C^*–algebras of directed graphs and group actions [*]

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Abstract

Given a free action of a group G on a directed graph E we show that the crossed product of $C^*(E)$, the universal C^*–algebra of E, by the induced action is strongly Morita equivalent to $C^*(E/G)$. Since every connected graph E may be expressed as the quotient of a tree T by an action of a free group G we may use our results to show that $C^*(E)$ is strongly Morita equivalent to the crossed product $C_0(\partial T) \times G$, where ∂T is a certain 0–dimensional space canonically associated to the tree.

Dedicated to Marc A. Rieffel on the occasion of his 60^{th} *birthday.*

1 Introduction

The purpose of this paper is to study free actions of countable groups on directed graphs and their associated C^*–algebras. In previous work we have shown that given a directed graph E there is a C^*–algebra $C^*(E)$ which

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satisfies a certain universal property (see [KPR, Theorem 1.2]). If a countable group G acts on E then by the universal property there is an induced action of G on $C^*(E)$. If the action is free we show that the resulting crossed product $C^*(E) \times G$ is strongly Morita equivalent to $C^*(E/G)$ where E/G is the quotient graph; more precisely $C^*(E) \times G \cong C^*(E/G) \otimes \mathcal{K}(\ell^2(G))$. Moreover, given a connected graph E, one can associate in a canonical way a universal covering tree T (cf. [H], [LS]) together with a group G which acts freely on T (and so is a free group cf. [Se]) such that $T/G \cong E$. Let ∂T be the boundary of T, then the action of G on T induces an action of G on ∂T; applying our earlier results and using the fact that $C^*(T)$ is strongly Morita equivalent to $C_0(\partial T)$ in an equivariant way, we then show that $C^*(E)$ is strongly Morita equivalent to $C_0(\partial T) \times G$.

We now briefly describe some of the basic notions we shall be using in this paper: A directed graph E consists of countable sets E^0 of vertices and E^1 of edges, together with maps $r, s : E^1 \to E^0$ describing the range and source of edges. The graph is row finite if every vertex emits only finitely many edges and locally finite if in addition each vertex receives only finitely many edges. To avoid technical difficulties, in this paper we shall assume that every vertex in E emits an edge (i.e. $s(E^1) = E^0$). Given a directed graph E, a representation of E consists of a set $\{P_v : v \in E^0\}$ of mutually orthogonal projections and a set $\{S_e : e \in E^1\}$ of partial isometries satisfying

$$S_e{}^* S_e \;=\; P_{r(e)} \;=\; \sum_{s(f)=r(e)} S_f S_f{}^* \quad \text{for each } e \in E^1, \tag{1}$$

(for more details see [KPR, §1]). In [KPR, Theorem 1.2] we showed that there is a universal C^*–algebra denoted $C^*(E)$ which is generated by non–zero partial isometries and projections satisfying (1). Much of our analysis of $C^*(E)$ is done by using a groupoid model for $C^*(E)$ which was developed in [KPRR]: Following [R, §III.2] in [KPRR, §2] we use the shift–tail equivalence relation on the space E^∞ of all infinite paths in a row–finite directed graph E to construct a locally compact, second countable, r–discrete groupoid $\mathcal{G}_E$ whose unit space $\mathcal{G}_E^{(0)}$ may be identified with E^∞. By [KPR, Remark 1.3] if E is row finite then $C^*(E) \cong C^*(\mathcal{G}_E)$ and we shall identify these C^*–algebras without comment throughout this paper.

Let $A \in M_n(\mathbf{Z}^+)$ with no zero row or column, then there is a graph E_A naturally associated to A such that the Cuntz–Krieger algebra $\mathcal{O}_A$ is isomorphic to $C^*(E_A)$. The class of graph C^*–algebras includes the Doplicher–

Roberts algebras of [DR, §3] and up to strong Morita equivalence all AF algebras as well (see [KPR, Theorem 2.4]).

This paper is organised as follows: in the second section we consider a labelling of the edges of a graph F by elements of a countable group G, that is, a function $c : F^1 \to G$. This allows us to define a skew–product graph $F(c)$ in analogy with a skew–product groupoid (see [R, I.1.6]). In the third section we discuss the notion of the free action of a countable group G on the vertices of a directed graph E, the quotient E/G then has the structure of a directed graph. These two constructions are linked in the following way: if $c : F^1 \to G$ is a function then G acts freely on $F(c)$ with $F(c)/G \cong F$, secondly if G acts freely on E then there is a function $c : (E/G)^1 \to G$ such that $(E/G)(c) \cong E$ (this isomorphism is G–equivariant). It may therefore be appropriate to regard E as the graph theoretical analog of a principal G–bundle over E/G (and c may be regarded as the analog of a G–valued cocycle that provides patching data). The action of G on E induces a natural action on the associated C^*–algebra so that the crossed product is strongly Morita equivalent to the C^*–algebra of the quotient graph (by analogy with Green's Theorem [G, Theorem 14]). Combining Corollaries 2.5, 3.9 and 3.10 we have the following result:

1.1 Theorem: *Let E be a locally finite directed graph and suppose that $\lambda : G \to Aut(E)$ is a free action of a countable group G on the vertices of E. Then*

$$C^*(E) \times_\lambda G \quad \cong \quad C^*(E/G) \otimes \mathcal{K}\left(\ell^2(G)\right),$$

where λ also denotes the induced action of G on $C^*(E)$; moreover if G is abelian then there is an action α of $\widehat{G}$ on $C^*(E/G)$ such that

$$C^*(E) \quad \cong \quad C^*(E/G) \times_\alpha \widehat{G},$$

and under this isomorphism λ is identified with $\widehat{\alpha}$.

In the final section we firstly show in 4.3 that the C^*–algebra of a row finite directed tree T is strongly Morita equivalent to $C_0(\partial T)$ where ∂T is the boundary of T, obtained as the quotient of the infinite path space T^∞ by shift tail equivalence. Then for a connected directed graph E by choosing a base vertex $\star \in E^0$ and considering the collection of undirected paths beginning at $\star$ we construct the universal covering tree $T = (E, \star)$ of E, in analogy with the universal covering space of a path–connected topological space. The fundamental group G of the graph E, which consists of undirected loops at

$\star$, acts freely on T in such a way that $T/G \cong E$. The action of G on T induces an action of G on ∂T, the boundary of T. Since the equivalence of $C^*(T)$ and $C_0(\partial T)$ is G–equivariant, we obtain (see Lemma 4.10 and Corollary 4.14):

1.2 Theorem: *Let E be connected row finite directed graph, then $C^*(E)$ is strongly Morita equivalent to $C_0(\partial T) \times G$ where ∂T is the boundary of the universal covering tree T of E and G is the fundamental group of E based at the vertex used to construct T. Moreover G is a free group, and if E^0 is finite then $G \cong \mathbf{F}_n$ where $n = |E^1| - |E^0| + 1$.*

Earlier results of this type are to be found in [ETW, Sp, SZ]: The Toeplitz extension of $\mathcal{O}_n$ arising in the Fock space construction was shown to be a crossed-product by a free semigroup in [ETW]. In [Sp], certain Cuntz-Krieger algebras are exhibited as crossed products of unital abelian C*-algebras by free products of cyclic groups — the action arises as a boundary action. Other results of this nature may be found in [SZ].

