

COVERINGS OF k -GRAPHS

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ABSTRACT. k -graphs are higher-rank analogues of directed graphs which were first developed to provide combinatorial models for operator algebras of Cuntz-Krieger type. Here we develop the theory of covering spaces for k -graphs, obtaining a satisfactory version of the usual topological classification in terms of subgroups of a fundamental group. We then use this classification to describe the C^* -algebras of covering k -graphs as crossed products by coactions of homogeneous spaces, generalizing recent results on the C^* -algebras of graphs.

1. INTRODUCTION

k -graphs are combinatorial structures which are k -dimensional analogues of (directed) graphs. They were introduced by Kumjian and the first author [17] to help understand work of Robertson and Steger on higher-rank analogues of the Cuntz-Krieger algebras [26]. The theory of k -graphs and their C^* -algebras parallels in many respects that of graphs and their Cuntz-Krieger algebras [17, 23]. Here we investigate to what extent there is an analogue for k -graphs of the theory of coverings of graphs, and the implications of this theory for the C^* -algebras of k -graphs.

A *covering* of a graph F is by definition a surjective graph morphism $p: E \rightarrow F$ which is a local isomorphism. As for coverings of topological spaces, the coverings of F are classified by the conjugacy classes of subgroups of the fundamental group $\pi_1(F)$, and every connected covering arises as a quotient of a universal covering (see [11] or [4], for example). This last theorem has interesting ramifications for the Cuntz-Krieger algebras $C^*(E)$ of covering graphs: if $p: E \rightarrow F$ is

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a covering, then there is a coaction of the fundamental group $\pi_1(F)$ on $C^*(F)$ and a subgroup H of $\pi_1(F)$ such that $C^*(E)$ is isomorphic to the crossed product $C^*(F) \times_{\delta} (\pi_1(F)/H)$ [4, Theorem 3.2]. This theorem has in turn been of considerable interest in nonabelian duality for C^* -algebras: it provided a family of crossed products by homogeneous spaces which we could analyse using our understanding of graph algebras, and this analysis inspired substantial improvements in Mansfield's Imprimitivity Theorem [13].

We seek, therefore, an analogue of this theory of covering graphs for k -graphs, and a generalisation of [4, Theorem 3.2] which describes the C^* -algebras of the covering k -graphs. Any theory of coverings must involve the fundamental group, and a majority of the authors prefer to use the whole fundamental groupoid. We showed in [19] that the fundamental groupoids of k -graphs do not behave as well as one might hope, and in particular that the path category need not embed faithfully in the fundamental groupoid when $k > 1$. So it is something of a relief that our final results on coverings mirror in every respect the classical topological theory.

Our approach is to exploit an equivalence between the coverings of a k -graph and actions of its fundamental groupoid, under which the connected coverings correspond to transitive actions. Thus we deduce many of our main theorems from a classification of the transitive actions of an arbitrary groupoid.

Because every small category is isomorphic to a quotient of a path category, it will be clear from the proofs that all our results carry over to arbitrary small categories; however, we eschew such a generalization since we have no useful applications.

After we completed this paper, we learned of the existence of [1, 3, 12, 15], which contain results similar to some of ours. In [1, Appendix], Bridson and Haefliger develop the elementary theory of the fundamental group and coverings of a small category and proves results similar to some of ours. Bridson and Haefliger concentrate on the fundamental *group* — indeed, they stop just short of defining the fundamental groupoid. In [3, 12], Brown and Higgins investigate coverings of groupoids, and prove the equivalence with groupoid actions. Our work was done completely independently of these other sources, and we believe our methods are of interest, especially our use of skew products. In [15], Kumjian develops, in the specific context of k -graphs, the fundamental groupoid and the existence of the universal covering, and

proves that, under reasonable hypotheses, the C^* -algebra of the universal covering k -graph is Rieffel-Morita equivalent to a commutative algebra. We thank Kumjian for bringing [1] to our attention.

We begin in Section 2 by introducing our notion of covering, and stating our main classification theorems. Analogues of these theorems for coverings of groupoids were proved in [3, Chapter 9]. In Section 3, we briefly discuss actions of groupoids on sets, and prove the equivalence between the category of coverings of a k -graph Λ and the category of actions of its fundamental groupoid $\mathcal{G}(\Lambda)$ (Theorem 3.5). The main theorem of this section is a technical result (Theorem 3.7) which implies that the connected coverings of Λ correspond to transitive actions of $\mathcal{G}(\Lambda)$, and that the fundamental groups of coverings of Λ can be identified with the stability groups of the corresponding actions of $\mathcal{G}(\Lambda)$. In Section 4, we state and prove analogues for groupoid actions of most of our main theorems, and then in 5 we prove the main theorems themselves. Many of them follow from the general results in the previous section, but when it seemed easier to prove a result about coverings directly, we did so.

In Section 6, we construct universal coverings using skew products. We also show that every connected covering is a *relative skew product* (Corollary 6.10), and prove a version of the Gross-Tucker Theorem which identifies the k -graphs which admit free actions of a group as skew products.

It seems to us that k -graphs are likely to be of interest in their own right, so we have been careful to limit our discussion of C^* -algebras to a final section on the applications of our theory. Our sought-after generalisation of [4, Theorem 3.2] is Corollary 7.2. The main idea in the proof of Corollary 7.2 comes from [14]: every group-valued cocycle η on a k -graph Λ induces a normal and maximal coaction δ_η of the group on $C^*(\Lambda)$, and every k -graph carries a suitable cocycle with values in the fundamental group. We also prove a decomposition theorem which generalises [4, Corollary 3.6], prove that the C^* -algebra of every k -graph is nuclear, and prove that the C^* -algebra of the skew product by the degree map is always AF.

2. MAIN RESULTS

For k -graphs and groupoids we adopt the conventions of [17, 19, 23]. Briefly, a k -graph is a small category Λ equipped with a functor $d: \Lambda \rightarrow \mathbb{N}^k$ satisfying the *factorization property*: for all $\alpha \in \Lambda$ and $n, l \in \mathbb{N}^k$ such that $d(\alpha) = n + l$ there exist unique $\beta, \gamma \in \Lambda$ such that $d(\beta) = n$, $d(\gamma) = l$, and $\alpha = \beta\gamma$. When $d(\alpha) = n$ we say α has *degree* n .

A *groupoid* is a small category in which every morphism has an inverse. If \mathcal{C} is either a k -graph or a groupoid, the *vertices* are the objects, and \mathcal{C}^0 denotes the set of vertices. For $\alpha \in \mathcal{C}$, the *source* $s(\alpha)$ is the domain, and the *range* $r(\alpha)$ is the codomain. For $u, v \in \mathcal{C}^0$ we write $u\mathcal{C} = r^{-1}(u)$, $\mathcal{C}v = s^{-1}(v)$, and $u\mathcal{C}v = u\mathcal{C} \cap \mathcal{C}v$. \mathcal{C} is *connected* if the equivalence relation on \mathcal{C}^0 generated by $\{(u, v) \mid u\mathcal{C}v \neq \emptyset\}$ is $\mathcal{C}^0 \times \mathcal{C}^0$; for a groupoid this just means $u\mathcal{C}v \neq \emptyset$ for all $u, v \in \mathcal{C}^0$. If Λ is a k -graph, $u, v \in \Lambda^0$, and $n \in \mathbb{N}^k$, we write $\Lambda^n = d^{-1}(n)$, $u\Lambda^n = u\Lambda \cap \Lambda^n$, and $\Lambda^n v = \Lambda v \cap \Lambda^n$. A *morphism* between k -graphs is a degree-preserving functor.

In general we often write composition of maps as juxtaposition, especially when we are chasing around commutative diagrams.

Definition 2.1. A *covering* of a k -graph Λ is a surjective k -graph morphism $p: \Omega \rightarrow \Lambda$ such that for all $v \in \Omega^0$, p maps $v\Omega$ 1-1 onto $p(v)\Lambda$ and $v\Omega$ 1-1 onto $p(v)\Lambda$. If (Ω, p) and (Σ, q) are coverings of Λ , a *morphism* from (Ω, p) to (Σ, q) is a k -graph morphism $\phi: \Omega \rightarrow \Sigma$ making the diagram

$$\begin{array}{ccc} \Omega & \xrightarrow{\phi} & \Sigma \\ & \searrow p & \swarrow q \\ & \Lambda & \end{array}$$

commute; we write $\phi: (\Omega, p) \rightarrow (\Sigma, q)$. A covering $p: \Omega \rightarrow \Lambda$ is *connected* if Ω , hence Λ , is connected.

Remark. If Λ is connected then surjectivity of p is implied by the other properties. Also, any functor $\phi: \Omega \rightarrow \Sigma$ making the above diagram commute automatically preserves degrees, hence is a morphism of coverings.

For every k -graph Λ there is a *fundamental groupoid* $\mathcal{G}(\Lambda)$, together with a *canonical functor* $i: \Lambda \rightarrow \mathcal{G}(\Lambda)$, having the universal property that for any functor T from Λ into a groupoid \mathcal{H} there exists a unique groupoid morphism T' making the diagram

$$\begin{array}{ccc} \Lambda & \xrightarrow{i} & \mathcal{G}(\Lambda) \\ & \searrow T & \downarrow T' \\ & & \mathcal{H} \end{array}$$

commute. $\Lambda \mapsto \mathcal{G}(\Lambda)$ is functorial from k -graphs to groupoids. The *fundamental group* of Λ at a vertex $x \in \Lambda^0$ is the isotropy group

$$\pi(\Lambda, x) := x\mathcal{G}(\Lambda)x.$$

(The subscript 1 in the standard notation seems redundant in this context.) By functoriality of $\Lambda \mapsto \mathcal{G}(\Lambda)$, a covering $p: \Omega \rightarrow \Lambda$ induces homomorphisms $p_*: \pi(\Omega, v) \rightarrow \pi(\Lambda, p(v))$.

Theorem 2.2. *Let (Ω, p) and (Σ, q) be connected coverings of a k -graph Λ . For all $x \in \Lambda^0$, the family $\{p_*\pi(\Omega, v) \mid p(v) = x\}$ is a conjugacy class of subgroups of $\pi(\Lambda, x)$. For all $x \in \Lambda^0$, $v \in p^{-1}(x)$, and $u \in q^{-1}(x)$, there is a morphism $(\Omega, p) \rightarrow (\Sigma, q)$ taking v to u if and only if $p_*\pi(\Omega, v) \subset q_*\pi(\Sigma, u)$. Consequently, $(\Omega, p) \cong (\Sigma, q)$ if and only if the subgroups $p_*\pi(\Omega, v)$ and $q_*\pi(\Sigma, u)$ of $\pi(\Lambda, x)$ are conjugate for some, hence every, $x \in \Lambda^0$, $v \in p^{-1}(x)$, and $u \in q^{-1}(x)$.*

For our next result we will need to enlarge our supply of morphisms:

Definition 2.3. If $p: \Omega \rightarrow \Lambda$ and $q: \Omega \rightarrow \Gamma$ are coverings, a *morphism* from (Ω, p) to (Ω, q) is a k -graph morphism $\phi: \Lambda \rightarrow \Gamma$ making the diagram

$$\begin{array}{ccc} & \Omega & \\ p \swarrow & & \searrow q \\ \Lambda & \xrightarrow{\phi} & \Gamma \end{array}$$

commute; we write $\phi: (\Omega, p) \rightarrow (\Omega, q)$.

From the context it is always clear which type of morphism of coverings we mean. There is an obvious notion of morphism which would unify the two kinds we've introduced, but since we have no use for it we omit it. In the above definition, observe that since p is surjective, there is at most one morphism $\phi: (\Omega, p) \rightarrow (\Omega, q)$.

