*–algebras of infinite graphs.* Illinois J. Math **46** (2002), 1159–1176.

[2] T. Bates, D. Pask, I. Raeburn and W. Szymański. *The C^*-algebras of row-finite graphs.* New York J. Math. **6** (2000), 307–324.

[3] T. Bates and D. Pask. *Flow equivalence of graph algebras.* Ergod. Th. & Dynam. Sys., **24** (2004), 367–382.

[4] T. Bates and D. Pask. *Flow equivalence of labelled graph algebras.* In preparation.

[5] O. Bratteli. *Inductive limits of finite dimensional C^*-algebras.* Trans. Amer. Math. Soc., **171** (1972), 195–234.

[6] T. Carlsen. *On C^*-algebras Associated with Sofic Shifts.* J. Operator Theory **49** (2003), 203–212.

[7] T. Carlsen. *Cuntz-Pimsner C^*-algebras associated with subshifts*, preprint NTNU, (2005).

[8] T. Carlsen and K. Matsumoto. *Some remarks on the C^*-algebras associated with subshifts.* Math. Scand. **95** (2004), 145–160.

[9] K. Deicke, D. Pask and I. Raeburn, *Coverings of directed graphs and crossed products of C^*-algebras by coactions of homogeneous spaces.* Internat. J. Math., **14** (2003), 773-789.

[10] A. an Huef and I. Raeburn. *The ideal structure of Cuntz-Krieger algebras.* Ergod. Th. & Dynam. Sys. **17** (1997), 611–624.

[11] W. Krieger. *Sofic Systems I.* Israel J. Math., **48**, (1984), 305-330.

[12] A. Kumjian, D. Pask and I. Raeburn. *Cuntz-Krieger algebras of directed graphs.* Pacific J. Math. **184** (1998), 161–174.

[13] D. Lind and B. Marcus. *An Introduction to Symbolic Dynamics and Coding*, CUP, 1995.

[14] K. Matsumoto. *On C^*-algebras associated with subshifts.* Internat. J. Math. **8**, (1997), 357-374.

[15] K. Matsumoto. *K-theory for C^*-algebras associated with subshifts.* Math. Scand. **82**, (1998), 237-255.

[16] K. Matsumoto. *Relations among generators of C^*-algebras associated with subshifts*, Internat. J. Math. **10** (1999), 385-405.

[17] K. Matsumoto. *Dimension groups for subshifts and simplicity of the associated C^*-algebras.* J. Math. Soc. Japan **51** (1999), 679-697.

[18] K. Matsumoto. *On automorphisms of C^*-algebras associated with subshifts.* J. Operator Theory **44**, (2000), 91-112.

[19] K. Matsumoto. *Bowen-Franks groups for subshifts and Ext-groups for C^*-algebras.* K-Theory **23**, (2001), 67-104.

[20] K. Matsumoto. *C^*-algebras associated with presentations of subshifts.* Documenta Math. **7**, (2002), 1-30.

[21] K. Matsumoto. *Stabilized C^*-algebras constructed from symbolic dynamical systems.* Ergodic Theory Dynam. Sys. **20**, (2000), 821-841.

[22] D. Pask and I. Raeburn, *On the K-theory of Cuntz-Krieger algebras.* Publ. RIMS Kyoto Univ. **32** (1996), 415–443.

[23] I. Raeburn and W. Szymański. *Cuntz-Krieger algebras of infinite graphs and matrices.* Trans. AMS, **356**, (2004) 39–59.

[24] J. Samuel. *C^*-algebras of Sofic Shifts.* PhD. Thesis Univ. Victoria, (1998).

[25] R.G. Taylor. *Models of computation and formal languages*, Oxford University Press, 1998.

[26] M. Tomforde. *A unified approach to Exel-Laca algebras and C^*-algebras associated to graphs.* J. Operator Theory **50** (2003), 345–368.

[27] M. Tomforde. *Simplicity of ultragraph algebras.* Indiana Univ. Math. J. **52** (2003), 901–926.

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