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2 Skew product graphs

In this section G shall always be a countable group, and E a row–finite directed graph unless otherwise stated. The set of paths μ in E of length $|\mu| = k$ is denoted E^k and the finite path space is denoted E^* the maps r, s extend naturally to E^*. A path $\mu \in E^*$ with $|\mu| \geq 1$ is said to be a loop if $r(\mu) = s(\mu)$. The shift tail equivalence relation for $x, y \in E^\infty$ is given by $x \sim_k y$ if and only if there is an $N \geq 1$ and $k \in \mathbf{Z}$ such that $x_i = y_{i-k}$ for $i \geq N$. The groupoid $\mathcal{G}_E$ then consists of triples (x, k, y) such that $x \sim_k y$ in E^∞, for more details see [KPRR, §2]. Note that our sign convention agrees with that used by Renault for the Cuntz groupoid (see [R, p. 140]) but it differs from that used in [KPRR, KPR] (it clearly does not affect the results contained therein).

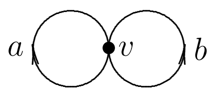
2.1 Definitions: *Let E be a directed graph, and $c : E^1 \to G$ be a function. We may then form the* **skew product** *graph $E(c) = (G \times E^0, G \times E^1, r, s)$ where*

$$r(g, e) \quad = \quad (gc(e), r(e)) \text{ and } \quad s(g, e) \quad = \quad (g, s(e)). \tag{2}$$

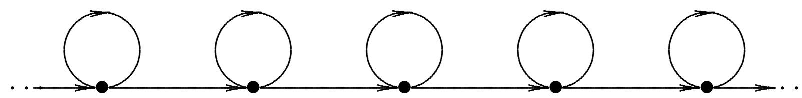
Let E, F be two directed graphs, the **cartesian product graph** *is defined to be $E \times F = (E^0 \times F^0, E^1 \times F^1, r, s)$ where $r(e,f) = (r(e), r(f))$ and $s(e,f) = (s(e), s(f))$.*

In the literature (see [GT, §2.2.1]) $E(c)$ is sometimes referred to as a "derived graph" (the graph together with labelling, $c : E^1 \to G$, is referred to as a "voltage graph"). Since every vertex in E emits an edge it follows from (2) that every vertex in $E(c)$ emits an edge and so by [KPR, Remark 1.3] we may identify $C^*(\mathcal{G}_{E(c)})$ with $C^*(E(c))$. If E is any directed graph, $G = \mathbf{Z}$ and $c(e) = 1$ for all $e \in E^1$ then $E(c)$ is identical to $Z \times E$ where Z is the graph shown below:

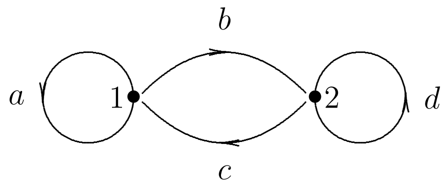
2.2 Examples: Suppose that F is the directed graph



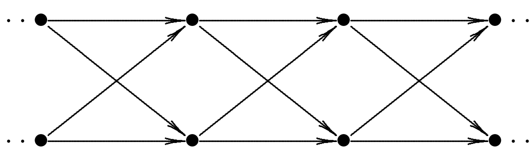
and define $c : F^1 \to G = \mathbf{Z}$ by $c(a) = 0$ and $c(b) = 1$, then $F(c)$ is given by



(see [KPR, Example 3.10]). If E is the graph



then $Z \times E$ is given by



If $c(e) = 1_G$ for all $e \in E^1$, then $E(c)$ is the disjoint union of $|G|$ isomorphic copies of E.

We may extend the definition of c to E^* by defining $c(w) = 1_G$ for $w \in E^0$ and $c(\mu) = c(\mu_1)\dots c(\mu_{|\mu|})$ where $\mu = (\mu_1, \dots, \mu_{|\mu|}) \in E^*$. Note that $c(\mu\nu) = c(\mu)c(\nu)$ for all $\mu, \nu \in E^*$ with $r(\mu) = s(\nu)$. Define $\tilde{c} : \mathcal{G}_E \to G$ by $\tilde{c}(x, k, y) = c(\mu)c(\nu)^{-1}$ where $x = \mu z$, $y = \nu z$, $|\mu| - |\nu| = k$ and $z \in E^\infty$. To see that our definition of $\tilde{c}$ is well–defined suppose that $(x, k, y) \in \mathcal{G}_E$ is interpreted as $x = \mu\gamma z'$, $y = \nu\gamma z'$ where $\mu, \nu, \gamma \in E^*$, $|\mu| - |\nu| = k$ and $z' \in E^\infty$, then

$$\tilde{c}(x, k, y) \;=\; c(\mu\gamma)c(\nu\gamma)^{-1} \;=\; c(\mu)c(\nu)^{-1}.$$

2.3 Lemma: *Let E be a directed graph and $c : E^1 \to G$ a function then $E(c)^n$ may be identified with $G \times E^n$ and $E(c)^\infty$ with $G \times E^\infty$. Under these identifications $x' \sim_k y'$ in $E(c)^\infty$ if and only if $x' = (g, x)$, $y' = (h, y)$ where $x = \mu z \sim_k \nu z = y$ in E^∞ and $hc(\nu) = gc(\mu)$ in G; moreover the function $\tilde{c} : \mathcal{G}_E \to G$ is a continuous 1–cocycle.*

Proof:

From (2) we may see that a sequence of edges $(g_i, e_i)_{i=1}^n$ belongs to $E(c)^n$ if and only if $r(e_i) = s(e_{i+1})$ and

$$g_{i+1} \;=\; g_i c(e_i) \;=\; g_1 c(e_1) \cdots c(e_i) \tag{3}$$

for $i = 1, \dots, n-1$. Set $\mu = (e_i)_{i=1}^n \in E^n$; it follows from (3) that g_i is completely specified by g_1 and $c(e_1), \dots, c(e_{i-1})$ so we may identify $E(c)^n$ with $G \times E^n$ by $(g_i, e_i)_{i=1}^n \mapsto (g_1, \mu)$, where $r(g, \mu) = (gc(\mu)), r(\mu))$ and $s(g, \mu) = (g, s(\mu))$. Continuing in this way we may identify $E(c)^\infty$ with $G \times E^\infty$ by $(g_i, x_i)_{i=1}^\infty \mapsto (g_1, (x_i)_{i=1}^\infty)$.

Suppose that $x' \sim_k y'$ in $E(c)^\infty$ then there is $N \in \mathbf{N}$ such that $x'_i = y'_{i-k}$ for $i \geq N$. Since

$$\begin{array}{lllll} x'_i & = & (g_i, x_i) & = & (g_1 c(x_1) \cdots c(x_{i-1}), x_i) \\ y'_{i-k} & = & (h_{i-k}, y_{i-k}) & = & (h_1 c(y_1) \cdots c(y_{i-k-1}), y_{i-k}) \end{array} \tag{4}$$

we may deduce that $x = \mu z \sim_k \nu z = y$ in E^∞ where $|\mu| = N$ and $g_1 c(\mu) = h_1 c(\nu)$ in G. Suppose that $x = \mu z \sim_k \nu z = y$ in E^∞ and $gc(\mu) = hc(\nu)$ if we define $x' = (g, x)$ and $y' = (h, y)$ then by (4) we may see that $x'_i = y'_{i-k}$ for $i \geq |\mu|$ and so $x' \sim_k y'$ in $E(c)^\infty$.