We will also need to know about quotients by group actions: let $\text{Aut}(\Omega, p)$ denote the automorphism group of a connected covering $p: \Omega \rightarrow \Lambda$. As we shall show in Section 5, the quotient map $\Omega \rightarrow \Omega / \text{Aut}(\Omega, p)$ gives rise to a commuting diagram

$$\begin{array}{ccc} \Omega & \longrightarrow & \Omega / \text{Aut}(\Omega, p) \\ p \downarrow & \swarrow & \\ \Lambda & & \end{array}$$

of connected coverings.

Corollary 2.4. *Let $p: \Omega \rightarrow \Lambda$ be a connected covering, $x \in \Lambda^0$, and $v \in p^{-1}(x)$. Then the following are equivalent:*

- (i) *the subgroup $p_*\pi(\Omega, v)$ of $\pi(\Lambda, x)$ is normal;*
- (ii) *$\text{Aut}(\Omega, p)$ acts transitively on $p^{-1}(x)$;*
- (iii) *the covering $\Omega / \text{Aut}(\Omega, p) \rightarrow \Lambda$ is an isomorphism;*
- (iv) *(Ω, p) is isomorphic to the covering $\Omega \rightarrow \Omega / \text{Aut}(\Omega, p)$.*

Theorem 2.5. *Let $p: \Omega \rightarrow \Lambda$ be a connected covering, $x \in \Lambda^0$, and $v \in p^{-1}(x)$. Then the normalizer $N(p_*\pi(\Omega, v))$ of $p_*\pi(\Omega, v)$ in $\pi(\Lambda, x)$ acts on (Ω, p) , and in fact*

$$\text{Aut}(\Omega, p) \cong N(p_*\pi(\Omega, v))/p_*\pi(\Omega, v).$$

Theorem 2.2 shows how isomorphism classes of connected coverings of a k -graph Λ are inversely related to conjugacy classes of subgroups of the fundamental group $\pi(\Lambda, x)$ (for any choice of vertex x). The identity map on Λ gives a minimal covering, corresponding to the improper subgroup $\pi(\Lambda, x)$. At the opposite extreme:

Definition 2.6. A covering $p: \Omega \rightarrow \Lambda$ is *universal* if it is connected and for every connected covering $q: \Sigma \rightarrow \Lambda$ there exists a morphism $(\Omega, p) \rightarrow (\Sigma, q)$.

For coverings of small categories, the following result is [1, Proposition A.19].

Theorem 2.7. *Every connected k -graph Λ has a universal covering. A connected covering $p: \Omega \rightarrow \Lambda$ is universal if and only if $p_*\pi(\Omega, v) = \{x\}$ for some, hence every, $x \in \Lambda^0$ and $v \in p^{-1}(x)$.*

The following result shows that every subgroup of $\pi(\Lambda, x)$ occurs in the form $p_*\pi(\Omega, v)$ for some connected covering (Ω, p) :

Theorem 2.8. *Let $p: \Omega \rightarrow \Lambda$ be a universal covering, $x \in \Lambda^0$, $v \in p^{-1}(x)$, and H a subgroup of $\pi(\Lambda, x)$. Let H act on (Ω, p) according to Theorem 2.5. Then the associated covering $q: \Omega/H \rightarrow \Lambda$ is connected, and*

$$H = q_*\pi(\Omega/H, vH).$$

Moreover, every connected covering of Λ is isomorphic to one of these coverings $\Omega/H \rightarrow \Lambda$.

3. COVERINGS AND ACTIONS

Definition 3.1. (cf. [3, 12]) An *action* of a groupoid \mathcal{G} on a set V is a functor T from \mathcal{G} to the category of sets such that V is the disjoint union of the sets $T(x)$ for $x \in \mathcal{G}^0$. Put:

- $V_x = T(x)$ for $x \in \mathcal{G}^0$;
- $\mathcal{G} * V = \{(a, v) \mid a \in \mathcal{G}, v \in V_{s(a)}\}$;
- $av = T(a)(v)$ for $(a, v) \in \mathcal{G} * V$.

The *transformation groupoid* is the set $\mathcal{G} * V$ with operations

$$s(a, v) = (s(a), v), \quad r(a, v) = (r(a), av), \quad (a, bv)(b, v) = (ab, v).$$

The *stability group* at $v \in V$ is

$$S_v := \{a \in \mathcal{G} \mid av = v\}.$$

The action (V, \mathcal{G}) is

- *transitive* if $V = \mathcal{G}v$ for some (hence every) $v \in V$;
- *free* if $S_v = \{x\}$ for all $x \in \mathcal{G}^0$ and $v \in V_x$.

Thus, for each object $x \in \mathcal{G}^0$ we have a set V_x , and for each $a \in x\mathcal{G}y$ we have a bijection $v \mapsto av$ from $V_y \rightarrow V_x$. Since we require the sets V_x to be pairwise disjoint, we have a bundle $V \rightarrow \mathcal{G}^0$, and \mathcal{G} acts as bijections among the fibers of this bundle.

Definition 3.2. If \mathcal{G} acts on both V and U , a *morphism* from (V, \mathcal{G}) to (U, \mathcal{G}) is a map $\phi: V \rightarrow U$ which is \mathcal{G} -equivariant in the sense that

$$\phi(av) = a\phi(v) \quad \text{for all } (a, v) \in \mathcal{G} * V.$$

Remark. Thus, a morphism between actions of \mathcal{G} is just a natural transformation between the functors.

We are ready to begin forging the connection between coverings and actions, which will take the form of an equivalence between the categories of coverings of a k -graph Λ (and their morphisms) and actions of $\mathcal{G}(\Lambda)$ (and their morphisms).

Proposition 3.3. *Let Λ be a k -graph, and let its fundamental groupoid $\mathcal{G}(\Lambda)$ act on a set V . Put*

$$\Lambda * V = \{(\alpha, v) \in \Lambda \times V \mid v \in V_{s(\alpha)}\}$$

and

$$\alpha v = i(\alpha)v \quad \text{for } (\alpha, v) \in \Lambda * V.$$

Then $\Lambda * V$ becomes a k -graph with operations

$$\begin{aligned} s(\alpha, v) &= (s(\alpha), v) & r(\alpha, v) &= (r(\alpha), \alpha v) \\ (\alpha, \beta v)(\beta, v) &= (\alpha\beta, v) & d(\alpha, v) &= d(\alpha), \end{aligned}$$

and the coordinate projection $p_\Lambda: \Lambda * V \rightarrow \Lambda$ is a covering.

Moreover, the assignments

$$(1) \quad (V, \mathcal{G}(\Lambda)) \mapsto (\Lambda * V, p_\Lambda) \quad \text{and} \quad \phi \mapsto id_\Lambda * \phi$$

give a functor from actions of $\mathcal{G}(\Lambda)$ to coverings of Λ .

Proof. Routine computations (similar to those showing $\mathcal{G}(\Lambda) * V$ is a groupoid) verify that $\Lambda * V$ is a category. Also, p_Λ is clearly a surjective

morphism. To see that it has the covering property, just note that for $(x, v) \in \Lambda^0 * V = (\Lambda * V)^0$ we have

$$\begin{aligned} (\Lambda * V)(x, v) &= \{(\alpha, v) \mid \alpha \in \Lambda x\} \\ (x, v)(\Lambda * V) &= \{(\alpha, \alpha v) \mid \alpha \in x\Lambda\}. \end{aligned}$$

The map $d: \Lambda * V \rightarrow \mathbb{N}^k$ is the composition of the functors $d: \Lambda \rightarrow \mathbb{N}^k$ and $p_\Lambda: \Lambda * V \rightarrow \Lambda$, so is a functor. We verify the factorization property: let $(\lambda, v) \in \Lambda * V$ and $n, l \in \mathbb{N}^k$ with $d(\lambda, v) = n + l$. Then $d(\lambda) = n + l$, so there exist $\mu \in \Lambda^n, \nu \in \Lambda^l$ such that $\lambda = \mu\nu$. Since

$$s(\mu) = r(\nu) = p(\nu\nu) \quad \text{and} \quad s(\nu) = s(\lambda) = p(v),$$

we have $(\mu, \nu\nu), (\nu, v) \in \Lambda * V$. Since

$$s(\mu, \nu\nu) = (s(\mu), \nu\nu) = (r(\nu), \nu\nu) = r(\nu, v),$$

we can multiply:

$$(\mu, \nu\nu)(\nu, v) = (\mu\nu, v) = (\lambda, v).$$

Since

$$d(\mu, \nu\nu) = d(\mu) = n \quad \text{and} \quad d(\nu, v) = d(\nu) = l,$$

the factorization has the right degrees. For uniqueness, if also

$$(\lambda, v) = (\alpha, \beta v)(\beta, v)$$

with

$$d(\alpha, \beta v) = n \quad \text{and} \quad d(\beta, v) = l,$$

then $\lambda = \alpha\beta$ with $d(\alpha) = n$ and $d(\beta) = l$, so we must have $\alpha = \mu$ and $\beta = \nu$, hence

$$(\alpha, \beta v) = (\mu, \nu\nu) \quad \text{and} \quad (\beta, v) = (\nu, v).$$

Thus $\Lambda * V$ is a k -graph. p_Λ preserves degrees by construction, hence is a covering.

For the other part, routine computations show that if $\phi: V \rightarrow U$ is a morphism of actions of $\mathcal{G}(\Lambda)$, then $\text{id}_\Lambda * \phi$ is a morphism of the corresponding coverings of Λ , and that (1) is functorial. \square

In the opposite direction:

Proposition 3.4. *Let $p: \Omega \rightarrow \Lambda$ be a covering of k -graphs. Then there exists a unique action of $\mathcal{G}(\Lambda)$ on Ω^0 such that*

$$i(\alpha)v = r(\lambda) \quad \text{if } \alpha = p(\lambda) \text{ and } v = s(\lambda),$$

where $i: \Lambda \rightarrow \mathcal{G}(\Lambda)$ is the canonical functor.

Moreover, the assignments

$$(2) \quad (\Omega, p) \mapsto (\Omega^0, \mathcal{G}(\Lambda)) \quad \text{and} \quad \phi \mapsto \phi|_{\Omega^0}$$

give a functor from coverings of Λ to actions of $\mathcal{G}(\Lambda)$.

Proof. For $\alpha \in x\Lambda y$, the range and source maps of Ω take $p^{-1}(\alpha)$ 1-1 onto $p^{-1}(x)$ and $p^{-1}(y)$, respectively, thus affording a bijection $T(\alpha) : p^{-1}(y) \rightarrow p^{-1}(x)$. Routine computations show that the resulting map T from Λ to the groupoid of bijections among the sets in the family $\{p^{-1}(x) \mid x \in \Lambda^0\}$ is functorial. Since T is a functor from Λ into a groupoid, it factors uniquely through a morphism from $\mathcal{G}(\Lambda)$ to the same groupoid, giving the desired action of $\mathcal{G}(\Lambda)$.

For the other part, a routine computation shows that if $\phi : (\Omega, p) \rightarrow (\Sigma, q)$ is a morphism of coverings of Λ , then

$$\phi(\alpha v) = \alpha\phi(v) \quad \text{for } (\alpha, v) \in \Lambda * V.$$

Thus if T and S are the functors giving the actions of $\mathcal{G}(\Lambda)$ on Ω^0 and Σ^0 , respectively, then ϕ gives a natural transformation from Ti to Si (where $i : \Lambda \rightarrow \mathcal{G}(\Lambda)$ is the canonical functor), hence a natural transformation from T to S by universality of i . Therefore $\phi|\Omega^0$ is a morphism of coverings. Routine computations show that (2) is functorial. \square

For coverings of groupoids, the following result is [3, Section 9.4, Exercise 9]. For coverings of small categories, it is similar to [1, Proposition A.23].

