ACTIONS OF SEMIGROUPS ON DIRECTED GRAPHS AND THEIR C^*-ALGEBRAS

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Abstract. A free action α of a group G on a row-finite directed graph E induces an action α_* on its Cuntz-Krieger C^*-algebra $C^*(E)$, and a recent theorem of Kumjian and Pask says that the crossed product $C^*(E) \times_{\alpha_*} G$ is stably isomorphic to the C^*-algebra $C^*(E/G)$ of the quotient graph. We prove an analogue for free actions of Ore semigroups. The main ingredients are a new generalisation of a theorem of Gross and Tucker, dilation theory for endomorphic actions of Ore semigroups on graphs and C^*-algebras, and the Kumjian-Pask Theorem itself.

The space of paths in a directed graph E can be modelled by systems of partial isometries on Hilbert space: one associates to each edge e a partial isometry S_e in such a way that the product S_eS_f is a nonzero partial isometry when ef is a path, and zero otherwise. Since partial isometries are the linear operators T satisfying $T = TT^*T$, the appropriate algebraic envelopes for such *Cuntz-Krieger systems* $\{S_e\}$ are C^*-algebras; the *graph algebra* $C^*(E)$ is the universal C^*-algebra generated by a Cuntz-Krieger system. For finite graphs, these algebras turn out to be precisely the Cuntz-Krieger algebras associated to Markov chains [3]. More recently, the algebras of infinite graphs have arisen in a variety of contexts, and the fundamental results of Cuntz and Krieger extend in an attractive and approachable manner (see [2] for more precise statements and references).

The general theory of C^*-algebras provides powerful tools for problems involving symmetry groups and representation theory, so it is natural to ask how group actions interact with graph algebras. Every action α of a group G on E induces an action α_* of G on $C^*(E)$. A recent theorem of Kumjian and Pask asserts that if α is free, then the crossed product $C^*(E) \times_{\alpha_*} G$ is stably isomorphic to the C^*-algebra $C^*(E/G)$ of the quotient graph [10, Corollary 3.10]; alternatively, the C^*-algebras $C^*(E) \times_{\alpha_*} G$ and $C^*(E/G)$ are Morita equivalent in the sense appropriate for C^*-algebras. This theorem is strikingly similar to a well-known result of Green about free and proper actions of groups on locally compact spaces [5], and suggests that graph algebras might be profitably viewed as noncommutative function algebras in the sense of Connes.

Here we prove an analogue of the Kumjian-Pask Theorem for endomorphic actions of semigroups on graphs, under mild hypotheses on the semigroup and the action. First, we restrict our attention to Ore semigroups S, which can be always be embedded in a group Γ in such a way that $\Gamma = S^{-1}S$. We do this because Laca has shown that one can then expect to dilate endomorphic actions of S to automorphic actions of Γ [12]. The class of Ore semigroups includes all generating subsemigroups of abelian groups as well as many interesting nonabelian semigroups (see [12, §1.1], for example). Second, we assume that the action of S has a fundamental domain: a collection of edges and their sources whose images under S cover E. There is always such a domain for group actions, but simple examples show that we need to assume it

Date: 8 November 1999.
1991 *Mathematics Subject Classification.* 20M20, 46L55, 54H15, 05C20.
Key words and phrases. directed graph, Ore semigroup, dilation, C^*-algebra, crossed product.
This research was supported by the Australian Research Council.

here and that the assumption is reasonable. When there is a fundamental domain, the action α induces an action α_* of S by endomorphisms of $C^*(E)$.

The object of the paper, then, is to prove the following theorem. The proof uses a variety of results in graph theory, dilation theory and graph C^*-algebras, and some of these results may be of independent interest.

Theorem. *Suppose S is an Ore semigroup and $\alpha : S \to \operatorname{End} E$ is a free action of S on a row-finite directed graph for which there is a fundamental domain. Then $C^*(E) \times_{\alpha_*} S$ is stably isomorphic to $C^*(E/S)$.*

The key ingredient in the Kumjian-Pask Theorem is a characterisation of the graphs admitting free actions as skew products, due to Gross and Tucker. We begin by developing an analogous characterisation of the graphs which admit free actions of Ore semigroups with fundamental domains (Theorem 1.8). We prove a slightly more general theorem than we need later: it is not necessary to assume that the edges in the fundamental domain are attached to the vertices in the domain (such a domain is called a transversal for the action rather than a fundamental domain). A preliminary version of this theorem was presented in [16].