Observe that if $x \sim_k y$ and $y \sim_\ell z$ then without loss of generality $x = \mu t$, $y = \nu t$, $z = \gamma t$, where $t \in E^\infty$, $|\mu| - |\nu| = k$ and $|\nu| - |\gamma| = \ell$, then

$$\tilde{c}((x, k, y)(y, \ell, z)) = \tilde{c}(x, k, z) = c(\mu)c(\gamma)^{-1} = c(\mu)c(\nu)^{-1}c(\nu)c(\gamma)^{-1} = \tilde{c}(x, k, y)\tilde{c}(y, \ell, z),$$

and so $\tilde{c}$ is a homomorphism. Since $\tilde{c}$ is locally constant it is evidently continuous. □

Note that if $c : E^1 \to \mathbf{Z}$ is defined by $c(e) = 1$ for all $e \in E^1$, then $\tilde{c}(x, k, y) = k$ for all $(x, k, y) \in \mathcal{G}_E$. From [R, Proposition II.3.8] it follows that the skew product groupoid $\mathcal{G}_E(\tilde{c})$ is r–discrete and amenable if E is locally finite (see [KPRR, Corollary 5.5]).

2.4 Theorem: *Let E be a directed graph and $c : E^1 \to G$ a function, then $\mathcal{G}_E(\tilde{c}) \cong \mathcal{G}_{E(c)}$.*

Proof:

Consider the map $\phi : \mathcal{G}_E(\tilde{c}) \to \mathcal{G}_{E(c)}$ given by $\phi([x, k, y], g) = (x', k, y')$ where $x' = (g, x)$ and $y' = (g\tilde{c}([x, k, y]), y)$; note that $x' \sim_k y'$ in $E(c)^\infty$ from 2.3, and so $(x', k, y') \in \mathcal{G}_{E(c)}$. For

$$(([x, k, y], g), ([y, \ell, z], g\tilde{c}([x, k, y]))) \in \mathcal{G}_E(\tilde{c})^{(2)}$$

we have

$$\begin{aligned} \phi([x, k, y], g)\phi([y, \ell, z], g\tilde{c}([x, k, y])) &= (x', k, y')(y', \ell, z') \text{ where } z' = (g\tilde{c}([x, k+\ell, z]), z) \in E(c \\ &= (x', k+\ell, z') \ = \ \phi([x, k+\ell, z], g), \end{aligned}$$

hence ϕ is multiplicative. It is then straightforward to show that ϕ is bijective and preserves composability, so it is an isomorphism of groupoids. □

2.5 Corollary: *If E is a directed graph and $c : E^1 \to G$ is a function where G is a abelian group then there is an action α of $\widehat{G}$ on $C^*(E)$ such that*

$$\alpha_\chi(S_e) = \langle \chi, c(e) \rangle S_e$$

for all $e \in E^1$ and $\chi \in \widehat{G}$; moreover,

$$C^*(E(c)) \ \cong \ C^*(E) \times_\alpha \widehat{G}.$$

Proof:

Since $C^*(E)$ is defined to be the universal C^*–algebra generated by the S_e subject to (1) and α preserves these relations it is clear that it defines an action of $\widehat{G}$ on $C^*(E)$. One checks that α is given by the cocycle $\tilde{c}$ in the sense that if $f \in C_c(\mathcal{G}_E) \subset C^*(E)$ then $(\alpha_\chi f)(\gamma) = \langle \chi, \tilde{c}(\gamma) \rangle f(\gamma)$. From 2.4 $C^*(\mathcal{G}_{E(c)}) \cong C^*(\mathcal{G}_E(\tilde{c}))$, it then suffices to show that this latter C^*–algebra is a crossed product of $C^*(\mathcal{G}_E)$ by $\widehat{G}$; but this follows by [R, II.5.7]. □

This result may to be used to show that $C^*(E)$ belongs to $\mathcal{N}$, the bootstrap class to which the UCT applies (see [RS, Theorem 1.17]). If $G = \mathbf{Z}$ and $c(e) = 1$ for all $e \in E^1$ then the associated action α of $\mathbf{T} = \hat{\mathbf{Z}}$ on $C^*(E)$ is referred to as the gauge action (see [E]).

2.6 Proposition: *Let E be a directed graph, then $C^*(Z \times E) \cong C^*(E) \times_\alpha \mathbf{T}$ is an AF algebra. Moreover, $C^*(E)$ belongs to the bootstrap class $\mathcal{N}$.*

Proof:

We claim that $Z \times E$ has no loops: by 2.3 any finite path $\mu' \in (Z \times E)^k$ is of the form $\mu' = (a, \mu)$ for some $a \in \mathbf{Z}$ and $\mu \in E^k$. Suppose $\mu' \in (Z \times E)^k$ is a loop then $r(\mu') = s(\mu')$ and so $a = a + k$, which means that $k = 0$ but $|\mu'| = k \geq 1$ which gives us a contradiction. Hence by [KPR, Theorem 2.4] and 2.5 $C^*(E) \times_\alpha \mathbf{T}$ is an AF algebra. By the Takesaki-Takai duality theorem (see [P, Theorem 7.9.3]) one has:

$$C^*(E) \otimes \mathcal{K}(L^2(\mathbf{T})) \quad \cong \quad (C^*(E) \times_\alpha \mathbf{T}) \times_{\hat{\alpha}} \mathbf{Z};$$

we see that $C^*(E)$ is strongly Morita equivalent to the crossed product of an AF algebra by a $\mathbf{Z}$–action. Since the bootstrap class $\mathcal{N}$ contains all type I C^*–algebras, is closed under inductive limits and crossed products by $\mathbf{Z}$, $C^*(E)$ is in $\mathcal{N}$. $\square$

2.7 Note: Let E be a directed graph and $c : E^1 \to G$ a function. Given $\mu \in E^*$, then (g, μ) is a loop in $E(c)$ if and only if μ is a loop in E and c satisfies a *Kirchoff condition* on μ, that is $c(\mu) = 1_G$. It follows that $C^*(E(c))$ is an AF algebra iff $c(\mu) \neq 1_G$ for every loop $\mu \in E^*$.

2.8 Proposition: *Let E be a directed graph such that every vertex receives an edge then $C^*(Z \times E)$ is strongly Morita equivalent to the fixed point algebra $C^*(E)^\alpha$.*

Proof:

Firstly we show that $C^*(\mathcal{G}_{Z \times E})$ is strongly Morita equivalent to $C^*(\mathcal{R})$ where $\mathcal{R}$ denotes the reduction of the groupoid $\mathcal{G}_{Z \times E}$ to the clopen set $N = \{(0, x) : x \in E^\infty\}$. It suffices to show that $\mathcal{R}$ is full reduction of $\mathcal{G}_{Z \times E}$ ([MRW, Theorem 2.8]); but this follows immediately from the fact that $S = \{(0, v) : v \in E^0\}$ is cofinal in $Z \times E$ since every vertex in E receives an edge.

Next we show that the groupoids $\mathcal{R}$ and $\tilde{c}^{-1}(0)$ are isomorphic where $\tilde{c} : \mathcal{G}_E \to \mathbf{Z}$ is given by $\tilde{c}(x,k,y) = k$. Since $(x',k,y') \in \mathcal{R}$ if and only if $x' = (0,x)$, $y' = (0,y)$ and $k = 0$ (see 2.3), we may define $\psi : \mathcal{R} \to \tilde{c}^{-1}(0)$, by $\psi(x',0,y') = (x,0,y)$, where $x,y \in E^\infty$. One easily checks that ψ is a groupoid isomorphism and hence induces an isomorphism of the corresponding C^*–algebras.

Finally, by adapting [PR, Lemma 2.2.3] we may also show that $C^*(\tilde{c}^{-1}(0))$ is isomorphic to $C^*(E)^\alpha$. $\square$

More generally, suppose that G is an abelian group, and $c : E^1 \to G$ a function. If for every $v \in E^0$ and $g \in G$ there is a path $\mu \in E^*$ with $r(\mu) = v$ and $c(\mu) = g$ then the fixed point algebra $C^*(E)^\alpha$ of the dual action $\widehat{G}$ on $C^*(E)$ is stably isompmorphic to $C^*(E) \times_\alpha \widehat{G} \cong C^*(E(c))$.