Theorem 3.5. *The functors described in the preceding two propositions give a category equivalence between coverings of Λ and actions of $\mathcal{G}(\Lambda)$.*

Proof. First, given a covering (Ω, p) , it follows from the definitions that the map $(p, s) : \Omega \rightarrow \Lambda * \Omega^0$ is bijective, and clearly $p = p_\Lambda(p, s)$. A computation using the identity $r(\lambda) = p(\lambda)s(\lambda)$ for $\lambda \in \Omega$ verifies that (p, s) is functorial. Thus $(\Omega, p) \cong (\Lambda * \Omega^0, p_\Lambda)$.

Next, given an action $(V, \mathcal{G}(\Lambda))$, it is obvious that the map from V to $\Lambda^0 * V = (\Lambda * V)^0$ taking $v \in V_x$ to (x, v) is bijective, and it follows straight from the definitions that it is Λ -equivariant, hence $\mathcal{G}(\Lambda)$ -equivariant. Therefore $(V, \mathcal{G}(\Lambda)) \cong (\Lambda^0 * V, \mathcal{G}(\Lambda))$.

If $\phi : (\Omega, p) \rightarrow (\Sigma, q)$ and $\psi : (V, \mathcal{G}(\Lambda)) \rightarrow (U, \mathcal{G}(\Lambda))$ are morphisms, then the diagrams

$$\begin{array}{ccc} (\Omega, p) & \xrightarrow{\cong} & (\Lambda * \Omega^0, p_\Lambda) \\ \phi \downarrow & & \downarrow \text{id}_\Lambda * \phi|\Omega^0 \\ (\Sigma, q) & \xrightarrow{\cong} & (\Lambda * \Sigma^0, p_\Lambda) \end{array}$$

and

$$\begin{array}{ccc} (V, \mathcal{G}(\Lambda)) & \xrightarrow{\cong} & (\Lambda^0 * V, \mathcal{G}(\Lambda)) \\ \psi \downarrow & & \downarrow \text{id}_{\Lambda^0} * \psi \\ (U, \mathcal{G}(\Lambda)) & \xrightarrow{\cong} & (\Lambda^0 * U, \mathcal{G}(\Lambda)) \end{array}$$

commute. Thus the isomorphisms of the preceding two paragraphs implement a natural equivalence between the functors of Propositions 3.3 and 3.4. \square

The above equivalence matches up the automorphism groups:

Corollary 3.6. *If $p: \Omega \rightarrow \Lambda$ is a covering then the map $\phi \mapsto \phi|_{\Omega^0}$ gives an isomorphism $\text{Aut}(\Omega, p) \cong \text{Aut}(\Omega^0, \mathcal{G}(\Lambda))$.*

Theorem 3.7. *Let $p: \Omega \rightarrow \Lambda$ be a covering, and let $\mathcal{G}(\Lambda)$ act on Ω^0 as in Proposition 3.4. Then the map*

$$(p_*, s): \mathcal{G}(\Omega) \rightarrow \mathcal{G}(\Lambda) * \Omega^0$$

is a groupoid isomorphism.

Before giving the proof of this theorem, let us use it to deduce the following two results, which are similar to [3, 9.4.2] in the case of coverings of groupoids, and to [1, Proposition A.22] in the case of coverings of small categories.

Corollary 3.8. *A covering $p: \Omega \rightarrow \Lambda$ is connected if and only if the corresponding groupoid action $(\Omega^0, \mathcal{G}(\Lambda))$ is transitive.*

Proof. Ω is connected if and only if $\mathcal{G}(\Omega)$, equivalently $\mathcal{G}(\Lambda) * \Omega^0$, is. For $(a, v) \in \mathcal{G}(\Lambda) * \Omega^0$, we have $s(a, v) = (s(a), v)$ and $r(a, v) = (r(a), av)$. It follows that $\mathcal{G}(\Omega)$ is connected if and only if for all $u, v \in \Omega^0$ there exists $a \in \mathcal{G}(\Lambda)$ such that $u = av$, i.e., if and only if $\mathcal{G}(\Lambda)$ acts transitively on Ω^0 . \square

Corollary 3.9. *Let $p: \Omega \rightarrow \Lambda$ be a covering, $x \in \Lambda^0$, and $v \in p^{-1}(x)$. Then p_* maps the fundamental group $\pi(\Omega, v)$ isomorphically onto the stability group S_v of the corresponding groupoid action $(\Omega^0, \mathcal{G}(\Lambda))$.*

Proof. The isomorphism of Theorem 3.7 takes the morphism p_* to the coordinate projection $\mathcal{G}(\Lambda) * \Omega^0 \rightarrow \mathcal{G}(\Lambda)$. It follows that p_* maps $\mathcal{G}(\Omega)v$ 1-1 onto $\mathcal{G}(\Lambda)x$. Thus p_* maps $\pi(\Omega, v)$ isomorphically onto some subgroup of $\pi(\Lambda, x)$. For $c \in \mathcal{G}(\Omega)v$ and $a = p(c)$ we have $av = r(c)$, so $c \in \pi(\Omega, v)$ if and only if $a \in S_v$. The result follows. \square

In [1, Proposition A.17] (for coverings of small categories), the above injectivity of p_* is asserted to follow “directly from the definition of a covering”, but to us it doesn’t seem so immediate.

Proof of Theorem 3.7. Our strategy is to present the fundamental groupoids of Ω and Λ as path categories of augmented graphs modulo cancellation relations and commuting squares, and match up the kernels. More precisely, we will build a commutative diagram

$$\begin{array}{ccc} \mathcal{P}(F^+) & \xrightarrow[\text{onto}]{R} & \mathcal{G}(\Omega) \\ (q_*,s) \downarrow \cong & & \downarrow (p_*,s) \\ \mathcal{P}(E^+) * \Omega^0 & \xrightarrow[Q*\text{id}]{\text{onto}} & \mathcal{G}(\Lambda) * \Omega^0 \end{array}$$

of functors, where the left-hand vertical isomorphism takes the equivalence relation determined by the top horizontal functor R onto the equivalence relation determined by the bottom horizontal functor $Q*\text{id}$. This will suffice to show that (p_*, s) is an isomorphism, since the horizontal functors are surjective.

Let E be the *1-skeleton* of Λ , that is, the graph whose vertices coincide with those of Λ and whose edges E^1 comprise all elements of Λ whose degree is a standard basis vector in \mathbb{N}^k . We need to recall a few things from [19]. A *diagram of type E* in a category \mathcal{C} is a map $D: E \rightarrow \mathcal{C}$ which is a morphism from E to the underlying graph of \mathcal{C} . There is a (small) *path category* $\mathcal{P}(E)$ and a *canonical diagram* $\Delta: E \rightarrow \mathcal{P}(E)$ with the universal property that for every diagram $D: E \rightarrow \mathcal{C}$ there is a unique functor T making the diagram

$$\begin{array}{ccc} E & \xrightarrow{\Delta} & \mathcal{P}(E) \\ & \searrow D & \downarrow T \\ & & \mathcal{C} \end{array}$$

commute. The assignment $E \mapsto \mathcal{P}(E)$ is functorial from graphs to small categories. A *relation* for E is a pair (α, β) of paths in $\mathcal{P}(E)$ with $s(\alpha) = s(\beta)$ and $r(\alpha) = r(\beta)$. If K is a set of relations for E , a diagram D of type E *satisfies* K if $T(\alpha) = T(\beta)$ for all $(\alpha, \beta) \in K$. Let S_Λ denote the set of all *commuting squares* for Λ , i.e., relations for E of the form (ef, gh) , where e and f are composable edges in E with orthogonal degrees and g and h are the unique edges such that $d(g) = d(f)$, $d(h) = d(e)$ and $ef = gh$.

The *augmented graph* $E^+ = E \cup E^{-1}$, where E^{-1} denotes the inverse edges, can be used to give a presentation of the fundamental groupoid; more precisely, letting C_E be the set $\{(e^{-1}e, s(e)) \mid e \in E^1 \cup E^{-1}\}$ of *cancellation relations* for E , there is a surjective functor, which we

denote for this proof by Q , making the diagram

$$\begin{array}{ccc} E^+ & \longleftarrow & E \\ \Delta \downarrow & & \downarrow i|_E \\ \mathcal{P}(E^+) & \xrightarrow[Q]{\text{onto}} & \mathcal{G}(\Lambda) \end{array}$$

commute, such that the associated equivalence relation on $\mathcal{P}(E^+)$ is generated by $C_E \cup S_\Lambda$, where $\Delta: E^+ \rightarrow \mathcal{P}(E^+)$ is the canonical diagram and $i: \Lambda \rightarrow \mathcal{G}(\Lambda)$ is the canonical functor. In particular, the diagram $Q\Delta: E^+ \rightarrow \mathcal{G}(\Lambda)$ satisfies C_E , i.e.,

$$Q\Delta(e^{-1}) = Q\Delta(e)^{-1} \quad \text{for all } e \in E^1.$$

Let $R: \mathcal{P}(F^+) \rightarrow \mathcal{G}(\Omega)$ be the corresponding surjective functor for the 1-skeleton F of the covering k -graph Ω .

Consider the diagram

$$\begin{array}{ccccc} \mathcal{P}(F^+) & \xrightarrow{R} & & \mathcal{G}(\Omega) & \\ & \swarrow \Delta & & \nearrow i & \\ & & F^+ \longleftarrow F \longrightarrow \Omega & & \\ & & \downarrow q \quad \downarrow p|_F \quad \downarrow p & & \downarrow p_* \\ & & E^+ \longleftarrow E \longrightarrow \Lambda & & \\ & \swarrow \Delta & & \searrow i & \\ \mathcal{P}(E^+) & \xrightarrow{Q} & & \mathcal{G}(\Lambda) & \end{array}$$

The Δ 's are canonical diagrams and the i 's are canonical functors. The right-hand quadrilateral commutes by functoriality of $\Lambda \mapsto \mathcal{G}(\Lambda)$. The restriction $p|_F$ is a graph morphism, and takes F onto E by definition of covering and 1-skeleton. The graph morphism $q: F^+ \rightarrow E^+$ is the extension of $p|_F$ defined by

$$q(f^{-1}) = p(f)^{-1}.$$

The inside squares commute by definition of $p|_F$ and q . The left-hand quadrilateral commutes by functoriality of $E \mapsto \mathcal{P}(E)$. The top and bottom 5-sided diagrams commute by construction of R and Q . Thus

$$p_* R \Delta|_F = Q q_* \Delta|_F.$$

Since $Q\Delta$ satisfies C_E , $Q\Delta q = Qq_*\Delta$ satisfies C_F ; since $R\Delta$ also satisfies C_F and p_* is a groupoid morphism, it follows that $p_* R \Delta = Qq_* \Delta$, hence $p_* R = Qq_*$ by universality of Δ .