One approach to the theory of semigroup crossed products uses dilation theory: given a semigroup crossed product $A \times_\beta S$, we construct an action β^∞ of the enveloping group Γ on a direct limit A_∞, and view $A \times_\beta S$ as a full corner in the ordinary crossed product $A_\infty \times_{\beta^\infty} \Gamma$. Any cancellative semigroup S can be partially ordered by setting $s \preceq_r t \iff t = qs$ for some $q \in S$; the Ore condition implies that S is directed by $\preceq_r$, and Laca proved in [12] that one can dilate actions of Ore semigroups (see [12] for further references). In §2 we introduce the notion of direct systems of graphs, show how to dilate endomorphic actions of S on graphs, and prove that dilating a semigroup skew product gives a group skew product. In §3, we consider the problem of lifting an action α of S on E to an action on $C^*(E)$, and prove that this can be done whenever there is a fundamental domain. We then prove that dilation is compatible with the construction of the graph algebra, in the sense that the C^*-algebra of the dilation is the dilation of the C^*-algebra. Because our graph algebras need not have identities, Laca's results do not apply directly; in §4, therefore, we show how to adapt [12, Theorem 2.2.1] to our situation. The main technical problem is to show that all the induced endomorphisms $(\alpha_*)_s$ extend to homomorphisms between multiplier algebras.

We prove our Theorem in §5 by showing that $C^*(E) \times_{\alpha_*} S$ is isomorphic to the C^*-algebra tensor product $C^*(E/S) \otimes \mathcal{K}(\ell^2(S))$, where $\mathcal{K}(\ell^2(S))$ denotes the algebra of compact operators on the Hilbert space $\ell^2(S)$. The idea is to write E as a skew product using the theory of §1, dilate to realise $C^*(E) \times S$ as a corner in $C^*((E/S) \times_c \Gamma) \times \Gamma$, use the Kumjian-Pask Theorem to move everything into $C^*(E/S) \otimes \mathcal{K}(\ell^2(\Gamma))$, and then identify the corner as the subalgebra $C^*(E/S) \otimes \mathcal{K}(\ell^2(S))$ of $C^*(E/S) \otimes \mathcal{K}(\ell^2(\Gamma))$. This last step involves chasing the projection defining the corner through the various isomorphisms, and we have found it helpful to use the concrete description of the Kumjian-Pask isomorphism from [9] rather than the original.

1. Skew-Product Graphs and Semigroup Actions

Every semigroup in this paper is countable and has an identity 1. A cancellative semigroup is said to be *Ore* if it is right-reversible; that is, if $Ss \cap St \neq \emptyset$ for all $s, t \in S$. There is an elegant characteristic of these semigroups; for a proof and further references see [12, Theorem 1.1.2].

Theorem 1.1. *A semigroup S can be embedded in a group Γ with $S^{-1}S = \Gamma$ if and only if it is an Ore semigroup. If so, the group Γ is determined up to canonical isomorphism, and every semigroup homomorphism ϕ from S into a group G extends uniquely to a group homomorphism $\varphi : \Gamma \to G$.*

A *directed graph* $E = (E^0, E^1, r, s)$ consists of countable sets E^0 of vertices and E^1 of edges together with maps $r, s : E^1 \to E^0$ describing the range and source of edges. If E and F are directed graphs, then a *graph morphism* ϕ is a pair of maps $\phi = (\phi^0 : E^0 \to F^0, \phi^1 : E^1 \to F^1)$ which preserve connectivity:

$$\phi^0(r_E(e)) = r_F(\phi^1(e)) \text{ and } \phi^0(s_E(e)) = s_F(\phi^1(e))$$

for all $e \in E^1$. The automorphisms of a graph E form a group $\operatorname{Aut} E$ and the injective endomorphisms form a unital semigroup $\operatorname{End} E$. An *action* of a semigroup S on a directed graph E is a homomorphism $\alpha : S \to \operatorname{End} E$ such that $\alpha(1)$ is the identity. The action is *free* if the action on E^0 is free; that is, if $\alpha_s v = \alpha_t v$ implies $s = t$. Since α preserves connectivity, a free action also acts freely on E^1. We say that two actions (E, S, α), (F, S, β) are isomorphic if there is an isomorphism $\phi : E \to F$ such that $\phi \circ \alpha_t = \beta_t \circ \phi$. The assumption of injectivity on our endomorphisms is not needed for some of our results, but we have included it to avoid situations which would turn out to be trivial.

Examples 1.2. The following graphs admit free actions of $\mathbb{N}$ in which $n \in \mathbb{N}$ shifts each vertex n vertices to the right. Although these examples seem quite similar, only one of them satisfies the hypotheses of our Theorem (see Example 1.6).

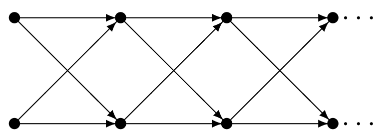

FIGURE 1

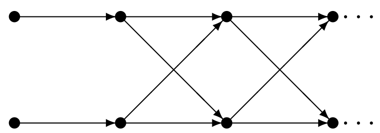

FIGURE 2

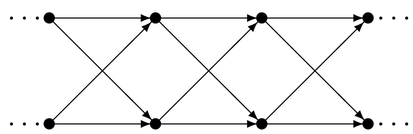

FIGURE 3

Let α be an action of an Ore semigroup S on a directed graph E, and define relations on E^i by $x \sim y$ iff there exist $s, t \in S$ such that $\alpha_s x = \alpha_t y$. Because S is right-reversible, these are equivalence relations, and the equivalence classes are the vertices and edges in the *quotient graph* $E/S := (E^0/S, E^1/S, r, s)$; the range and source maps are defined by requiring the *quotient map* $q : E \to E/S$ to be a graph morphism (see [16, Proposition 2.9]).