Cayley graphs of finitely generated groups (see [GT, §1.2.4], [St, §0.5.7]) form a key class of examples of skew products:

2.9 Definition: *Let G be a group with a finite set of generators $g_1, \ldots, g_n$. The (right)* **Cayley graph of G with respect to** *$g_1, \ldots, g_n$ is the directed graph E_G where $E_G^0 = G$, $E_G^1 = G \times \{g_1, \ldots, g_n\}$ with range and source maps given by $r(h, g_i) = hg_i$ and $s(h, g_i) = h$ for $i = 1, \ldots, n$.*

2.10 Examples: (1) Let G be a group with generators $g_1, \ldots, g_n$, let B_n be the directed graph with a single vertex and edges $\{1, \ldots, n\}$ if $c : B_n \to G$ is given by $c(k) = g_k$ for $k = 1, \ldots, n$, then $B_n(c) \cong E_G$ (see [GT, Theorem 2.2.3]).
(2) In 2.2 graph E is the Cayley graph for $\mathbf{Z}_2 = \{0,1\}$ with respect to the generators $0, 1$. Graph $F(c)$ is the Cayley graph of $\mathbf{Z}$ with respect to the generators $0, 1$.

Let E be a directed graph, $c : E^1 \to G$ a function. E is said to be c–**cofinal** if for every $x \in E^\infty$, $v \in E^0$ and $g \in G$ there is an $\mu \in E^*$ and $n \geq 1$ such that $s(\mu) = v$ and $r(\mu) = s(x_n)$ and if $\nu = (x_1, \ldots, x_{n-1})$ then $c(\nu)c(\mu)^{-1} = g$ (if $n = 1$ we put $\nu = s(x)$). The following result is an immediate consequence of the definition of cofinality (see [KPRR, Corollary 6.8]) and c–cofinality.

2.11 Proposition: *Let E be a directed graph and $c : E^1 \to G$ a function then $E(c)$ is cofinal if and only if E is c–cofinal.*

2.12 Notes: (1) By [R, I.4.14] it follows that E is c–cofinal if and only if E is cofinal and the asymptotic range of the induced cocycle $\tilde{c}$ on $\mathcal{G}_E$ at x exhausts G (i.e. $R^x_\infty(\tilde{c}) = G$) for all $x \in E^\infty$.
(2) If E^0 is finite, every vertex receives an edge and $c(e) = 1$ for all $e \in E^1$ then c–cofinality is equivalent to E being aperiodic, in the sense that there is a $k \geq 1$ such that for every $u, v \in E^0$ we have $\mu \in E^k$ such that $s(\mu) = u$, $r(\mu) = v$. No graph with E^0 infinite can be aperiodic (recall that we have assumed that E is row-finite).
(3) Now suppose that E satisfies condition (L) of [KPR, §3] (that is, each loop has an exit); it follows that $E(c)$ also satisfies condition (L). Hence, by [KPRR, Corollary 6.8] $C^*(E(c))$ is simple iff E is c–cofinal (note that if a directed graph is cofinal, condition (L) is equivalent to condition (K) of [KPRR, §6]). In particular, if $c(e) = 1$ for all $e \in E^1$ and E is c–cofinal then $C^*(Z \times E)$ is simple, as is the fixed point algebra $C^*(E)^\alpha$ (cf. [CK, p.253]).

3 Groups acting on directed graphs

The following ideas and concepts are adapted from [GT, §1.1.6] or [Se, Definition 1]: let E, F be two directed graphs, then a **graph morphism** $f : E \to F$ is a pair of maps $f = (f^0, f^1)$ where $f^i : E^i \to F^i$ for $i = 0, 1$ are such that $f^0(r(e)) = r(f^1(e))$ and $f^0(s(e)) = s(f^1(e))$, for all $e \in E^1$. To keep our notation simple we will often omit the superscript on graph morphisms. A graph morphism $f : E \to F$ is said to have the **unique path lifting property** if given $u \in E^0$ and $e \in F^1$ with $s(e) = f(u)$, then there is a unique $e' \in E^1$ such that $s(e') = u$ and $f(e') = e$. There is a natural notion of isomorphism of directed graphs; the group of automorphisms of a graph E is denoted $\mathrm{Aut}\,(E)$.

Let E be a directed graph and G a countable group, then G **acts on** E if there is a group homomorphism $g \mapsto \lambda_g \in \mathrm{Aut}(E)$. The action λ of G on E is called **free** if λ acts freely on the vertices, that is if $\lambda_g v = v$ for any $v \in E^0$ then $g = 1_G$. Note that in this case the action of G is also free on the edges of E.

3.1 Lemma: *Let E be a row finite directed graph and $\lambda : G \to Aut(E)$ be an action, then there is an induced action (which we also denote λ) of G on $\mathcal{G}_E$ by homeomorphisms which is free if λ is.*

Proof:

Let $\lambda : G \to \mathrm{Aut}\,(E)$, then G acts on E^* and E^∞ by defining $(\lambda_g x)_i = \lambda_g x_i$ for all i. It is easy to check that the action of G on E^∞ preserves shift tail equivalence, since $x \sim_k y$ if and only if $\lambda_g x \sim_k \lambda_g y$ for $x, y \in E^\infty$. Hence G acts on $\mathcal{G}_E$ by defining $\lambda_g(x,k,y) = (\lambda_g, k, \lambda_g y)$, and then $\lambda_g^{-1} = \lambda_{g^{-1}}$; moreover for each $g \in G$, λ_g is a homeomorphism since $\lambda_g Z(\mu,\nu) = Z(\lambda_g \mu, \lambda_g \nu)$ for $\mu, \nu \in E^*$. $\square$

The **quotient graph** $E/G = ((E/G)^0, (E/G)^1, r, s)$ consists of the equivalence classes of vertices and edges under the action of G, together with range and source maps $r([e]) = [r(e)]$, and $s([e]) = [s(e)]$ which one may check are well–defined. The quotient map $q^0 : v \mapsto [v]$, $q^1 : e \mapsto [e]$ is a surjective graph morphism $q : E \to E/G$ which has the unique path lifting property.

An action $\lambda : G \to \mathrm{Aut}\,(E)$ as above also induces an action of G on $C^*(E)$ which, to simplify notation, we denote λ. One has $\lambda_g(S_e) = S_{\lambda_g(e)}$. We show below (see 3.10) that if G acts freely on E, then $C^*(E) \times_\lambda G \;\cong\; C^*(E/G) \otimes \mathcal{K}\left(\ell^2(G)\right)$.

3.2 Examples: (1) If G is a group with generators $g_1, \ldots, g_n$, then G acts naturally on its Cayley graph E_G by: $\beta_g^0(h) = gh$ and $\beta_g^1(h, g_i) = (gh, g_i)$, for all $g, h \in G$ and $i = 1, \ldots, n$. The action is clearly free, and transitive on the vertices of E_G (see [GT, Theorem 1.2.5]). In fact it is easy to see that $E_G/G \cong B_n$; which from 2.10(1) is a special case of (2) below.
(2) Let E be a directed graph and $c : E^1 \to G$ a function then there is a natural free action γ of G on $E(c)$ defined by

$$\gamma_g^0(h,v) \;=\; (gh, v), \qquad \gamma_g^1(h,e) \;=\; (gh, e), \tag{5}$$

moreover $E(c)/G \cong E$ (see [GT, Theorem 2.2.1]).

3.3 Note: Conversely, if G acts freely on E then by [GT, Theorem 2.2.2] there is a function $c : (E/G)^1 \to G$ such that $(E/G)(c) \cong E$ in an equivariant way (the function c will be unique up to a “coboundary”).