Thus the diagram

$$\begin{array}{ccc} \mathcal{P}(F^+) & \xrightarrow{R} & \mathcal{G}(\Omega) \\ q_* \downarrow & & \downarrow p_* \\ \mathcal{P}(E^+) & \xrightarrow{Q} & \mathcal{G}(\Lambda) \end{array}$$

commutes. It is easy to see that slightly enlarged diagram

$$\begin{array}{ccc} \mathcal{P}(F^+) & \xrightarrow{R} & \mathcal{G}(\Omega) \\ (q_*, s) \downarrow & & \downarrow (p_*, s) \\ \mathcal{P}(E^+) * \Omega^0 & \xrightarrow{Q * \text{id}} & \mathcal{G}(\Lambda) * \Omega^0 \end{array}$$

also commutes — the sources just come along for the ride. This is the diagram indicated at the beginning of the proof. $q_*: \mathcal{P}(F^+) \rightarrow \mathcal{P}(E^+)$ is a 1-graph covering, and the map (q_*, s) is the associated isomorphism from the proof of Theorem 3.5.

The horizontal functors R and $Q * \text{id}$ are surjective, so it remains to prove that (q_*, s) takes the equivalence relation determined by R , which is generated by the cancellation relations C_E and the commuting squares S_Ω , onto the equivalence relation determined by $Q * \text{id}$, which is generated by the pairs of the form $((\alpha, v), (\beta, v))$, where $(\alpha, \beta) \in C_E \cup S_\Lambda$.

Let $(f^{-1}f, s(f)) \in C_F$, and put $e = q_*(f) \in E^+$. We have

$$q_*(f^{-1}f) = q_*(f)^{-1}q_*(f) = e^{-1}e,$$

and $s(f^{-1}f) = s(f)$, so

$$(q_*, s)(f^{-1}f) = (e^{-1}e, s(f)).$$

On the other hand, $q_*(s(f)) = s(e)$, so

$$(q_*, s)(s(f)) = (s(e), s(f)).$$

Thus

$$(3) \quad (q_*, s)(f^{-1}f, s(f)) = ((e^{-1}e, s(f)), (s(e), s(f)));$$

note that $(e^{-1}e, s(e))$ is a typical element of C_E .

Now let $(ab, cd) \in S_\Omega$, so that $a, b, c, d \in F^1$, the common degree of a and d is orthogonal to the common degree of b and c , and $ab = cd$ in the k -graph Ω . Put

$$e = q(a), \quad f = q(b), \quad g = q(c), \quad \text{and} \quad h = q(d).$$

Then

$$q_*(ab) = ef \quad \text{and} \quad q_*(cd) = gh,$$

and $ef = gh$ in the k -graph Λ because the diagram

$$\begin{array}{ccc} F & \xrightarrow{c} & \Omega \\ p|F \downarrow & & \downarrow p \\ E & \xrightarrow{c} & \Lambda \end{array}$$

commutes. On the other hand, $s(ab) = s(cd)$, so

$$(4) \quad (q_*, s)(ab, cd) = ((ef, s(ab)), (gh, s(ab)));$$

since $q : F \rightarrow E$ is a covering, (ef, gh) is a typical element of S_Λ .

Together, Equations (3) and (4) show that (q_*, s) takes $C_F \cup S_\Omega$ onto the set $\{((\alpha, v), (\beta, v)) \mid (\alpha, \beta) \in C_E \cup S_\Lambda\}$, which suffices. \square

It follows from Theorem 3.7 that if $p : \Omega \rightarrow \Lambda$ is a k -graph covering, then $p_* : \mathcal{G}(\Omega) \rightarrow \mathcal{G}(\Lambda)$ is a *groupoid covering* in the sense of [3, 12].

4. CLASSIFICATION OF TRANSITIVE GROUPOID ACTIONS

By the results of the preceding section, to classify connected coverings we only need the well-known classification of transitive groupoid actions. The results we state in this section are elementary and we claim no originality. We supply the proofs for the convenience of the reader.

Proposition 4.1. *Let (V, \mathcal{G}) be a transitive groupoid action and $x \in \mathcal{G}^0$. Then the family $\{S_v \mid v \in V_x\}$ is a conjugacy class of subgroups of $x\mathcal{G}x$.*

Proof. Just note that $S_{av} = aS_v a^{-1}$ for $(a, v) \in \mathcal{G} * V$. \square

Proposition 4.2. *Let a groupoid \mathcal{G} act transitively on both V and U , and let $x \in \mathcal{G}^0$, $v \in V_x$, and $u \in U_x$. Then there is a morphism $(V, \mathcal{G}) \rightarrow (U, \mathcal{G})$ taking v to u if and only if $S_v \subset S_u$.*

Proof. If $\phi : V \rightarrow U$ is equivariant and $\phi(v) = u$, then

$$\phi(av) = a\phi(v) = au \quad \text{for all } a \in \mathcal{G}x,$$

so $S_v \subset S_u$. Conversely, assume $S_v \subset S_u$, and define $\phi : V \rightarrow U$ by

$$\phi(av) = au \quad \text{for } a \in \mathcal{G}x.$$

This is well-defined because if $a, b \in \mathcal{G}x$ and $av = bv$, then

$$b^{-1}a \in S_v \subset S_u,$$

so $au = bu$. Clearly ϕ is equivariant. \square

Proposition 4.3. *Let (V, \mathcal{G}) be a transitive groupoid action, $x \in \mathcal{G}^0$, and $v \in V_x$. Then the normalizer $N(S_v)$ of S_v in $x\mathcal{G}x$ acts on the right of the action (V, \mathcal{G}) by automorphisms, and in fact*

$$\text{Aut}(V, \mathcal{G}) \cong N(S_v)/S_v.$$

Proof. The computations in the proof of Proposition 4.2 show that every automorphism of (V, \mathcal{G}) is of the form $av \mapsto acv$, where $c \in x\mathcal{G}x$ satisfies

$$S_v = S_{cv} = cS_v c^{-1},$$

i.e., $c \in N(S_v)$, and conversely every such c gives rise to an automorphism of (V, \mathcal{G}) in this manner. Define $(av)c = acv$. Then $N(S_v)$ acts on the right, since for $c, d \in N(S_v)$ we have

$$((av)c)d = (acv)d = acdv = (av)cd.$$

Clearly $(av)c = av$ if and only if $c \in S_v$. Thus $N(S_v)/S_v$ acts freely on the right of (V, \mathcal{G}) . The result follows. \square

Our next result is the groupoid-action analogue of Theorem 2.7. However, rather than merely asserting the existence of a certain kind of action of a groupoid, we give more detail, because this will be useful when we apply it to the analogue for coverings. First, we need:

Definition 4.4. A *cocycle* on a groupoid \mathcal{G} is a functor $\eta: \mathcal{G} \rightarrow G$ where G is a group. The *cocycle action* of \mathcal{G} on the Cartesian product $\mathcal{G}^0 \times G$ is given by

$$a(s(a), g) = (r(a), \eta(a)g).$$

We write $\mathcal{G}^0 \times_{\eta} G$ to indicate $\mathcal{G}^0 \times G$ equipped with the cocycle action.

Proposition 4.5. *Let \mathcal{G} be a connected groupoid and $x \in \mathcal{G}^0$. There is a cocycle $\eta: \mathcal{G} \rightarrow x\mathcal{G}x$ such that the associated cocycle action $(\mathcal{G}^0 \times_{\eta} x\mathcal{G}x, \mathcal{G})$ is free and transitive.*

Proof. For each $y \in \mathcal{G}^0$ pick $t_y \in y\mathcal{G}x$, with $t_x = x$. Then $\eta(a) = t_{r(a)}^{-1} a t_{s(a)}$ defines a surjective cocycle $\mathcal{G} \rightarrow x\mathcal{G}x$ which is the identity map on $x\mathcal{G}x$.

For $a \in y\mathcal{G}z$ and $g \in x\mathcal{G}x$ we have $a(z, g) = (y, \eta(a)g)$. The action is transitive, because if $y \in \mathcal{G}^0$ and $g \in x\mathcal{G}x$ we have $t_y g(x, x) = (y, g)$. By transitivity, to show that the action is free it suffices to check the stability group at (x, x) : for $a \in y\mathcal{G}x$ we have $a(x, x) = (y, \eta(a))$, so if $a \in S_{(x,x)}$ then $y = x$ and $\eta(a) = x$, so $a \in x\mathcal{G}x$, hence $\eta(a) = a$, thus $a = x$. \square

We next give a groupoid-action analogue of Theorem 2.8. First note that if a group G acts on (the right of) a groupoid action (V, \mathcal{G}) by automorphisms, then \mathcal{G} acts on the quotient set V/G by $a(vG) = (av)G$.

Proposition 4.6. *Let (V, \mathcal{G}) be a free transitive groupoid action, $x \in \mathcal{G}^0$, $v \in V_x$, and H a subgroup of $x\mathcal{G}x$. Let H act on (V, \mathcal{G}) according to Proposition 4.3. Then the associated action $(V/H, \mathcal{G})$ is transitive, and $H = S_{vH}$.*

Proof. The action of \mathcal{G} on V/H is transitive since it is a quotient of the transitive action on V . We have $V = \mathcal{G}v$, H acts on V by $(av)h = ahv$, and \mathcal{G} acts on V/H by $a(vH) = avH$. Thus $a(vH) = vH$ if and only if $av \in Hv$, equivalently $a \in H$ by freeness. \square

5. PROOFS OF MAIN RESULTS

Proof of Theorem 2.2. The corresponding groupoid actions $(\Omega^0, \mathcal{G}(\Lambda))$ and $(\Sigma^0, \mathcal{G}(\Lambda))$ are transitive, by Corollary 3.8. We have $\pi(\Lambda, x) = x\mathcal{G}(\Lambda)x$, and for $v \in p^{-1}(x)$ we have $p_*\pi(\Omega, v) = S_v$, so the first statement follows from Proposition 4.1.

By Theorem 3.5 a morphism $\phi: (\Omega, p) \rightarrow (\Sigma, q)$ with $\phi(v) = u$ corresponds to a morphism $\psi: (\Omega^0, \mathcal{G}(\Lambda)) \rightarrow (\Sigma^0, \mathcal{G}(\Lambda))$ with $\psi(v) = u$, and we have $p_*\pi(\Omega, v) = S_v$ and $q_*\pi(\Sigma, u) = S_u$, so the second statement follows from Proposition 4.2.

The last statement now follows quickly from the above. \square

We need to know about quotients of k -graphs by group actions:

Proposition 5.1. *Let G a group acting freely by automorphisms on the right of a k -graph Ω . Then the quotient set Ω/G becomes a k -graph with operations*

$$\begin{aligned} s(\lambda G) &= s(\lambda)G & r(\lambda G) &= r(\lambda)G \\ (\lambda G)(\mu G) &= (\lambda\mu)G & d(\lambda G) &= d(\lambda), \end{aligned}$$

and the quotient map $\Omega \rightarrow \Omega/G$ is a covering.

Proof. Routine computations using freeness of the action show that Ω/G is a category, and the quotient map is then a surjective functor. Because G acts by automorphisms, all elements of any orbit αG have the same degree, which we define to be $d(\alpha G)$. This gives a functor $d: \Omega/G \rightarrow \mathbb{N}^k$, and by construction the quotient map intertwines the two “ d ”s.

We verify the factorization property: let $\lambda G \in \Omega/G$ and $n, l \in \mathbb{N}^k$ with $d(\lambda G) = n + l$. Then $d(\lambda) = n + l$, so there exist unique $\mu \in \Omega^n$

and $\nu \in \Omega^l$ such that $\lambda = \mu\nu$, and then

$$d(\mu G) = n, \quad d(\nu G) = l, \quad \text{and} \quad \lambda G = \mu G \nu G.$$

For the uniqueness of λG and μG , suppose

$$d(\alpha G) = n, \quad d(\beta G) = l, \quad \text{and} \quad \lambda G = \alpha G \beta G.$$

Then $d(\alpha) = n$ and $d(\beta) = l$, and we can adjust β in the G -orbit so that $s(\alpha) = r(\beta)$. Then $\alpha G \beta G = (\alpha\beta)G$, so there exists $g \in G$ such that

$$(\alpha g)(\beta g) = (\alpha\beta)g = \lambda.$$

But $d(\alpha g) = n$ and $d(\beta g) = l$, so we must have $\alpha g = \mu$ and $\beta g = \nu$, hence $\alpha G = \mu G$ and $\beta G = \nu G$.