Definition 1.3. Let E be a directed graph, S a semigroup, and $c, d : E^1 \to S$ functions. The *skew-product graph* $E \times_{c,d} S$ has vertex set $E^0 \times S$, edge set $E^1 \times S$, and range and source maps defined by

$$r(e, t) = (r(e), tc(e)) \text{ and } s(e, t) = (s(e), td(e)).$$

There is a free action τ of S by left translation on $E \times_{c,d} S$ defined by

$$\tau_s^0(v,t) = (v, st) \text{ and } \tau_s^1(e,t) = (e, st).$$

The skew-product graphs studied in [11, 16] use just one function $c : E^1 \to S$ and were denoted $E(c)$; in the present notation, $E(c)$ is $E \times_{c,1} S$, where 1 is the constant function $e \mapsto 1$. We have changed the definition to get a more general Theorem 1.8, and to eliminate possible ambiguity concerning the range of the functions c and d. For groups the extra generality dissolves:

Lemma 1.4. *Let E be a directed graph and let $c, d : E^1 \to \Gamma$ be functions. Then*

$$(E \times_{c,d} \Gamma, \Gamma, \tau) \cong (E \times_{d^{-1}c,1} \Gamma, \Gamma, \sigma),$$

where $d^{-1}c$ is defined by $d^{-1}c(e) = d(e)^{-1}c(e)$, and τ and σ are the left-translation actions on the respective skew products.

Proof. The isomorphism $\phi : E \times_{c,d} \Gamma \to E \times_{d^{-1}c,1} \Gamma$ is given by $\phi(v,g) = (v,g)$ and $\phi(e,g) = (e, gd(e))$. □

Definition 1.5. Let $\alpha : S \to \operatorname{End} E$ be an action of an Ore semigroup on a directed graph. A set $F \subset E^0 \cup E^1$ is an *S-transversal* for α if for every $u \in E^0 \cup E^1$ there is a unique $c \in F$ such that $u \in \{\alpha_t c : t \in S\}$. The S-transversal F is a *fundamental domain* if $s(e) \in F^0$ for every $e \in F^1$.

When S is a group, the definition of S-transversal is the same as that of [4, Chapter I]. An argument similar to the first part of the proof of [4, Chapter I, Proposition 2.6] shows that every action of a group has a fundamental domain.

Examples 1.6. The set $F = \{(e,1) : e \in E^1\} \cup \{(v,1) : v \in E^0\}$ is an S-transversal for $(E \times_{c,d} S, S, \tau)$; F is a fundamental domain if and only if $d = 1$. The shift actions of $\mathbb{N}$ on Figures 1 and 2 have $\mathbb{N}$-transversals which are illustrated below. The shift action on Figure 3, on the other hand, has no $\mathbb{N}$-transversal. The transversal in Figure 4 is a fundamental domain; the shift action on Figure 2 does not have a fundamental domain.

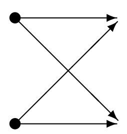

FIGURE 4. An $\mathbb{N}$-transversal for Figure 1

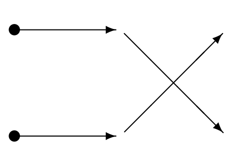

FIGURE 5. An $\mathbb{N}$-transversal for Figure 2

Proposition 1.7. *Let $\alpha : S \to \operatorname{End} E$ be an action of an Ore semigroup on a directed graph. If α has an S-transversal F, then for each $x \in (E/S)^0$ and $y \in (E/S)^1$, there are unique elements $v_x, e_y \in F$ such that $q^{-1}(x) = \{\alpha_t v_x : t \in S\}$ and $q^{-1}(y) = \{\alpha_t e_y : t \in S\}$.*

Proof. We will prove this for edges; the argument for vertices is similar. Given $y \in (E/S)^1$, choose $e \in E^1$ such that $q(e) = y$. Then there is a unique $e_y \in F$ with $e = \alpha_t e_y$ for some $t \in S$. Certainly $q(\alpha_s e_y) = q(e_y) = q(e) = y$ for $s \in S$, so $\{\alpha_t e_y : t \in S\} \subset q^{-1}(y)$. On the other hand, if $q(f) = y = q(e)$ for some $f \in E^1$, then there are $r, s \in S$ such that $\alpha_r f = \alpha_s e = \alpha_{st} e_y$

and a unique $u \in F$ such that $f = \alpha_p u$ for some $p \in S$. But now we have $u, e_y \in F$ and $\alpha_{rp} u = \alpha_{st} e_y$, which is only possible if $u = e_y$. □

Theorem 1.8. *Let $\alpha : S \to \operatorname{End} E$ be a free action of an Ore semigroup on a directed graph. Then α has an S-transversal if and only if there are functions $c, d : (E/S)^1 \to S$ such that $(E, S, \alpha) \cong (E/S \times_{c,d} S, S, \tau)$.*

Proof. Suppose there are functions $c, d : (E/S)^1 \to S$ and an isomorphism ψ of (E, S, α) onto $(E/S \times_{c,d} S, S, \tau)$. Then $F := \{\psi^0(x, 1) : x \in (E/S)^0\} \cup \{\psi^1(y, 1) : y \in (E/S)^1\}$ is an S-transversal of α.