3.4 Proposition: *Let E be a locally finite graph such that $C^*(E)$ is purely infinite, then*

(i) if $f : E \to F$ is a surjective graph morphism which has the unique path lifting property then $C^*(F)$ is purely infinite;

(ii) if G is a countable group and $c : E^1 \to G$ is a function such that every vertex $v \in E^0$ connects to a loop $\mu \in E^*$ with $c(\mu) = 1_G$, then $C^*(E(c))$ is purely infinite.

Proof:

By [KPR, Theorem 3.9] $C^*(E)$ is purely infinite if and only if E satisfies condition (L), i.e. every loop has an exit, and every vertex in E connects to a loop. To show that $C^*(F)$ is purely infinite it suffices to show that every vertex connects to a loop with exit (for then every loop has an exit). Given $u \in F^0$, then since f is surjective there is $v \in E^0$ with $f(v) = u$. By hypothesis $C^*(E)$ is purely infinite so v connects via $\nu \in E^*$ to a loop $\mu \in E^*$ with exit; without loss of generality we may assume that μ has an exit $d \neq \ mu_1$ at $s(\mu)$. Since f is a graph morphism $f(\mu)$ is a loop in F^*; moreover, by the unique path lifting property $f(d) \neq f(\mu_1)$. Hence the loop $f(\mu) \in F^*$ has an exit and so u connects via $f(\nu)$ to a loop $f(\mu)$ with exit.

Recall from 2.7 that every loop in $E(c)$ is of the form (g, μ) where $\mu \in E^*$ is a loop with $c(\mu) = 1_G$. By hypothesis $\mu \in E^*$ has an exit, and so from (2) $(g, \mu) \in E(c)^*$ has an exit. Since every $w \in E^0$ connects via $\nu \in E^*$ to a loop μ we may see that every $(g, w) \in E(c)^0$ connects via $(g, \nu) \in E(c)^*$ to a loop $(gc(\nu), \mu) \in E(c)^*$. Hence by [KPR, Theorem 3.9] we may deduce that $C^*(E(c))$ is purely infinite. □

3.5 Corollary: *Let E be a locally finite graph such that $C^*(E)$ is purely infinite*

(i) if a countable group G acts freely on E, then $C^*(E/G)$ is purely infinite;

(ii) if $c : E^1 \to G$ is a function where G is a countable group in which every element has finite order, then $C^*(E(c))$ is purely infinite.

Proof:

For (i) observe that the quotient map $q : E \to E/G$ is a surjective graph morphism with the unique path lifting property. For (ii) observe that if $\mu \in E^*$ is a loop and $c(\mu)$ has order n in G then $c(\mu^n) = 1_G$ and so every vertex in E connects to a loop $\nu \in E^*$ with $c(\nu) = 1_G$. □

3.6 Notes: (1) The pair F, and $c : F^1 \to \mathbf{Z}$ given in 2.2 satisfy the hypotheses of 3.4(ii).
(2) From 3.2(2) and 3.4(i) given a directed graph E, and a function $c : E^1 \to G$ such that $C^*(E(c))$ is purely infinite then $C^*(E)$ is purely infinite. However the converse is not true: if $C^*(E)$ is purely infinite then $C^*(E(c))$ is not necessarily purely infinite – for any graph E, $Z \times E$ has no loops (see graphs E and $Z \times E$ in 2.2).

For our next result we need the following fact about groupoids which is certainly well known; as we were unable to find an explicit reference we provide a proof (cf. [R, Proposition I.1.8 (i)]):

3.7 Proposition: *Let $\mathcal{G}$ be an amenable r-discrete groupoid and $c : \mathcal{G} \to G$ be a continuous 1–cocycle, where G is a countable group, and let β be the action of G on $\mathcal{G}(c)$ given by (see [R, p.9]):*

$$\beta_a(x, b) \quad = \quad (x, ab), \tag{6}$$

where $x \in \mathcal{G}$ and $a, b \in G$. Then $\mathcal{G}(c)$ such that

$$C^*(\mathcal{G}(c)) \times_\beta G \quad \cong \quad C^*(\mathcal{G}) \otimes \mathcal{K}\left(\ell^2(G)\right),$$

where β also denotes the action induced on $C^*(\mathcal{G}(c))$.

Proof:

By standard arguments $C^*(\mathcal{G}(c)) \times_\beta G \;\cong\; C^*(\mathcal{G}(c) \times_\beta G)$ where $\mathcal{G}(c) \times_\beta G$ is the groupoid semi-direct product (see [R, I.1.7]). Firstly we show that $\mathcal{G}(c) \times_\beta G \cong \mathcal{G} \times \mathcal{I}$ where $\mathcal{I}$ is the principal transitive groupoid arising from the equivalence relation on G where all elements are equivalent. The elements $[(x, a), b], [(z, p), q] \in \mathcal{G}(c) \times_\beta G$ are composable if and only if $(z, p) = (y, b^{-1}ac(x))$ where $(x, y) \in \mathcal{G}^{(2)}$. In this case

$$[(x, a), b)][(y, b^{-1}ac(x)), q] \quad = \quad [(x, a)(y, ac(x)), bq] \quad = \quad [(xy, a), bq]$$

and so if we define $\phi : \mathcal{G}(c) \times_\beta G \to \mathcal{G} \times \mathcal{I}$ by $\phi([(x, a), b]) = [x, (a, b^{-1}ac(x))]$ then

$$\begin{aligned} \phi([(x, a), b)])\phi\left(\left[\left(y, b^{-1}ac(x)\right), q\right]\right) &= \left[x, \left(a, b^{-1}ac(x)\right)\right]\left[y, \left(b^{-1}ac(x), q^{-1}b^{-1}ac(x)c(y)\right)\right] \\ &= \left[xy, \left(a, (bq)^{-1}ac(xy)\right)\right] \\ &= \phi([(xy, a), bq]). \end{aligned}$$

Thus ϕ (preserves composability and) is multiplicative. One checks that that ϕ is a homeomorphism and that it preserves groupoid structure; hence, it is a groupoid isomorphism.

From [MS, Proposition 6.11] we then have $C_r^*(\mathcal{G} \times \mathcal{I}) \cong C_r^*(\mathcal{G}) \otimes C_r^*(\mathcal{I})$ where $\otimes$ denotes the minimal or spatial tensor product. Since $\mathcal{G}$ is amenable and $\mathcal{I}$ is an AF groupoid by [MS, Proposition 6.6] we may conclude that

$$C^*(\mathcal{G} \times \mathcal{I}) \quad \cong \quad C^*(\mathcal{G}) \otimes \mathcal{K}\left(\ell^2(G)\right),$$

which completes the proof. $\square$

3.8 Note: By [M, Theorem 3.2], the continuous 1–cocycle $c : \mathcal{G} \to G$ induces a coaction of G on $C^*(\mathcal{G})$ and $C^*(\mathcal{G}(c))$ is the resulting crossed product. The action β may be regarded as dual to this coaction; the above can then be interpreted as a special case of a standard duality result. Recall that $c : E^1 \to G$ induces a 1–cocycle $\tilde{c} : \mathcal{G}_E \to G$ and by 2.4 one has $\mathcal{G}_E(\tilde{c}) \cong \mathcal{G}_{E(c)}$. Hence, $C^*(E(c))$ is a crossed product of $C^*(E)$ by a coaction of G.