Thus Ω/G is a k -graph, and the quotient map is a k -graph morphism. For $v \in \Omega^0$, the quotient map takes Ωv onto $(\Omega/G)(vG)$ by construction; we must show that it is injective on this set. Let $\lambda, \mu \in \Omega v$ such that $\lambda G = \mu G$. Then there exists $g \in G$ such that $\lambda = \mu g$. Thus

$$v = s(\lambda) = s(\mu g) = s(\mu)g = v g.$$

Since G acts freely, we must have $g = e$, hence $\lambda = \mu$. Similarly for $v\Omega$ and $(vG)(\Omega/G)$. \square

Proposition 5.2. *Let G be a group acting freely on the right of a k -graph covering $p: \Omega \rightarrow \Lambda$. Then the map $\lambda G \mapsto p(\lambda): \Omega/G \rightarrow \Lambda$ is a covering.*

Proof. We could deduce this from a corresponding groupoid-action result, but it's faster to prove this one directly. We certainly have a commuting diagram

$$\begin{array}{ccc} \Omega & \twoheadrightarrow & \Omega/G \\ p \downarrow & \swarrow & \\ \Lambda & & \end{array}$$

of surjective functors, where $\Omega \rightarrow \Omega/G$ is the covering from the preceding proposition. It is easy to verify that, whenever we have such a commuting diagram of surjective functors, if two of the maps are coverings, then so is the third. \square

For the proof of our next main result, we wish to apply the above to the automorphism group of a connected covering. For this we need to know that this group acts freely:

Proposition 5.3. *Every automorphism of a connected k -graph covering acts freely.*

Proof. Since the covering is connected, the corresponding groupoid action is transitive, and it is straightforward to verify that every automorphism of a transitive groupoid action acts freely. \square

Proof of Corollary 2.4. The equivalence (i) \Leftrightarrow (ii) follows quickly from Theorem 2.2.

The quotient k -graph $\Omega/\text{Aut}(\Omega, p)$ is connected since Ω is, hence the covering $q: \Omega/\text{Aut}(\Omega, p) \rightarrow \Lambda$ is connected. Thus this covering is an isomorphism if and only if $q^{-1}(x) = \{vG\}$. Since the set $q^{-1}(x)$ coincides with the set of $\text{Aut}(\Omega, p)$ -orbits of elements of $p^{-1}(x)$, we have (ii) \Leftrightarrow (iii).

Finally, for (iii) \Leftrightarrow (iv), just note that the covering $\Omega/\text{Aut}(\Omega, p) \rightarrow \Lambda$ is the unique morphism from the covering $\Omega \rightarrow \Omega/\text{Aut}(\Omega, p)$ to the given covering $\Omega \rightarrow \Lambda$. \square

Proof of Theorem 2.5. Passing to the corresponding groupoid action $(\Omega^0, \mathcal{G}(\Lambda))$, the result follows from Proposition 4.3. \square

Later we will need a precise description of the action of $N(p_*\pi(\Omega, v))$ on (Ω, p) corresponding to the action of $N(S_v)$ on $(\Omega^0, \mathcal{G}(\Lambda))$, and we record this here: let $\lambda \in \Omega$ and $c \in N(p_*\pi(\Omega, v))$. Since the covering (Ω, p) is connected, by Corollary 3.8 $\mathcal{G}(\Lambda)$ acts transitively on Ω^0 , so there exists $a \in \mathcal{G}(\Lambda)$ such that $s(\lambda) = av$. Then λc is the unique element of Ω such that

$$p(\lambda c) = p(\lambda) \quad \text{and} \quad s(\lambda c) = acv.$$

This is well-defined because c normalizes $p_*\pi(\Omega, v)$.

Proof of Theorem 2.7. By [19, Proposition 5.9], the fundamental groupoid $\mathcal{G}(\Lambda)$ is connected since Λ is. Proposition 4.5 gives a certain free and transitive action $(V, \mathcal{G}(\Lambda))$. Let $p: \Omega \rightarrow \Lambda$ be the corresponding covering, which is connected by Corollary 3.8, and let $x \in \Lambda^0$ and $v \in p^{-1}(x)$. Then

$$p_*(\Omega, v) = S_v = \{x\},$$

so the covering (Ω, p) is universal by Theorem 2.2.

Moreover, again by Theorem 2.2, if (Σ, q) is any universal covering of Λ , then because there is a morphism $(\Sigma, q) \rightarrow (\Omega, p)$, we must have $q_*\pi(\Sigma, u) = \{x\}$ for all $u \in q^{-1}(x)$. \square

It will be useful to record the following alternative characterization of universal coverings. But first:

Definition 5.4. A k -tree is a connected k -graph Ω with $\pi(\Omega, v) = \{v\}$ for some, hence every, vertex $v \in \Omega^0$.

Corollary 5.5. *If $p: \Omega \rightarrow \Lambda$ is a connected covering, then the following are equivalent:*

- (i) *the covering (Ω, p) is universal;*
- (ii) *the corresponding groupoid action $(\Omega^0, \mathcal{G}(\Lambda))$ is free;*
- (iii) *Ω is a k -tree.*

Proof. This follows from Theorem 2.7 since p_* is injective and $S_v = p_*\pi(\Omega, v)$. □

Remark. A 1-tree is the path category (modulo conventions regarding composition) of a graph which is a tree in the usual sense. In a 1-tree, between any 2 vertices there is at most 1 undirected path, hence certainly at most 1 directed path; this does not generally hold in k -trees, as illustrated by one of our basic examples [19, Example 7.2] of a k -graph Λ which does not embed faithfully in its fundamental groupoid. This was a 2-graph with 4 vertices, 4 horizontal edges, and 6 vertical edges. All multiple edges between vertices collapsed under the canonical functor i , making the fundamental groupoid an equivalence relation on 4 objects, thus the fundamental groups were all trivial. Hence this is an example of a 2-tree with multiple morphisms with the same source and range. This is unfortunate, because it means that in practice we have no effective algorithm for determining whether a given k -graph is a k -tree, short of computing the fundamental group.

Proof of Theorem 2.8. First of all, since the covering is universal, we have $p_*\pi(\Omega, v) = \{x\}$. Thus the action of H on (Ω, p) guaranteed by Theorem 2.5 is free, hence by Proposition 5.2 we really do have a covering $q: \Omega/H \rightarrow \Lambda$. Moreover, this covering is connected since it is a quotient of the connected covering (Ω, p) . By Corollary 5.5 the corresponding groupoid action $(\Omega^0, \mathcal{G}(\Lambda))$ is free. We have $x\mathcal{G}(\Lambda)x = \pi(\Lambda, x)$, so H acts on $(\Omega^0, \mathcal{G}(\Lambda))$ according to Proposition 4.3. By Proposition 4.6 we have $H = S_{vH}$. Since $q_*\pi(\Omega/H, vH) = S_{vH}$ by Corollary 3.9, we have shown the first part of the theorem. The other part now follows immediately from Theorem 2.2. □

6. SKEW PRODUCTS

Our statement of Theorem 2.7 merely asserted the existence of universal coverings, and the proof merely showed how this existence followed from the analogous Proposition 4.5 for groupoid actions. However, Proposition 4.5 gave a specific construction of the desired groupoid action using cocycles. The analogue for k -graph coverings is a skew product:

Definition 6.1. A *cocycle* on a k -graph Λ is a functor from Λ to a group.

Observation 6.2. *Since a cocycle $\eta: \Lambda \rightarrow G$ is a functor into a group(oid), there is a unique cocycle κ making the diagram*

$$\begin{array}{ccc} \Lambda & \xrightarrow{\eta} & G \\ i \downarrow & \nearrow \kappa & \\ \mathcal{G}(\Lambda) & & \end{array}$$

commute. In fact, this gives a 1-1 correspondence between cocycles on Λ and on $\mathcal{G}(\Lambda)$.

Proposition 6.3. *Let $\eta: \Lambda \rightarrow G$ be a k -graph cocycle. Then the Cartesian product $\Lambda \times G$ becomes a k -graph with operations*

$$\begin{aligned} s(\alpha, g) &= (s(\alpha), g) & r(\alpha, g) &= (r(\alpha), \eta(\alpha)g) \\ (\alpha, \eta(\beta)g)(\beta, g) &= (\alpha\beta, g) & d(\alpha, g) &= d(\alpha), \end{aligned}$$

and the coordinate projection $\Lambda \times G \rightarrow \Lambda$ is a covering.

Proof. Let $\eta': \mathcal{G}(\Lambda) \rightarrow G$ be the corresponding groupoid cocycle. The associated cocycle action of $\mathcal{G}(\Lambda)$ is on the set $\Lambda^0 \times G$, and the covering k -graph corresponding to this groupoid action is $\Lambda * (\Lambda^0 \times G)$. The map

$$(\alpha, (x, g)) \mapsto (\alpha, g): \Lambda * (\Lambda^0 \times G) \rightarrow \Lambda \times G$$

is bijective, transforms the k -graph operations on $\Lambda * (\Lambda^0 \times G)$ into the operations on $\Lambda \times G$ indicated in the proposition, and transforms the corresponding covering $\Lambda * (\Lambda^0 \times G) \rightarrow \Lambda$ into the coordinate projection $\Lambda \times G \rightarrow \Lambda$. \square

The following definition appears in [17, Definition 5.1]:

Definition 6.4. The *skew product k -graph* associated to a cocycle $\eta: \Lambda \rightarrow G$ is $\Lambda \times G$ with the operations from Proposition 6.3. We write $\Lambda \times_{\eta} G$ to indicate this k -graph. The *skew-product covering* is the coordinate projection $p_{\Lambda}: \Lambda \times_{\eta} G \rightarrow \Lambda$.

Note that in the proof of the above proposition, the skew-product covering $\Lambda \times_{\eta} G \rightarrow \Lambda$ was not exactly the same as the covering corresponding to the cocycle action $(\mathcal{G}^0 \times G, \mathcal{G})$, rather these coverings were merely isomorphic — nevertheless for convenience we regard the skew-product covering as *corresponding* to the cocycle action, thus committing a mild abuse.