Conversely, assume that α has an S-transversal F. By Proposition 1.7, for each $x \in (E/S)^0$ and $y \in (E/S)^1$ there are unique elements $v_x \in F^0$ and $e_y \in F^1$ such that $q(v_x) = x$ and $q(e_y) = y$. For $y \in (E/S)^1$ we have $r(e_y) = \alpha_s v_{r(y)}$ and $s(e_y) = \alpha_t v_{s(y)}$ for some $s, t \in S$; we define $c(y) = s$ and $d(y) = t$. The uniqueness of s and t follows from the freeness of the action. Define $\phi^i : (E/S \times_{c,d} S)^i \to E^i$ by $\phi^0(x, t) = \alpha_t v_x$ and $\phi^1(y, t) = \alpha_t e_y$. For $y \in (E/S)^1$ and $t \in S$ we have

$$\phi^0(s(y, t)) = \phi^0(s(y), td(y)) = \alpha_t \alpha_{d(y)} v_{s(y)} = s(\alpha_t e_y) = s(\phi^1(y, t)),$$

and similarly $\phi^0(r(y, t)) = r(\phi^1(y, t))$. Thus ϕ is a graph morphism. If $\phi^0(w, s) = \phi^0(x, t)$, then $\alpha_s v_w = \alpha_t v_x$, $v_w = v_x$ by the uniqueness property of F, $w = x$, and $s = t$ by freeness. Similarly, if $\phi^1(y, s) = \phi^1(z, t)$, then $y = z$ and $s = t$, so ϕ is injective. To see that ϕ is surjective, let $v \in E^0$ and $e \in E^1$, and note that $v = \alpha_s v_{q(v)} = \phi^0(q(v), s)$ and $e = \phi^1(q(e), t)$ for some $s, t \in S$. Equivariance follows from the definition of ϕ. □

Remark 1.9. When F is a fundamental domain, d is the constant function $y \mapsto 1$, the skew-product graph is that of [16, Definition 2.3], and Theorem 1.8 becomes [16, Theorem 2.19]. When S is a group (and hence the action has a fundamental domain), Theorem 1.8 becomes [6, Theorem 1.2.5]. If every edge in the S-transversal F has its *range* vertex in F, the function c is identically 1. However, it will be important that we used source vertices when defining fundamental domains: sources and ranges play different roles in the definition of the graph algebra $C^*(E)$.

2. Direct Limits

A binary relation $\leq$ on a set X is a *preorder* if it is reflexive and transitive. A preordered set $(X, \leq)$ is *directed* if for every $x, y \in X$, there exists $z \in X$ such that $x \leq z$ and $y \leq z$, and a subset Y of X is *cofinal* if for each $x \in X$ there exists $y \in Y$ such that $x \leq y$.

Let $\{E_x : x \in X\}$ be a family of directed graphs in which the index set X is directed by $\leq$. Suppose that for every $x \leq y$ there is a graph morphism $\pi^x_y : E_x \to E_y$ such that $\pi^x_x = \mathrm{id}$ and $\pi^y_z \circ \pi^x_y = \pi^x_z$ whenever $x \leq y \leq z$. Then $(\{E_x : x \in X\}, \{\pi^x_y\})$ is a *direct system* of directed graphs with *linking maps* π^x_y.

Proposition 2.1. *Let $(\{E_x : x \in X\}, \{\pi^x_y\})$ be a direct system of directed graphs. Then there are a directed graph F and graph morphisms $\pi^x : E_x \to F$ satisfying $\pi^x = \pi^y \circ \pi^x_y$ whenever $x \leq y$, which have the following property: whenever G is a directed graph and $\psi^x : E_x \to G$ are graph morphisms satisfying $\psi^x = \psi^y \circ \pi^x_y$, there is a unique graph morphism $\psi : F \to G$ such that $\psi \circ \pi^x = \psi^x$. Moreover:*

1. *if $\bigcup_{x \in X} \psi^x(E^i_x) = G^i$, then ψ is surjective; and*
2. *if each ψ^x is injective, then ψ is injective.*

Proof. We define a relation $\sim$ on $\bigcup_{x \in X} E^i_x$ by $E^i_x \ni p \sim q \in E^i_y$ if and only if there exists $z \in X$ with $\pi^x_z(p) = \pi^y_z(q)$. The property of the linking maps shows $\sim$ to be an equivalence relation.

The vertex and edge sets of F will be $F^i = \bigcup_{x\in X} E^i_x/\sim$. A short calculation shows that there are well-defined maps $r_F, s_F : F^1 \to F^0$ such that $r_F([e]) = [r(e)]$ and $s_F([e]) = [s(e)]$, so F is a directed graph; the maps $e \mapsto [e]$ and $v \mapsto [v]$ then form graph morphisms $\pi^x : E_x \to F$, and it follows from the definition of $\sim$ that $\pi^x = \pi^y \circ \pi^x_y$.