3.9 Corollary: *Let E be a locally finite directed graph and $c : E^1 \to G$ be a function then*

$$C^*(E(c)) \times_\gamma G \quad \cong \quad C^*(E) \otimes \mathcal{K}\left(\ell^2(G)\right)$$

where γ denotes the action on $C^*(E(c))$ induced by the natural action of G on $E(c)$ (see eq. (5) above). Moreover, if G is abelian then $\gamma = \widehat{\alpha}$ where α is given in 2.5.

Proof:

Comparing (5) and (6) one sees that β and the action induced by γ are identical on $\mathcal{G}_{E(c)}$ and thus on $C^*(E(c))$; the first result follows by applying 3.7. Careful examination of the isomorphism between $C^*(\mathcal{G}_E) \times_\alpha \widehat{G}$ and $C^*(\mathcal{G}_E(\tilde{c}))$ described in the proof of [R, Theorem II.5.7] reveals that the dual action $\widehat{\alpha}$ of G on the crossed product translates to the action β of G on $C^*(\mathcal{G}_E(\tilde{c})) \cong C^*(\mathcal{G}_{E(c)})$. The second assertion then follows since we have identified β with γ. $\square$

The following result could perhaps be interpreted as an analogue of Green's Theorem [G, Theorem 14] for graphs.

3.10 Corollary: *Let E be a locally finite directed graph and $\lambda : G \to Aut(E)$ a free action then*

$$C^*(E) \times_\lambda G \quad \cong \quad C^*(E/G) \otimes \mathcal{K}\left(\ell^2(G)\right).$$

Proof:

From [GT, Theorem 2.2.2], there is a function $c : (E/G)^1 \to G$ such that $(E/G)(c) \cong E$. The result then follows by applying 3.9. □

4 The universal covering tree of a graph

Here the notation is adapted from [St, §2.1], [GT, §1.2.1]: let E be a directed graph, for $e \in E^1$ we formally denote the reverse edge by $\overline{e}$ where $s(\overline{e}) = r(e)$ and $r(\overline{e}) = s(e)$. The set of reverse edges is denoted $\overline{E}^1$; it is then natural to define $\overline{\overline{e}} = e$ for $\overline{e} \in \overline{E}^1$. A **walk** in E is then a sequence $a = (a_1, \ldots, a_n)$, which we write $a = a_1 \cdots a_n$, where $a_i \in E^1 \cup \overline{E^1}$ are such that $r(a_i) = s(a_{i+1})$ for $i = 1, \ldots, n-1$; we write $s(a) = s(a_1)$ and $r(a) = r(a_n)$. It will be convenient to regard a vertex as a trivial walk. A walk $a = a_1 \cdots a_n$ is said to be **reduced** if it does not contain the subword $a_i a_{i+1} = e\overline{e}$ for any $e \in E^1 \cup \overline{E}^1$. The set of reduced walks in E is denoted E^{rw}. For $a = a_1 \cdots a_n \in E^{rw}$ the reverse walk is written $\overline{a} := \overline{a_n} \cdots \overline{a_1}$; henceforth, for $a, b \in E^{rw}$ with $r(a) = s(b)$, ab will be understood to be the reduced walk obtained by concatenation and cancellation (for example if $a = b\overline{e}$ for some $e, b \in E^{rw}$ then $ae = b$). We adopt the convention that reduced walks in E are denoted by $a, b, \ldots$, whereas paths in E are denoted $\mu, \nu, \ldots$.

The directed graph E is said to be **connected** if given any two distinct vertices in E, there is a reduced walk between them. A directed graph T is a **tree** if and only if there is precisely one reduced walk between any two vertices. We define the boundary of a tree to be the quotient of the infinite path space by shift tail equivalence:

4.1 Definitions: *Let T be a tree; then $\partial T = T^\infty / \sim$ endowed with the quotient topology is called the* **boundary** *of T; denote the quotient map, $x \in T^\infty \mapsto [x] \in \partial T$, by ϕ. For each $v \in T^0$ define*

$$Y(v) \quad = \quad \{\, [x] \in \partial T \;:\; s(x) = v \,\}.$$

As there is another notion of boundary applicable to *undirected* graphs it might be preferable to use the term directed boundary for the notion introduced here; but since we deal exclusively with directed graphs there should be no risk of confusion. Note that if $v, w \in T^0$ are distinct vertices, then $Y(v) \cap Y(w)$ is nonempty iff the unique reduced walk from v to w is of the form $\mu\overline{\nu}$ with $\mu, \nu \in T^*$ in which case $Y(v) \cap Y(w) = Y(u)$ where $u = r(\mu) = r(\nu)$.

4.2 Lemma: *Let T be a row finite tree, then $\{Y(v) : v \in T^0\}$ forms a basis of compact open sets for the quotient topology on ∂T, moreover $\phi : T^\infty \to \partial T$ is a local homeomorphism and ∂T is Hausdorff.*

Proof:

We claim that $\phi(Z(\mu)) = Y(r(\mu))$ for all $\mu \in T^*$. To see this let $y \in Z(\mu)$, then $y = \mu z$ for some $z \in T^\infty$ and so $\phi(y) = [y] = [z] \in Y(r(\mu))$. On the other hand if $[z] \in Y(r(\mu))$ then $z \sim \mu z$ and so $[z] = [\mu z] \in \phi(Z(\mu))$. Since the topology of T^∞ is generated by sets of the form $Z(\mu)$ for $\mu \in T^*$ it follows that the $Y(v)$'s generate the quotient topology on ∂T; hence, $\{Y(v) : v \in E^0\}$ forms a basis.

For all $\mu \in T^*$ we claim that the restriction $\phi\big|_{Z(\mu)}$ of ϕ to $Z(\mu)$ is injective. To see this suppose that $\phi(x) = \phi(y)$, for $x, y \in Z(\mu)$ then $[x] = [y]$ that is $x = \kappa z \sim \nu z = y$ for some $\kappa, \nu \in E^*$ and $z \in T^\infty$. Since $s(\kappa) = s(\nu) = s(\mu)$ the walk $a = \kappa\overline{\nu}$ is a reduced loop in T, hence $\kappa = \nu$ and then $x = y$. Since $Z(\mu)$ for $\mu \in T^*$ cover T^∞ we may deduce that ϕ is a local homeomorphism.

Let $v \in T^0$, since T is row finite, $Z(v)$ is compact and since $\phi(Z(v)) = Y(v)$ we may deduce that $Y(v) \subseteq \partial T$ is compact and ∂T is Hausdorff. $\square$

If $x \sim_k y$ in T^∞ then there is only one such $k \in \mathbf{Z}$ with this property since T is a tree and so we may write the elements of $\mathcal{G}_T$ as (x, y). Since the equivalence relations: $\sim$ corresponding to shift tail equivalence and $R(\phi)$ corresponding to the local homeomorphism $\phi : T^\infty \to \partial T$, are identical on T^∞ we may obtain the next result from [K, §4]. We include the details since they will be useful later.