We can now apply this to construct universal coverings:

Corollary 6.5. *Let Λ be a connected k -graph and $x \in \Lambda^0$. Then there is a cocycle $\eta: \Lambda \rightarrow \pi(\Lambda, x)$ such that the skew-product covering $\Lambda \times_\eta \pi(\Lambda, x) \rightarrow \Lambda$ is universal.*

Proof. In the proof of Proposition 4.5 we constructed a groupoid cocycle $\mathcal{G}(\Lambda) \rightarrow \pi(\Lambda, x)$; let $\eta: \Lambda \rightarrow \pi(\Lambda, x)$ be the associated k -graph cocycle. The skew-product covering corresponds to the cocycle action of $\mathcal{G}(\Lambda)$, and the proof of Theorem 2.7 showed that this corresponding covering is universal. \square

Proposition 6.6. *Let $\eta: \Lambda \rightarrow G$ be a k -graph cocycle. Then G acts freely on the skew-product covering $\Lambda \times_\eta G \rightarrow \Lambda$ via*

$$(\lambda, g)h = (\lambda, gh) \quad \text{for } \lambda \in \Lambda, g, h \in G.$$

Proof. This can be checked directly without pain, but it is even easier to note that G acts on the corresponding groupoid action $(\Lambda^0 \times G, \mathcal{G}(\Lambda))$ by

$$(x, g)h = (x, gh) \quad \text{for } x \in \Lambda^0, g, h \in G,$$

and then apply Proposition 3.6. It is obvious that the action is free. \square

For a connected k -graph Λ , Corollary 6.5 gives a specific construction of a universal covering $\Lambda \times_\eta \pi(\Lambda, x) \rightarrow \Lambda$, so Theorem 2.5 gives an action of $\pi(\Lambda, x)$ on this covering. In the following result we verify that this action coincides with the one guaranteed by Proposition 6.6:

Proposition 6.7. *Let Λ be a connected k -graph, and let $\Lambda \times_\eta \pi(\Lambda, x) \rightarrow \Lambda$ be the universal covering as in Corollary 6.5. Then the action of $\pi(\Lambda, x)$ on the skew product $\Lambda \times_\eta \pi(\Lambda, x)$ given by Theorem 2.5 agrees with the action given by Proposition 6.6.*

Proof. The corresponding action of $\mathcal{G}(\Lambda)$ is on $(\Lambda \times_\eta \pi(\Lambda, x))^0 = \Lambda^0 \times \pi(\Lambda, x)$. Let's see how the proof of Theorem 2.5 tells us $\pi(\Lambda, x)$ acts on $\Lambda \times_\eta \pi(\Lambda, x)$: denoting the skew product covering map $\Lambda \times_\eta \pi(\Lambda, x) \rightarrow \Lambda$ by p_Λ , we must start by choosing a vertex $v \in p_\Lambda^{-1}(x)$. Then the action of an element $h \in \pi(\Lambda, x)$ on an element $(\lambda, g) \in \Lambda \times_\eta \pi(\Lambda, x)$ is computed as follows: find $b \in \mathcal{G}(\Lambda)$ such that $bv = s(\lambda, g)$, and then $(\lambda, g)h$ is the unique element of $\Lambda \times_\eta \pi(\Lambda, x)$ such that both

$$\begin{aligned} p_\Lambda((\lambda, g)h) &= p_\Lambda(\lambda, g) = \lambda & \text{and} \\ s((\lambda, g)h) &= bhv. \end{aligned}$$

For v we choose (x, x) . By definition of the cocycle action, for any $b \in \mathcal{G}(\Lambda)$ we have

$$b(x, x) = (r(b), \eta(b)).$$

Given (λ, g) , put $y = s(\lambda)$. Then $s(\lambda, g) = (y, g)$, so we want $b \in \mathcal{G}(\Lambda)$ such that

$$(r(b), \eta(b)) = (y, g).$$

The cocycle η constructed in the proof of Proposition 4.5 takes $y\mathcal{G}(\Lambda)x$ onto $\pi(\Lambda, x)$, so such an element b exists. Then for such a b we have

$$bh(x, x) = (r(bh), \eta(bh)) = (r(b), \eta(b)\eta(h)) = (y, gh).$$

Therefore $(\lambda, g)h = (\lambda, gh)$. as desired, since

$$p_\Lambda(\lambda, gh) = \lambda \quad \text{and} \quad s(\lambda, gh) = (y, gh). \quad \square$$

Theorem 2.8 concerns an action of a subgroup H of the fundamental group $\pi(\Lambda, x)$ on a universal covering of Λ ; we want to see how this looks when Λ is a skew product $\Lambda \times_\eta \pi(\Lambda, x)$ as in Corollary 6.5. It is cleaner to do it in the abstract: let $\eta: \Lambda \rightarrow G$ be a cocycle and H a subgroup of G , and let H act on the skew product $\Lambda \times_\eta G$ according to Proposition 6.6. Since this action is free, we can form the associated covering $(\Lambda \times_\eta G)/H \rightarrow \Lambda$. The map

$$(\lambda, g)H \mapsto (\lambda, gH): (\Lambda \times_\eta G)/H \rightarrow \Lambda \times (G/H)$$

is bijective, transforms the k -graph operations on the quotient $(\Lambda \times_\eta G)/H$ into

$$\begin{aligned} s(\lambda, gH) &= (s(\lambda), gH) & r(\lambda, gH) &= (r(\lambda), \eta(\lambda)gH) \\ (\lambda, \eta(\mu)gH)(\mu, gH) &= (\lambda\mu, gH) & d(\lambda, gH) &= d(\lambda), \end{aligned}$$

and transforms the covering $(\Lambda \times_\eta G)/H \rightarrow \Lambda$ into the coordinate projection $\Lambda \times G/H \rightarrow \Lambda$.

Definition 6.8. If $\eta: \Lambda \rightarrow G$ is a k -graph cocycle and H is a subgroup of G , the *relative skew product k -graph*, denoted $\Lambda \times_\eta G/H$, is the Cartesian product $\Lambda \times G/H$ with the above operations, and the *relative skew product covering* is the coordinate projection $\Lambda \times_\eta G/H \rightarrow \Lambda$.

We should point out that this concept is not new: a version of relative skew products for graphs appears in, for example, [4, 11]. While we did not need relative skew products for the general theory of coverings — for us they arose as just a particular case of quotients of skew products — they will be important for us in our application to C^* -coactions.

Let's formalize the above discussion:

Proposition 6.9. *If $\eta: \Lambda \rightarrow G$ is a k -graph cocycle and H a subgroup of G , then the associated covering $(\Lambda \times_\eta G)/H \rightarrow \Lambda$ is isomorphic to the relative skew product covering $\Lambda \times_\eta G/H \rightarrow \Lambda$ via the map*

$$(a, g)H \mapsto (a, gH).$$

The value of the above definition is that it captures all connected coverings, as we show in the following result, a graph version of which appeared in [4, Proposition 2.2]:

Corollary 6.10. *Every connected covering is isomorphic to a relative skew-product covering.*

Proof. Let $p: \Omega \rightarrow \Lambda$ be a connected covering, $x \in \Lambda^0$, and $v \in p^{-1}(x)$. It follows from Theorem 2.8, Corollary 6.5, and Propositions 6.7 and 6.9 that (Ω, p) is isomorphic to a relative skew product covering $\Lambda \times_\eta \pi(\Lambda, x)/p_*\pi(\Omega, v)$. \square

Gross-Tucker Theorem. If the subgroup H of G is normal, then a relative skew product $\Lambda \times_\eta G/H$ may be regarded as an ordinary skew product associated to the cocycle $\Lambda \rightarrow G/H$ obtained from η by composing with the quotient homomorphism $G \rightarrow G/H$. In particular, with the notation from the above proof, if the subgroup $p_*\pi(\Omega, v)$ of $\pi(\Lambda, x)$ is normal, then the given connected covering (Ω, p) is isomorphic to a skew-product covering. From Corollary 2.4, we know that this will happen if and only if $\text{Aut}(\Omega, p)$ acts transitively on $p^{-1}(x)$. One situation where this is obviously true is for a covering $\Omega \rightarrow \Omega/G$, where G is a group acting freely on a connected k -graph Ω .

While all this is a nice application of the general theory of connected coverings, it cheats us out of the full truth: connectedness of Ω is unnecessary, as we'll show in the following result, a version of the Gross-Tucker Theorem (for the graph version, see [11, Theorem 2.2.2]). In the disconnected case it is more efficient to give a “bare-hands” proof. Actually, this result appears in [17, Remark 5.6], but we prove it here for the convenience of the reader, since we will need this more general result for C^* -coactions:

Theorem 6.11 (Gross-Tucker Theorem). *Let G be a group acting freely on a k -graph Σ . Then the covering $\Sigma \rightarrow \Sigma/G$ given by the quotient map is isomorphic to a skew product covering $(\Sigma/G) \times_\eta G \rightarrow \Sigma/G$.*

Proof. The corresponding groupoid-action result, Lemma 6.12 below, is easier, so here we merely indicate how the Gross-Tucker Theorem will follow. Put $\Lambda = \Sigma/G$, and let $t: \Sigma \rightarrow \Lambda$ be the quotient map. Then G is a subgroup of $\text{Aut}(\Sigma, t)$ acting freely and transitively on each set $t^{-1}(x)$ for $x \in \Lambda^0$. By the elementary Lemma 6.12 below, the associated groupoid action $(\Sigma^0, \mathcal{G}(\Lambda))$ is isomorphic to a cocycle action $(\Lambda^0 \times_\eta G, \mathcal{G}(\Lambda))$, so the covering $t: \Sigma \rightarrow \Lambda$ is isomorphic to a skew-product covering $\Lambda \times_\eta G \rightarrow \Lambda$. \square

We must pay the debt we incurred in the above proof:

Lemma 6.12. *Let G a group acting on the right of a groupoid action (V, \mathcal{G}) , freely and transitively on each set V_x for $x \in \mathcal{G}^0$. Then (V, \mathcal{G}) is isomorphic to a cocycle action.*

Proof. We begin by choosing a cross-section of the map $V \rightarrow \mathcal{G}^0$: for each $x \in \mathcal{G}^0$ pick $v_x \in V_x$. Let $x, y \in \mathcal{G}^0$ and $a \in x\mathcal{G}y$. Then both av_y and v_x are in V_x , so by hypothesis there exists a unique element $\eta(a)$ of G such that $v_x\eta(a) = av_y$. We verify that the resulting map $\eta: \mathcal{G} \rightarrow G$ is a cocycle: if $x, y, z \in \mathcal{G}^0$, $a \in x\mathcal{G}y$, and $b \in y\mathcal{G}z$, then

$$v_x\eta(a)\eta(b) = av_y\eta(b) = abv_z = v_x\eta(ab),$$

so $\eta(a)\eta(b) = \eta(ab)$ since G acts freely.

Define $\phi: \mathcal{G}^0 \times_{\eta} G \rightarrow V$ by $\phi(x, g) = v_xg$. To see that ϕ is injective, let $(x, g), (y, h) \in \mathcal{G}^0 \times G$, and assume $\phi(x, g) = \phi(y, h)$. Then

$$v_yhg^{-1} = v_x \in V_x.$$

Since $v_y \in V_y$, so is v_yhg^{-1} . Thus we must have $x = y$, hence $g = h$ since G acts freely. To see that ϕ is surjective, let $x \in \mathcal{G}^0$ and $v \in V_x$. Since G acts transitively on V_x , we can choose $g \in G$ such that $v = v_xg$, and then $v = \phi(x, g)$. For $x, y \in \mathcal{G}^0$, $a \in x\mathcal{G}y$, and $g \in G$ we have

$$\phi(a(y, g)) = \phi(x, \eta(a)g) = v_x\eta(a)g = av_yg = a\phi(y, g),$$

so ϕ intertwines the cocycle action and the given action. Therefore $\phi: (\mathcal{G}^0 \times_{\eta} G, \mathcal{G}) \rightarrow (V, \mathcal{G})$ is an isomorphism. \square

7. COACTIONS

In this section we discuss the implications of our results for the C^* -algebras of k -graphs. There are by now several different classes of k -graphs Λ whose C^* -algebras $C^*(\Lambda)$ admit a satisfactory structure theory, and our results apply to all of them. Indeed, of the usual theory we need to know only that the core is AF and that the Gauge-Invariant Uniqueness Theorem holds. Thus the results of this section apply to, in increasing order of generality, the row-finite k -graphs without sources of [17], the locally convex row-finite k -graphs of [23], and the finitely aligned k -graphs of [24].