Now suppose we have graph morphisms $\psi^x : E_x \to G$ satisfying $\psi^x = \psi^y \circ \pi^x_y$. Given $u \in F^i$, choose $x \in X$ and $p \in E^i_x$ such that $\pi^x(p) = u$, and define $\psi(u) = \psi^x(p)$. If $\pi^x(p) = \pi^y(q)$ for some $q \in E^i_y$, then there exists $z \in X$ such that $\pi^x_z(p) = \pi^y_z(q)$, so $\psi^x(p) = \psi^z \circ \pi^x_z(p) = \psi^z \circ \pi^y_z(q) = \psi^y(q)$. Thus ψ is well-defined. It is a graph morphism because each ψ^x is a graph morphism. The identity $\psi \circ \pi^x = \psi^x$ is then true by definition and the uniqueness of ψ follows, as does (1).

For (2), we will show ψ is injective on edges; a similar argument works for vertices. Suppose $e, f \in F^1$ and $\psi(e) = \psi(f)$. There are $x, y \in X$, $e' \in E^1_x$ and $f' \in E^1_y$ such that $\pi^x(e') = e$ and $\pi^y(f') = f$. Since X is directed, there exists $z \in X$ such that $\psi^z \circ \pi^x_z(e') = \psi^z \circ \pi^y_z(f')$. By injectivity of ψ^z, this yields $\pi^x_z(e') = \pi^y_z(f')$, so $e = f$, and ψ is injective. □

We call $(F, \{\pi^x : x \in X\})$ the *direct limit* of the system, and denote it $\varinjlim E_x$; we also write $\varinjlim \psi^x$ for the graph morphism ψ of the Proposition 2.1. The next lemma is implicit in [12].

Lemma 2.2. *Let S be an Ore semigroup with enveloping group Γ, and define $\preceq_r$ on Γ by $g \preceq_r h$ if and only if $hg^{-1} \in S$. Then $\preceq_r$ is a right-invariant preorder which directs Γ, and for any $t \in S$, St is cofinal in S.*

Proof. It is routine to check that $\preceq_r$ is a right-invariant preorder. Theorem 1.1 and right-reversibility of S imply that $\preceq_r$ directs Γ. For cofinality, fix $s \in S$. We want $x \in St$ such that $s \preceq_r x$. Since S is right-reversible, there are $u, v \in S$ such that $ut = vs$. Then $s \preceq_r vs = ut \in St$, so $x = ut$ will do. □

Theorem 2.3. *Let $\alpha : S \to \operatorname{End} E$ be an action of an Ore semigroup on a directed graph. Then the family of directed graphs $\{E_t := E : t \in S\}$, together with the graph morphisms $\alpha^s_t : E_s \to E_t$ for $s \preceq_r t$ defined by $\alpha^s_t = \alpha_{ts^{-1}} : E = E_s \to E = E_t$, forms a direct system. Moreover, there is an action α^∞ of Γ on $E_\infty := \varinjlim E_x$ such that*

1. *if $s \in St$ then $\alpha^\infty_t \circ \alpha^s = \alpha^{st^{-1}}$ on $E_s = E = E_{st^{-1}}$; and*
2. *α^∞ dilates α in the sense that, for any $t \in S$, $\alpha^\infty_t \circ \alpha^1 = \alpha^1 \circ \alpha_t$ on $E = E_1$.*

Proof. The argument follows that of [12, Theorem 2.1.1].

We have $\alpha^s_t \circ \alpha^r_s = \alpha_{ts^{-1}} \circ \alpha_{sr^{-1}} = \alpha_{tr^{-1}} = \alpha^r_t$, so $(\{E_t\}, \{\alpha^s_t\})$ is a direct system. For each $t \in S$, consider the subsystem $\{E_s : s \in St\}$ with the same morphisms. Since St is cofinal in S, E_∞ is also the direct limit of the subsystem; we define graph morphisms $\beta^s : E_s \to E_\infty$ for $s \in St$ by $\beta^s = \alpha^{st^{-1}}$. For $a, b \in St$ with $a \preceq_r b$, we have

$$\begin{aligned}\beta^b \circ \alpha^a_b &= \alpha^{bt^{-1}} \circ \alpha_{ba^{-1}}\\ &= \alpha^{bt^{-1}} \circ \alpha_{bt^{-1}ta^{-1}}\\ &= \alpha^{bt^{-1}} \circ \alpha^{at^{-1}}_{bt^{-1}} = \alpha^{at^{-1}} = \beta^a.\end{aligned}$$

Thus by Proposition 2.1 there is a unique graph morphism $\alpha^\infty_t := \varinjlim \beta^s : E_\infty \to E_\infty$ such that $\alpha^\infty_t \circ \alpha^s = \beta^s$ for $s \in St$. Surjectivity and injectivity of α^∞_t follow from Proposition 2.1, so α^∞_t is an automorphism. We defined β^t to ensure that the action α^∞ satisfies (1). Since $\alpha^1 = \alpha^t \circ \alpha^1_t$ and $\beta^s = \alpha^{st^{-1}}$, choosing $s = t$ gives $\alpha^\infty_t \circ \alpha^1 = \beta^t \circ \alpha^1_t = \alpha^1 \circ \alpha_t$. □