4.3 Proposition: *Let T be a row finite tree then $C^*(T)$ is strongly Morita equivalent to $C_0(\partial T)$.*

Proof:

As in [K, §2] we may regard $C_c(T^\infty)$ as a right $C_c(\partial T)$–module by defining for $x \in T^\infty$

$$(gh)(x) \quad = \quad g(x)h([x]),$$

where $g \in C_c(T^\infty)$ and $h \in C_c(\partial T)$. There is a $C_c(\partial T)$–valued inner product on $C_c(T^\infty)$ defined by

$$\langle f, g\rangle_{C_0(\partial T)}([x]) \quad = \quad \sum_{y\in[x]} \overline{f(y)}g(y) \tag{7}$$

for $f, g \in C_c(T^\infty)$ and $[x] \in \partial T$. It is verified in [K, Proposition 2.2] that the inner product has all the requisite properties. We may regard $C_c(T^\infty)$ as a left $C_c(\mathcal{G}_T)$–module by defining for $x \in T^\infty$

$$(fg)(x) \quad = \quad \sum_{y\in[x]} f(x, y)g(y),$$

where $f \in C_c(\mathcal{G}_T)$ and $g \in C_c(T^\infty)$. There is a $C_c(\mathcal{G}_T)$–valued inner product on $C_c(T^\infty)$ defined for $(x, y) \in \mathcal{G}_T$ by

$$\langle f, g\rangle_{C_c(\mathcal{G}_T)}(x, y) \quad = \quad f(x)\overline{g(y)} \tag{8}$$

for $f, g \in C_c(T^\infty)$. One may also check that this inner product has all the necessary properties. Let X be the completion of $C_c(T^\infty)$ in the norm arising from $\langle\cdot,\cdot\rangle_{C_0(\partial T)}$; it follows that X is a $C^*(T) - C_0(\partial T)$ equivalence bimodule. □

Now we construct a "universal cover" T of a connected directed graph E (cf. [H]):

4.4 Definition: *Let E be a connected directed graph, fix a base point $\star \in E^0$ and let $E^{rw}(\star)$ denote the set of all reduced walks in E starting at $\star$. Define a directed graph $T = T(E, \star)$ as follows: let $T^0 = E^{rw}(\star)$, $T^1 = \{(a, e) \in E^{rw}(\star) \times E^1 : r(a) = s(e)\}$ and put $s(a, e) = a$, $r(a, e) = ae$. For $(a, e) \in T^1$ we identify $\overline{(a, e)}$ with $(ae, \overline{e})$.*

4.5 Lemma: *Let E be a connected directed graph then T is a tree and the isomorphism class of $T = T(E, \star)$ is independent of the choice of base point $\star \in E^0$.*

Proof:

Let $T = T(E, \star)$ and $a, b \in T^0$ with $a \neq b$ then

$$c = (a_1 \ldots a_{|a|}, \overline{a_{|a|}}) \ldots (a_1, \overline{a_1})(\star, b_1) \ldots (b_1 \ldots b_{|b|-1}, b_{|b|}) \tag{9}$$

is a reduced walk in T from a to b. Suppose that d is another walk in T from a to b. Since $a, b \in E^{rw}(\star)$ in order that $s(d) = a$ and $r(d) = b$ the walk d must at least contain the undirected edges in the reduced form of the right hand side of (9), so either $d = c$ or d is not reduced, hence T is a tree.

If we form the graph $W = T(E, \star')$ using a different base point $\star'$, then we claim that $W \cong T$. Since E is connected there is a reduced walk $a \in E^{rw}$ from $\star$ to $\star'$. Define a map $f : W \to T$ by $f^0(b) = ab$ and $f^1(b, e) = (ab, e)$ where $a \in E^{rw}(\star)$, $b \in E^{rw}(\star')$ and $e \in E^1$ then f is a graph morphism. If we define $f^{-1} : T \to W$ using the reverse walk $\overline{a}$, then we may see that f is an isomorphism. $\square$

4.6 Example: Let F be the graph in 2.2 then if we choose our base point to be the vertex v, then $T = T(F, v)$ is isomorphic to the Cayley graph $E_{\mathbf{F}_2}$ of the free group $\mathbf{F}_2$.

A modified version of the following definition is to be found in [St, §2.2.1]:

4.7 Definition: *Let E, F be directed graphs and $p : E \to F$ be a graph morphism then p is a* **covering map** *if*

(i) p is onto, that is p^0, p^1 are surjective,

(ii) for $u \in E^0$ the maps $p^1 : s^{-1}(u) \to s^{-1}(p^0(u))$ and $p^1 : r^{-1}(u) \to r^{-1}(p^0(u))$ are bijections.

If one extends the definition of p to reverse edges in the natural way: $p^1(\overline{e}) = \overline{p^1(e)}$, condition (ii) may be rephrased to assert that for every $u \in E^0$ and $f \in F^1 \cup \overline{F}^1$ with $s(f) = p(u)$, there is a unique $e \in E^1 \cup \overline{E}^1$ so that $s(e) = u$ and $p(e) = f$. It follows that if p is a covering map then it has the unique walk lifting property (see [GT, Theorem 2.1.1]): given $u \in E^0$ and $a \in F^{rw}$ with $s(a) = p(u)$, then there is a unique $\tilde{a} \in E^{rw}$ such that $s(\tilde{a}) = u$ and $p(\tilde{a}) = a$. Note that for any directed graph E and function $c : E^1 \to G$ the natural projection $E(c) \to E$ is a covering map. Equivalently, if G acts freely on E then the quotient map $q : E \to E/G$ is a covering map.

4.8 Lemma: *Let E be a connected directed graph and T as above then $p : T \to E$ defined by $p^0(a) = r(a)$ and $p^1(a,e) = e$ for $e \in E^1$ is a covering map. Moreover, T is a universal cover of E in the sense that if $q : W \to E$ is another covering map, then there is a graph morphism $\varphi : T \to W$ such that $p = q\varphi$. Moreover, φ is surjective if and only if W is connected.*

Proof:

Fix $\star \in E^0$; then $T = T(E, \star)$. The surjectivity of p follows immediately from the fact that E is connected (each vertex in E^0 may be connected to $\star$ via a reduced walk). Let $a \in T^0$ then $s^{-1}(a) = \{(a,e) : s(e) = r(a)\}$ and $s^{-1}(p^0(a)) = \{e : s(e) = r(a)\}$. For $(a,e) \in s^{-1}(a)$, one has $p^1(a,e) = e \in s^{-1}(r(a))$, so $p^1 : s^{-1}(a) \to s^{-1}(r(a))$ is clearly a bijection; similarly, one checks that $p^1 : r^{-1}(a) \to r^{-1}(r(a))$ is a bijection (recall that $\overline{(a,e)} = (ae, \overline{e})$) and hence $p : T \to E$ is a covering map.

Now let $q : W \to E$ be another covering map. Fix $\star' \in W^0$ such that $q(\star') = \star$. For $a \in T^0 = E^{rw}(\star)$ let a' be the unique reduced walk in W such that $s(a') = \star'$ and $q(a') = a$. Define $\varphi^0 : T^0 \to W^0$ by $\varphi^0(a) = r(a')$. Let $(a,e) \in T^1$, then $s(e) = r(a) = q^0(r(a'))$; hence there is a unique $e' \in W^1$ such that $s(e') = r(a')$ and $q^1(e') = e$ and so we may define $\varphi^1(a,e) = e'$. Then φ is the desired morphism. $\square$

4.9 Example: Now in 2.2, E is a covering graph for F and so by the above lemma and 4.6 it has the same universal covering tree, namely $E_{\mathbf{F}_2}$.

Hence we may refer to T as the universal covering tree of E. Let $G = \{a \in E^{rw}(\star) : r(a) = \star\}$ then G forms a countable group under concatenation. G acts naturally on T by defining $\tau_g^0 a = ga$, and $\tau_g^1(a,e) = (ga, e)$ for $g \in G$. We refer to G as the fundamental group of E (if E is connected the isomorphism class of $G = \pi_1(E, \star)$ is independent of the choice of basepoint).

4.10 Lemma: *With notation as above the map $\tau : G \to AutT$ defines a free action of G on T such that $T/G \cong E$. Moreover, G is a free group and if E^0 is finite then $G \cong \mathbf{F}_n$ where $n = |E^1| - |E^0| + 1$.*

Proof:

It is routine to check that the τ_g^i for $i = 0, 1$ and $g \in G$ satisfy the requisite properties and so $\tau(G) \subseteq \mathrm{Aut}\,(T)$. Suppose $a \in T^0$ is such that $ga = a$, then as $a, g \in E^{rw}$ we must have that $g = 1_G = r(\star) \in E^{rw}$, and so the action of G on T is free.