The main point of [14] is that a labelling of a graph gives rise to a coaction on the graph C^* -algebra, and that moreover the coaction crossed product is isomorphic to the C^* -algebra of the skew-product graph. Here we adapt this to k -graphs.

For C^* -coactions we adopt the conventions of [5, 6, 21, 22]. A *coaction* of a group G on a C^* -algebra A is an injective unital homomorphism $\delta: A \rightarrow A \otimes C^*(G)$ satisfying the *coaction identity* $(\text{id} \otimes \delta_G)\delta = (\delta \otimes \text{id})\delta$. For $g \in G$ the associated *spectral subspace* of A is $A_g := \{a \in A \mid$

$\delta(a) = a \otimes g\}$, and the *fixed point algebra* is $A^\delta := A_e$. The disjoint union $\mathcal{A} := \bigsqcup_{g \in G} A_g$ is a *Fell bundle* in the sense that $A_g A_h \subset A_{gh}$ and $A_g^* = A_{g^{-1}}$, and the linear span $\Gamma_c(\mathcal{A})$ is a dense $*$ -subalgebra of A . Define $\rho: \mathcal{A} \rightarrow G$ by $\rho(a) = g$ if $a \in A_g$. The coaction δ is called *maximal* if the norm of A is the largest C^* -norm on the $*$ -algebra $\Gamma_c(\mathcal{A})$ [5], and *normal* if $(\text{id} \otimes \lambda)\delta$ is injective, where λ is the left regular representation of G [21]. The Cartesian product $\mathcal{A} \times G/H$ is a Fell bundle over the transformation groupoid $G \times G/H$, with operations

$$(a, \rho(b)gH)(b, gH) = (ab, gH) \quad \text{and} \quad (a, gH)^* = (a^*, \rho(a)gH),$$

and the linear span $\Gamma_c(\mathcal{A} \times G/H)$ is a $*$ -algebra, whose completion $A \times_{\delta|} G/H$ in the largest C^* -norm is the *restricted crossed product* of A by δ . When $H = \{e\}$ the *dual action* of G on the crossed product $A \times_{\delta} G$ is given by $\widehat{\delta}_h(a, g) = (a, gh^{-1})$.

Let $\eta: \Lambda \rightarrow G$ be a k -graph cocycle. The right action of G on $\Lambda \times_{\eta} G$ discussed in Proposition 6.6 induces an action γ of G on $C^*(\Lambda \times_{\eta} G)$ such that

$$\gamma_h(s_{(\lambda, g)}) = s_{(\lambda, gh^{-1})}.$$

Theorem 7.1. *Let $\eta: \Lambda \rightarrow G$ be a k -graph cocycle and H a subgroup of G . Then:*

(i) *there exists a unique coaction $\delta = \delta_{\eta}$ of G on $C^*(\Lambda)$ such that*

$$\delta(s_{\lambda}) = s_{\lambda} \otimes \eta(\lambda) \quad \text{for } \lambda \in \Lambda;$$

(ii) *$C^*(\Lambda \times_{\eta} G/H) \cong C^*(\Lambda) \times_{\delta|} G/H$;*

(iii) *if $H = \{e\}$ the above isomorphism is equivariant for the action γ of G on $C^*(\Lambda \times_{\eta} G)$ and the dual action $\widehat{\delta}$ on $C^*(\Lambda) \times_{\delta} G$;*

(iv) *the coaction δ is both maximal and normal.*

Our desired extension of [4, Theorem 3.2] follows immediately from Theorem 7.1 and Corollary 6.10:

Corollary 7.2. *Let $p: \Omega \rightarrow \Lambda$ be a connected covering, $x \in \Lambda^0$, and $v \in p^{-1}(x)$. Then there exists a coaction δ of $\pi(\Lambda, x)$ on $C^*(\Lambda)$ such that*

$$C^*(\Omega) \cong C^*(\Lambda) \times_{\delta|} \pi(\Lambda, x)/p_*\pi(\Omega, v).$$

Proof of Theorem 7.1. (i) It is routine to verify that the assignment $\lambda \mapsto s_{\lambda} \otimes \eta(\lambda)$ gives a Cuntz-Krieger Λ -family in $C^*(\Lambda) \otimes C^*(G)$, and hence determines a unique homomorphism

$$\delta: C^*(\Lambda) \rightarrow C^*(\Lambda) \otimes C^*(G).$$

δ is obviously unital and satisfies the coaction identity, and the Gauge-Invariant Uniqueness Theorem shows that δ is injective.

(ii) Define $\theta: \Lambda \times_\eta G/H \rightarrow C^*(\Lambda) \times_{\delta|} G/H$ by

$$\theta(\lambda, gH) = (s_\lambda, gH).$$

It is routine to verify that this gives a Cuntz-Krieger $(\Lambda \times_\eta G/H)$ -family in the C^* -algebra $C^*(\Lambda) \times_{\delta|} G/H$, hence determines a homomorphism

$$\theta: C^*(\Lambda \times_\eta G/H) \rightarrow C^*(\Lambda) \times_{\delta|} G/H.$$

The Gauge-Invariant Uniqueness Theorem shows that θ is injective, and it is obviously surjective.

(iii) For the equivariance,

$$\theta\gamma_h(s_{(\lambda,g)}) = \theta(s_{(\lambda,gh^{-1})}) = (s_\lambda, gh^{-1}) = \widehat{\delta}_h(s_\lambda, g) = \widehat{\delta}_h\theta(s_{(\lambda,g)}).$$

(iv) We first show that the coaction δ is maximal. Let \mathcal{A} be the Fell bundle associated to the coaction δ . Since the spectral subspaces are linearly independent in $C^*(\Lambda)$, $\Gamma_c(\mathcal{A})$ sits inside $C^*(\Lambda)$ as a $*$ -subalgebra, giving an obvious representation of \mathcal{A} in $C^*(\Lambda)$, which in turn extends uniquely to a homomorphism $\pi: C^*(\mathcal{A}) \rightarrow C^*(\Lambda)$. For maximality it suffices, by [5, Proposition 4.2], to show that π is injective. The inclusion $\Lambda \hookrightarrow \mathcal{A}$ gives a map $\rho_0: \Lambda \rightarrow C^*(\mathcal{A})$; the image is a Cuntz-Krieger Λ -family, because the Cuntz-Krieger relations can be expressed within the Fell bundle \mathcal{A} . Thus there is a unique homomorphism $\rho: C^*(\Lambda) \rightarrow C^*(\mathcal{A})$ such that $\rho(s_\lambda) = \rho_0(\lambda)$ for $\lambda \in \Lambda$. For $\lambda \in \Lambda$ we have $\pi\rho_0(\lambda) = \lambda$. It follows that π is injective since ρ is a right inverse.

The homomorphism $\pi := (\text{id} \otimes \lambda)\delta$ intertwines the gauge action α and the tensor-product action $\gamma \otimes \text{id}$, and $\pi(p_v) \neq 0$ for every vertex v , so the Gauge-Invariant Uniqueness Theorem implies that π is faithful. Thus δ is normal. \square

Corollary 7.3. *For the Fell bundle \mathcal{A} of the coaction $\delta: C^*(\Lambda) \rightarrow C^*(\Lambda) \otimes C^*(G)$, we have $C^*(\mathcal{A}) = C_r^*(\mathcal{A})$ (so that \mathcal{A} is amenable in the sense of Exel [9]).*

Proof. The maximality of δ says that $C^*(\mathcal{A}) = C^*(\Lambda)$, and the normality that the regular representation $(\text{id} \otimes \lambda) \circ \delta$ is an isomorphism of $C^*(\Lambda)$ onto $C_r^*(\mathcal{A}) := \text{range}((\text{id} \otimes \lambda) \circ \delta)$. \square

Decomposition. We apply Theorem 7.1 to give an analogue for k -graphs of Green's Decomposition Theorem [10] for twisted crossed products, except we don't require the subgroup to be normal. For graphs the following is [4, Corollary 3.6] (and in fact our result follows from [4, Theorem 4.2], although the proof of that more general decomposition theorem is much harder):

Corollary 7.4. *Let $\eta: \Lambda \rightarrow G$ be a k -graph cocycle, $\delta = \delta_\eta$ the associated coaction of G on $C^*(\Lambda)$, and H a subgroup of G . Then there is a coaction ε of H on the restricted crossed product $C^*(\Lambda) \times_{\delta|} G/H$ such that*

$$C^*(\Lambda) \times_{\delta|} G/H \times_\varepsilon H \cong C^*(\Lambda) \times_\delta G,$$

equivariantly for the dual action $\widehat{\varepsilon}$ and the restricted dual action $\widehat{\delta}|_H$.

Proof. Since our aim is to apply Theorem 7.1, we need a cocycle. H acts freely on $\Lambda \times_\eta G$, and we have

$$(\Lambda \times_\eta G)/H \cong \Lambda \times_\eta G/H.$$

Thus by the Gross-Tucker Theorem 6.11 (twice!) there is a cocycle $\kappa: \Lambda \times_\eta G/H \rightarrow H$ such that

$$\Lambda \times_\eta G/H \times_\kappa H \cong \Lambda \times_\eta G.$$

By Theorem 7.1, letting δ_κ denote the corresponding coaction of H on $C^*(\Lambda \times_\eta G/H)$, we have

$$C^*(\Lambda \times_\eta G/H) \times_{\delta_\kappa} H \cong C^*(\Lambda) \times_\delta G,$$

equivariantly for $\widehat{\delta}_\kappa$ and $\widehat{\delta}|_H$. Appealing to the Gross-Tucker Theorem once more we have $C^*(\Lambda) \times_\eta G/H \cong C^*(\Lambda) \times_{\delta|} G/H$; this isomorphism is equivariant for a unique coaction ε of H on $C^*(\Lambda) \times_{\delta|} G/H$, and the result follows. \square

When $H = G$ the following result is [17, Theorem 5.7]:

Corollary 7.5. *With the above hypotheses,*

$$C^*(\Lambda \times_\eta G) \times_{\gamma|} H \cong C^*(\Lambda \times_\eta G/H) \otimes \mathcal{K}(l^2(H)).$$

In the above statement, γ denotes the action of G on $C^*(\Lambda \times_\eta G)$ described before Theorem 7.1.

Proof. We have:

$$\begin{aligned} C^*(\Lambda \times_\eta G) \times_{\gamma|} H &\cong C^*(\Lambda) \times_\delta G \times_{\widehat{\delta}} H \\ &\cong (C^*(\Lambda) \times_{\delta|} G/H) \times_\varepsilon H \times_{\widehat{\varepsilon}} H \\ &\cong (C^*(\Lambda) \times_{\delta|} G/H) \otimes \mathcal{K}(l^2(H)) \\ &\cong C^*(\Lambda \times_\eta G/H) \otimes \mathcal{K}(l^2(H)), \end{aligned}$$

where we successively applied: Theorem 7.1, Corollary 7.4, crossed-product duality, and Theorem 7.1 again. \square

Our use of [4, Theorem 4.2] brings up a natural question:

Question 7.6. Is there a decomposition theorem for maximal coactions? More precisely, if δ is a maximal coaction of G on a C^* -algebra A , and H a subgroup of G , is there a coaction ε of H on the restricted crossed product $A \times_{\delta|} G/H$ such that

$$(A \times_{\delta|} G/H) \times_{\varepsilon} H \cong A \times_{\eta} G?$$

Cohomology. The theories of both graphs and groupoids (see, e.g., [11, 20, 25]), contain a notion of cohomology of cocycles. This is easily adapted to k -graphs, and has ramifications for the associated coverings and coactions: we call cocycles $\eta, \kappa: \Lambda \rightarrow G$ *cohomologous* if there exists a map $x \mapsto \tau_x: \Lambda^0 \rightarrow G$ such that

$$\tau_x \eta(a) = \kappa(a) \tau_y \quad \text{for all } a \in x\Lambda y.$$

If we regard η and κ as functors then the map τ is just a natural isomorphism from η to κ . It is routine to verify that the map $(x, g) \mapsto (x, \tau_x g)$ gives a k -graph isomorphism $\Lambda \times_{\eta} G \cong \Lambda \times_{\kappa} G$ which is equivariant for the associated actions of G , and the unitary multiplier $\sum_{x \in \Lambda^0} (x \otimes \tau_x)$ implements an exterior equivalence between the associated coactions δ_{η} and δ_{κ} .