When the graph E is a skew product and α is the action by left translation (or equivalently, when the action is free and has an S-transversal), the direct limit is itself a skew product and α^∞ is left translation. More formally, in the notation of Theorem 2.3:

Proposition 2.4. *Suppose $(E, S, \alpha) = (F \times_{c,d} S, S, \tau)$ is a skew product with left-translation action. Then there is an isomorphism $\psi : E_\infty \to F \times_{c,d} \Gamma$ such that $\psi \circ \tau^\infty = \sigma \circ \psi$, where σ is the action of Γ by left translation on $F \times_{c,d} \Gamma$.*

Proof. For each $t \in S$, define $\psi^t : E_t \to F \times_{c,d} \Gamma$ on edges and vertices by $\psi^t(u, s) = (u, t^{-1}s)$. For $t \in S$ and $(e, a) \in E^1$ we have

$$\psi^t(s(e, a)) = \psi^t(s(e), ad(e)) = (s(e), t^{-1}ad(e)) = s(e, t^{-1}a) = s(\psi^t(e, a)),$$

and similarly $\psi^t(r(e, a)) = r(\psi^t(e, a))$. Thus ψ^t is a graph morphism. One can check $\psi^t \circ \tau^s_t = \psi^s$, so Proposition 2.1 gives a unique graph morphism $\psi := \varinjlim \psi^t : E_\infty \to F \times_{c,d} \Gamma$ satisfying $\psi \circ \tau^t = \psi^t$ for all $t \in S$. The injectivity and surjectivity of ψ follow from Proposition 2.1, so ψ is an isomorphism of E_∞ onto $F \times_{c,d} \Gamma$.

To show equivariance of ψ, we want $\psi \circ \tau^\infty_g = \sigma_g \circ \psi$ for all $g \in \Gamma$. We will check this on the edge sets. Fix $t \in S$. Since $\bigcup_{s \in St} \tau^s(E^1_s) = E^1_\infty$, consider $(e, a) \in E^1 = E^1_s$ with $s \in St$. Then

$$\begin{aligned}\psi \circ \tau^\infty_t(\tau^s(e, a)) &= \psi \circ \tau^{st^{-1}}(e, a) \quad \text{by Theorem 2.3}\\ &= (e, ts^{-1}a)\\ &= \sigma_t \circ \psi^s(e, a)\\ &= \sigma_t \circ \psi(\tau^s(e, a)).\end{aligned}$$

Now for $g \in \Gamma$ we write $g = s^{-1}t$, and

$$\psi \circ \tau^\infty_g = \psi \circ (\tau^\infty_s)^{-1} \circ \psi^{-1} \circ \psi \circ \tau^\infty_t = \sigma_g \circ \psi,$$

as required. □

3. Graph Algebras and Direct Limits

A direct system of C^*-algebras consists of C^*-algebras $\{A_x\}_{x \in X}$ indexed by a directed set together with homomorphisms $\{\pi^x_y : A_x \to A_y : x \le y\}$ satisfying $\pi^y_z \circ \pi^x_y = \pi^x_z$ whenever $x \le y \le z$. We assume the linking maps π^x_y are injective to ensure that trivial cases do not arise. We let $\varinjlim A_x$ denote the direct limit C^*-algebra, and write π^x for the canonical embeddings $A_x \hookrightarrow \varinjlim A_x$. We denote by $\varinjlim \psi^x : A_x \to B$ the homomorphism induced by homomorphisms $\psi^x : A_x \to B$ satisfying $\psi^y \circ \pi^x_y = \psi^x$ for $x \le y$ (see [17, Appendix L] for details).

Lemma 3.1. *In the above notation:*

1. *if $\bigcup_{x \in X} \psi^x(A_x)$ is dense in B, then $\varinjlim \psi^x$ is surjective;*
2. *if each ψ^x is injective, then $\varinjlim \psi^x$ is injective.*

Proof. (1) The range of $\varinjlim \psi^x$ contains $\bigcup_{x \in X} \psi^x(A_x)$; since the range of any homomorphism is closed, it must be all of B.