Let $[a] \in (T/G)^0$ define $\psi^0 : (T/G)^0 \to E^0$ by $\psi^0([a]) = r(a)$, then ψ^0 is well–defined since $r(ga) = r(a)$. Similarly, for $[(a,e)] \in (T/G)^1$ define $\psi^1(T/G)^1 \to E^1$ by $\psi^1([a,e]) = e$, then ψ^1 is clearly also well–defined. Note that $\psi : T/G \to E$ is a surjective graph morphism (recall that $p : T \to E$ is a covering map). Injectivity follows from the fact that if $a, b \in E^{rw}(\star)$ with $r(a) = r(b)$ then $g = a\bar{b} \in G$ and $a = gb$; hence ψ is an isomorphism. Since G acts freely on a tree it must be a free group, moreover if E^0 is finite then G is a free group of order $|E^1| - |E^0| + 1$ (see [Se, §3.3, Theorem 4'], [LS, Proposition 2.2]). $\square$

4.11 Example: Let E be the graph in 2.2 then from 4.9, $T = T(E, 1)$ is isomorphic to $E_{\mathbf{F}_2}$, and in this case $G \cong \mathbf{F}_3$ with generators $a, bc, bd\bar{b} \in E^{rw}(1)$.

4.12 Note: The action τ of G on T extends naturally to T^∞ and induces an action of G on $C_0(T^\infty)$: $\tau_g h = h \circ \tau_{g^{-1}}$. The induced action on $C^*(T)$ restricts to $C_c(\mathcal{G}_T)$ in a natural way: $\tau_g(f)(x,y) = f(\tau_{g^{-1}}x, \tau_{g^{-1}}y)$. Similarly there is an action $\tilde{\tau}$ of G on ∂T defined by $\tilde{\tau}_g[x] = [\tau_g x]$ (this action is clearly well-defined). We also let $\tilde{\tau}$ denote the induced action of G on $C_0(\partial T)$.

4.13 Theorem: *Let E be a row finite connected directed graph, let $T = T(E, \star)$ be its universal covering tree, ∂T the boundary of T, and G the fundamental group of E at $\star$ then $C^*(T) \times_\tau G$ is strongly Morita equivalent to $C_0(\partial T) \times_{\tilde{\tau}} G$.*

Proof:

By 4.3 $C_0(\partial T)$ and $C^*(T)$ are strongly Morita equivalent, and so by [CMW, Theorem 1] (see also [Co]) it suffices to check that the action of G on X is compatible with the given actions on $C_0(\partial T)$ and $C^*(T)$; that is, we must show that the inner products (7), (8) given in 4.3 satisfy two equivariance conditions:

For $h, k \in C_c(T^\infty)$, $[x] \in \partial T$ and $g \in G$ we check

$$\begin{aligned}
\langle \tau_g h, \tau_g k \rangle_{C_0(\partial T)}([x]) &= \sum_{y \in [x]} \overline{\tau_g h}(y) \tau_g k(y) \\
&= \sum_{z \in [\tau_{g^{-1}} x]} \overline{h(z)} k(z) \text{ since } \tau_{g^{-1}} x \sim \tau_{g^{-1}} y \text{ iff } x \sim y, \\
&= \langle h, k \rangle_{C_0(\partial T)}([\tau_{g^{-1}} x]) \;=\; \tilde{\tau} \langle h, k \rangle_{C_0(\partial T)}([x]),
\end{aligned}$$

and for $(x,y) \in \mathcal{G}_T$

$$\begin{aligned}\langle \tau_g h, \tau_g k\rangle_{C_c(\mathcal{G}_T)}(x,y) &= h(\tau_{g^{-1}}x)\overline{k(\tau_{g^{-1}}y)},\\ &= \langle h,k\rangle_{C_c(\mathcal{G}_T)}(\tau_{g^{-1}}x,\tau_{g^{-1}}y) \;=\; \tau_g\langle h,k\rangle_{C_c\mathcal{G}_T}(x,y).\end{aligned}$$

Hence we may deduce that $C^*(T) \times_\tau G$ is strongly Morita equivalent to $C_0(\partial T) \times_{\tilde{\tau}} G$ as required. $\square$

4.14 Corollary: *Let E be a row finite connected directed graph then $C^*(E)$ is strongly Morita equivalent to $C_0(\partial T) \times_{\tilde{\tau}} G$ where ∂T is the boundary of the universal covering tree $T = T(E,\star)$ of E and G is the fundamental group of E at $\star$.*

Proof:
From 4.10 we have that $T/G \cong E$; applying 3.10 we may see that $C^*(E) \otimes \mathcal{K}(\ell^2(G)) \cong C^*(T) \times_\tau G$ and the result then follows from 4.13. $\square$

4.15 Remarks: (1) An equivalent result which does not use the universal covering tree construction can be proved as follows: let E be a graph and Γ be the free group with generators γ_e for $e \in E^1$ then we may define $c : E^1 \to \Gamma$ by $c(e) = \gamma_e$ (cf. [QR, §6]). The skew product graph $F = E(c)$ is then a forest (i.e. a disjoint union of trees). As shown in 4.3 $C^*(F)$ is strongly Morita equivalent to $C_0(\partial F)$, the result then follows since the natural equivalence bimodule is equipped with a Γ-action which is compatible with the Γ-action on F and the induced action on ∂F.
(2) We may also use Theorem 2.8 of [MRW] to prove the above Corollary directly — we sketch the proof that T^∞ may be endowed with the structure of an equivalence between the transformation groupoid $\partial T \times G$ and $\mathcal{G}_E$ (see [MRW, Definition 2.1]). First, we require maps from T^∞ to the unit spaces of the two groupoids: for the first map take the quotient map $\phi : T^\infty \to \partial T$ and for the second take $\sigma : T^\infty \to E^\infty$ to be the quotient map by the action of G (note $\sigma(x) = (p(x_1), p(x_2), \ldots)$ where $p : T \to E$ is the covering map). We endow T^∞ with a left action by $\partial T \times G$ so that it becomes a left principal $\partial T \times G$–space over E^∞ (via σ) and a right action by $\mathcal{G}_E$ so that it becomes a right principal $\mathcal{G}_E$–space over ∂T (via ϕ). The left action

$$(\partial T \times G) * T^\infty \to T^\infty$$

is defined by $(([x],g),x) \mapsto gx$ where $[x] = \phi(x)$ (this is really just the action of G on T^∞) and the right action $T^\infty * \mathcal{G}_E \to T^\infty$ is defined via the unique walk lifting property: if $x \in T^\infty$ and $(\alpha z, |\alpha| - |\beta|, \beta z) \in \mathcal{G}_E$ with $\sigma(x) = \alpha z$ there are unique $\tilde{\alpha}$ and $\tilde{z}$ so that $p(\tilde{\alpha}) = \alpha$, $\sigma(\tilde{z}) = z$ and $x = \tilde{\alpha}\tilde{z}$; moreover, there is a unique $\tilde{\beta}$ so that $p(\tilde{\beta}) = \beta$, and $r(\tilde{\beta}) = r(\tilde{\alpha})$. The right action is defined by $(x, (\alpha z, |\alpha| - |\beta|, \beta z)) \mapsto \tilde{\beta}\tilde{z}$. In a natural sense this action may be regarded as equivalent to the right action of $\mathcal{G}_T$ on its unit space and so the quotient by this action is then the orbit space ∂T.

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