The gauge coaction. We can view the degree functor as a cocycle $d: \Lambda \rightarrow \mathbb{Z}^k$. By Theorem 7.1 there is a unique coaction $\delta = \delta_d$ of \mathbb{Z}^k on $C^*(\Lambda)$ such that

$$\delta(s_{\lambda}) = s_{\lambda} \otimes d(\lambda) \quad \text{for } \lambda \in \Lambda.$$

We call δ the *gauge coaction* because the corresponding action of $\mathbb{T}^k = \widehat{\mathbb{Z}^k}$ is the usual gauge action.

When Λ is row-finite and has no sources, the following result is contained in [17, Theorem 5.5].

Theorem 7.7. $C^*(\Lambda)$ is nuclear, and $C^*(\Lambda \times_d \mathbb{Z}^k)$ is AF.

Proof of nuclearity. This follows from [22, Corollary 2.17], since the fixed-point algebra $C^*(\Lambda)^{\delta}$ is nuclear (because it is AF) and the group \mathbb{Z}^k is amenable.

We will prove that $C^*(\Lambda \times_d \mathbb{Z}^k)$ is AF by proving that the isomorphic algebra $C^*(\Lambda) \times_{\delta} \mathbb{Z}^k$ is AF. The proof would not be hard if we had *saturation* (see below for the definition), for then the crossed product would be Morita-Rieffel equivalent to the fixed-point algebra. However, in the general case we require a digression.

Recall from [22] that an *ideal property* is a property \mathcal{P} of C^* -algebras such that: (1) every C^* -algebra has a largest ideal with \mathcal{P} , (2) \mathcal{P} is

inherited by ideals, and (3) \mathcal{P} is invariant under Morita-Rieffel equivalence. The motivation for this definition was then, and remains for us here, that if δ is a coaction of a discrete group G on a C^* -algebra A , then for any ideal property \mathcal{P} , the crossed product $A \times_\delta G$ has \mathcal{P} if and only if the fixed-point algebra A^δ does. It is shown in [22] that nuclearity is an ideal property, and it is well-known that liminality and postliminality are ideal properties.

Proposition 7.8. *Among separable C^* -algebras, AF is an ideal property.*

Proof. For invariance under Morita-Rieffel equivalence, let $A \sim B$ with A being AF. Since A and B are separable, we have $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$. Since A is AF, so is $A \otimes \mathcal{K}$, hence $B \otimes \mathcal{K}$. Thus the hereditary subalgebra B is also AF, by [8, Theorem 3.1]. The same result of Elliott shows that AF is inherited by ideals.

We finish by showing that every C^* -algebra A has a largest AF ideal, i.e., an AF ideal which contains every AF ideal. Claim: if I and J are AF ideals of A , then the ideal $I + J$ is AF. Since

$$(I + J)/I \cong I/(I \cap J),$$

the quotient $(I + J)/I$ is AF. Thus the extension $I + J$ of $(I + J)/I$ by I is AF, by results of Brown [2] and Elliott [8]. We pause to make this reference more precise, because the required result must be pieced together: Elliott proved in [8, Corollary 3.3] that AF is closed under extensions provided projections lift, and Brown proved that projections do indeed lift (from an AF quotient by an AF ideal) — actually, the full proof of Brown’s result is in [7, Section 9].

Now let I be the closed span of all AF ideals of A . Then I is certainly an ideal of A . By the above, I is the closure of an upward-directed union of AF ideals. Therefore I is AF. By construction, every AF ideal of A is contained in I . \square

We’re now ready to prove that the crossed product by the gauge coaction is AF:

Back to the proof of Theorem 7.7. Since $C^*(\Lambda)^\delta$ is AF and $C^*(\Lambda) \times_\delta \mathbb{Z}^k \cong C^*(\Lambda \times_d \mathbb{Z}^k)$, the result follows from [22, Corollary 2.17], because AF is an ideal property. \square

When $k = 1$, the skew-product graph has no cycles, so if the graph is also row-finite, the C^* -algebra of the skew-product graph is AF by [18, Theorem 2.4]. If we had a corresponding result concerning cycles for $k > 1$, this would give an alternate proof of the second part of Theorem 7.7 — but we don’t.

Saturation. Let $\delta = \delta_d$ be the gauge coaction of \mathbb{Z}^k on $C^*(\Lambda)$. Recall from [22] that δ is called *saturated* if

$$C^*(\Lambda)_{n+m} = \overline{\text{span}} C^*(\Lambda)_n C^*(\Lambda)_m \quad \text{for every } n, m \in \mathbb{Z}^k,$$

or equivalently if

$$C^*(\Lambda)^\delta = \overline{\text{span}} C^*(\Lambda)_n C^*(\Lambda)_n^* \quad \text{for every } n \in \mathbb{Z}^k.$$

If δ is saturated then $C^*(\Lambda)^\delta$ is Morita-Rieffel equivalent to $C^*(\Lambda) \times_\delta \mathbb{Z}^k$ [22].

Recall that a vertex v of Λ is called a *source* if $v\Lambda^n = \emptyset$ for some $n \in \mathbb{N}^k$, and similarly a *sink* if some $\Lambda^n v$ is empty. The following result generalizes [16, Proposition 2.8].

Proposition 7.9. *Let δ be the gauge coaction of \mathbb{Z}^k on $C^*(\Lambda)$.*

- (i) *If δ is surjective, in particular if Λ has either no sources or no sinks, every spectral subspace $C^*(\Lambda)_n$ for $n \in \mathbb{Z}^k$ is nontrivial.*
- (ii) *If Λ is row-finite and has neither sources nor sinks, then δ is saturated.*

Proof. (i) Let $n \in \mathbb{Z}^k$. Choose $l \in \mathbb{N}^k$ with $n+l \geq 0$, then $j \in \mathbb{N}^k$ with $j \geq n+l$ and $j \geq l$, and then $\lambda \in \Lambda^j$. We can factor

$$\lambda = \mu\nu = \alpha\beta \quad \text{with} \quad d(\nu) = n+l \quad \text{and} \quad d(\beta) = l.$$

Thus

$$0 \neq s_\lambda s_\lambda^* = s_\mu s_\nu s_\beta^* s_\alpha^*,$$

so that $s_\nu s_\beta^*$ is a nonzero element of

$$C^*(\Lambda)_{n+l} C^*(\Lambda)_l^* \subset C^*(\Lambda)_n.$$

(ii) Now assume that Λ is row-finite and has neither sources nor sinks. Let $l \in \mathbb{Z}^k$. To see that δ is saturated, we must show that

$$C^*(\Lambda)^\delta \subset \overline{\text{span}} C^*(\Lambda)_l C^*(\Lambda)_l^*.$$

By Lemma 7.10 below, we have

$$C^*(\Lambda)^\delta = C^*(\Lambda)_0 = \overline{\text{span}}\{s_\lambda s_\mu^* \mid d(\lambda) = d(\mu), s(\lambda) = s(\mu)\},$$

so it suffices to show that if

$$d(\lambda) = d(\mu) = n \quad \text{and} \quad s(\lambda) = s(\mu) = v,$$

then $s_\lambda s_\mu^* \in \text{span } C^*(\Lambda)_l C^*(\Lambda)_l^*$. Choose $m \in \mathbb{N}$ with $m \geq l$ and $m \geq n$. Since Λ is row-finite and has no sources,

$$p_v = \sum_{\alpha \in v\Lambda^{m-n}} s_\alpha s_\alpha^*.$$

Since Λ has no sinks, for each $\alpha \in v\Lambda^{m-n}$ we can choose $\nu_\alpha \in \Lambda^{m-l}s(\alpha)$. Then $p_{s(\alpha)} = s_{\nu_\alpha}^* s_{\nu_\alpha}$, so

$$\begin{aligned} s_\lambda s_\mu^* &= s_\lambda p_v s_\mu^* = \sum_{\alpha \in v\Lambda^{m-n}} s_\lambda s_\alpha s_{\nu_\alpha}^* s_{\nu_\alpha} s_\alpha^* s_\mu^* \\ &\in \text{span } C^*(\Lambda)_n C^*(\Lambda)_{m-n} C^*(\Lambda)_{m-l}^* C^*(\Lambda)_{m-l} C^*(\Lambda)_{m-n}^* C^*(\Lambda)_n^* \\ &\subset \text{span } C^*(\Lambda)_l C^*(\Lambda)_l^*. \quad \square \end{aligned}$$

In the above proof, we applied the following characterization of spectral subspaces for the gauge coaction:

Lemma 7.10. *Let $\eta: \Lambda \rightarrow G$ be a cocycle on the k -graph Λ , and let $\delta = \delta_\eta$ be the associated coaction of G on $C^*(\Lambda)$. Then for all $g \in G$,*

$$C^*(\Lambda)_g = \overline{\text{span}}\{s_\lambda s_\mu^* \mid \eta(\lambda)\eta(\mu)^{-1} = g\}.$$

Proof. Obviously a product $s_\lambda s_\mu^*$ is in $C^*(\Lambda)_g$ if and only if $\eta(\lambda)\eta(\mu)^{-1} = g$, so the left hand side contains the right.

Recall from [22] that there is a bounded linear projection

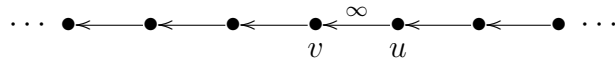
$$E_g = (\text{id} \otimes \chi_{\{g\}})\delta: C^*(\Lambda) \rightarrow C^*(\Lambda)_g,$$

where here the characteristic function $\chi_{\{g\}}$ is regarded as a linear functional on $C^*(G)$. Any $a \in C^*(\Lambda)_g$ can be approximated by a linear combination $\sum_1^n c_i s_{\lambda_i} s_{\mu_i}^*$, and then

$$\begin{aligned} a &= E_g(a) \approx E_g\left(\sum_1^n c_i s_{\lambda_i} s_{\mu_i}^*\right) = \sum_1^n c_i E_g(s_{\lambda_i} s_{\mu_i}^*) \\ &= \sum \{c_i s_{\lambda_i} s_{\mu_i}^* \mid \eta(\lambda_i)\eta(\mu_i)^{-1} = g\}, \end{aligned}$$

which is in the right hand side. The result follows. \square

Example 7.11. Row-finiteness is necessary in Proposition 7.9 (ii): let Λ be the 1-graph



in which there are infinitely many edges from u to v (and the graph extends indefinitely to the right and left). The projection p_v is in the fixed-point algebra $C^*(\Lambda)^\delta$, and cannot be approximated in norm by a linear combination of products $s_e s_f^*$ for edges $e, f \in v\Lambda u$. It follows that

$$C^*(\Lambda)^\delta \neq \overline{\text{span}} C^*(\Lambda)_1 C^*(\Lambda)_1^*,$$

so δ is not saturated.

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