(2) Each ψ^x is norm-preserving, and since $\varinjlim \psi^x \circ \pi^x = \psi^x$ it follows that $\varinjlim \psi^x$ is norm-preserving on the range of each π^x. Thus $\varinjlim \psi^x$ is norm-preserving on $\bigcup_{x \in X} \pi^x(A_x)$, and $\varinjlim \psi^x$ extends uniquely to an isometric homomorphism on $\varinjlim A_x$. □

We now recall some definitions from [11] and set up notation which will be used throughout the remainder of the paper. A directed graph E is *row-finite* if each vertex emits at most finitely many edges. A *Cuntz-Krieger E-family* consists of a set $\{p_v : v \in E^0\}$ of mutually orthogonal projections and a set $\{s_e : e \in E^1\}$ of partial isometries satisfying the *Cuntz-Krieger relations*

$$s^*_e s_e = p_{r(e)} \text{ for } e \in E^1, \quad \text{and} \quad p_v = \sum_{\{e : s(e) = v\}} s_e s^*_e \text{ for } v \in s(E^1).$$

It is proved in [11, Theorem 1.2] that there is a C^*-algebra $C^*(E)$ generated by a universal Cuntz-Krieger E-family $\{s_e, p_v\}$; moreover, we then have

$$C^*(E) = \overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu \text{ and } \nu \text{ are paths in } E \text{ with } r(\mu) = r(\nu)\},$$

where $s_\mu := s_{\mu_1} \dots s_{\mu_n}$ for every path μ of edges in E. There is a strongly continuous *gauge action* γ of $\mathbb{T}$ on $C^*(E)$ such that $\gamma_z(s_e) = z s_e$ and $\gamma_z(p_v) = p_v$.

Restricting a Cuntz-Krieger F-family to a subgraph E need not give a Cuntz-Krieger E-family: dropping a partial isometry s_e removes a term from the relation for $p_{s(e)}$. More generally, we need to impose conditions on a graph morphism $\pi : E \to F$ to ensure that it induces a homomorphism of $C^*(E)$ into $C^*(F)$.

Definition 3.2. A subgraph E of a directed graph F is *saturated* in F if, for each vertex $v \in E^0$, either all edges $e \in F^1$ with $s(e) = v$ lie in E^1 or none do; that is, if $s^{-1}(s(E^1)) = E^1$.

Proposition 3.3. *Let E and F be row-finite directed graphs and suppose $\pi : E \to F$ is an injective graph morphism such that $\pi(E)$ is saturated in F. Then there is an isomorphism π_* of $C^*(E)$ into $C^*(F)$ such that $\pi_*(s_e) = t_{\pi(e)}$ and $\pi_*(p_v) = q_{\pi(v)}$. The assignment $\pi \mapsto \pi_*$ is covariant in the sense that $\rho_* \circ \pi_* = (\rho \circ \pi)_*$ whenever possible.*

Proof. Let $C^*(E)$ be generated by the Cuntz-Krieger E-family $\{s_e, p_v\}$ and $C^*(F)$ by the Cuntz-Krieger F-family $\{t_f, q_w\}$. The hyothesis of saturation on $\pi(E)$ is precisely what is need to ensure that $\{t_{\pi(e)}, q_{\pi(v)} : e \in E^1, v \in E^0\}$ is a Cuntz-Krieger E-family. The universal property of $C^*(E)$ then gives a homomorphism $\pi_* : C^*(E) \to C^*(F)$ such that $\pi_*(s_e) = t_{\pi(e)}$ and $\pi_*(p_v) = q_{\pi(v)}$. The gauge action $\gamma^F : \mathbb{T} \to C^*(F)$ leaves $C^*(t_{\pi(e)}, q_{\pi(v)})$ invariant, so it restricts to an action on $C^*(t_{\pi(e)}, q_{\pi(v)})$, and $\gamma^F_z \circ \pi_* = \pi_* \circ \gamma^E_z$ for all $z \in \mathbb{T}$. Therefore [2, Theorem 2.1] implies that π_* is an isomorphism. One can check that $\rho_* \circ \pi_* = (\rho \circ \pi)_*$ on generators. □

Lemma 3.4. *Let $(\{E_x : x \in X\}, \{\pi^x_y\})$ be a direct system of directed graphs with injective linking maps. If $\pi^x_y(E_x)$ is saturated in E_y whenever $x \leq y$, then $\pi^y(E_y)$ is saturated in $\varinjlim E_x$ for all $y \in X$.*

Proof. Given $x \in X$ and $v \in \pi^x(E^0_x)$, suppose there are $e, f \in (\varinjlim E)^1$ with $s(e) = s(f) = v$, $e \neq f$, and $e = \pi^x(e') \in \pi^x(E_x)$. We know there exist $y \in X$ and $e'', f'' \in E^1_y$ such that $x \leq y$, $\pi^y(e'') = e$ and $\pi^y(f'') = f$. But then $\pi^x_y(e') = e''$ by injectivity of π^y, so there exists an $f' \in E^1_x$ such that $\pi^x_y(f') = f''$ since π^x_y is saturated in E_y. Then $\pi^x(f') = f$, and $\pi^x(E_x)$ is saturated in $\varinjlim E_x$. □

Theorem 3.5. *Let $(\{E_x : x \in X\}, \{\pi^x_y\})$ be a direct system of row-finite directed graphs with injective linking maps, and suppose that $\pi^x_y(E_x)$ is saturated in E_y for every $x \leq y$. Then $(\{C^*(E_x)\}, \{(\pi^x_y)_*\})$ is a direct system of C^*-algebras, and $\varinjlim (\pi^x)_*$ is an isomorphism of $\varinjlim C^*(E_x)$ onto $C^*(\varinjlim E_x)$.*

Proof. The homomorphisms $(\pi^x_y)_* : C^*(E_x) \to C^*(E_y)$ exist by Proposition 3.3. Suppose $x, y, z \in X$ satisfy $x \leq y$ and $y \leq z$. Then $(\pi^y_z)_* \circ (\pi^x_y)_* = (\pi^y_z \circ \pi^x_y)_* = (\pi^x_z)_*$ by Proposition 3.3, so $(\{C^*(E_x)\}, \{(\pi^x_y)_*\})$ forms a direct system of C^*-algebras. By Lemma 3.4, the graph morphisms π^x induce injective homomorphisms $(\pi^x)_* : C^*(E_x) \to C^*(\varinjlim E_x)$. Proposition 3.3 implies that $(\pi^y)_* \circ (\pi^x_y)_* = (\pi^y \circ \pi^x_y)_* = (\pi^x)_*$, so for each $x \in X$ we have an injective homomorphism $\varinjlim (\pi^x)_* : \varinjlim C^*(E_x) \to C^*(\varinjlim E_x)$. Since $\bigcup_{x \in X} \pi^x(E^i_x) = (\varinjlim E_x)^i$,

$$\bigcup_{x \in X} \{(\pi^x)_*(s_e), (\pi^x)_*(p_v) : e \in E^1_x, v \in E^0_x\} = \{t_f, q_w : f \in (\varinjlim E_x)^1, w \in (\varinjlim E_x)^0\};$$

thus $\bigcup_{x\in X}\{(\pi^x)_*(C^*(E_x))\}$ is dense in $C^*(\varinjlim E_x)$, and Lemma 3.1 implies that $\varinjlim(\pi^x)_*$ is surjective. □

We shall now see that the existence of a fundamental domain guarantees that all graph morphisms in sight are saturated. (The example in Figure 2 shows that the existence of transversals would not suffice.)

Lemma 3.6. *If a free action $\alpha : S \to \operatorname{End} E$ of a semigroup on a directed graph has a fundamental domain, then $\alpha_t(E)$ is saturated in E for $t \in S$.*

Proof. Given $t \in S$, let $e \in E^1$ and $v \in E^0$ satisfy $s(e) = v$. Suppose $f \in E^1$ satisfies $s(f) = \alpha_t v$; we want to show f is in the range of α_t. Since α has a fundamental domain F, there exists a unique $f' \in F^1$ such that $s(f') \in F$ and $\alpha_a f' = f$ for some $a \in S$, and then $\alpha_a s(f') = \alpha_t v$. We also know there is a unique $v' \in F^0$ such that $\alpha_b v' = v$ for some $b \in S$, thus $\alpha_a s(f') = \alpha_t\alpha_b v' = \alpha_{tb} v'$. Since F is a fundamental domain, we then have $v' = s(f')$ and, since the action is free, $a = tb$. Therefore $f = \alpha_{tb} f' = \alpha_t(\alpha_b f')$, as required. □

Proposition 3.7. *Let $\alpha : S \to \operatorname{End} E$ be a free action of a semigroup on a row-finite directed graph and suppose α has a fundamental domain. Then $\alpha_* : t \mapsto (\alpha_*)_t := (\alpha_t)_*$ is an action of S by injective endomorphisms of $C^*(E)$.*

Proof. Lemma 3.6 says that $\alpha_t(E)$ is saturated in E for every $t \in S$, so Proposition 3.3 says that there is an injective homomorphism $(\alpha_t)_*$. The covariance of $\alpha_t \mapsto (\alpha_t)_*$ ensures that α_* is a homomorphism on S. □

If $\beta : S \to \operatorname{End} A$ is an action of an Ore semigroup on a *unital* C^*-algebra A, it is shown in the proof of [12, Theorem 2.1.1] that $(\{A_t := A : t \in S\}, \{\beta^s_t := \beta_{ts^{-1}}\})$ is a direct system of C^*-algebras, and that there is an action β^∞ of Γ on the direct limit A_∞ such that

$$\beta^\infty_t(\beta^s(b)) = \beta^{st^{-1}} \text{ when } s \in St \text{ and } b \in A = A_t = A_{st^{-1}}, \text{ and} \tag{3.1}$$
$$\beta^\infty_t \circ \beta^1 = \beta^1 \circ \beta_t.$$

Laca's arguments also work for non-unital C^*-algebras: just use [17, Theorem L.2.1] instead of [8, Proposition 11.4.1]. We can therefore apply his construction to the action α_* of Proposition 3.7, to obtain an action $(\alpha_*)^\infty$ of Γ on the direct limit $C^*(E)_\infty$ of the system $(\{C^*(E)_t := C^*(E)\}, \{(\alpha_*)_{ts^{-1}}\})$. Fortunately, the isomorphism $C^*(E)_\infty \cong C^*(E_\infty)$ from Theorem 3.5 carries this into $(\alpha^\infty)_*$:

Proposition 3.8. *Let S be an Ore semigroup and $\alpha : S \to \operatorname{End} E$ a free action on a row-finite directed graph E with a fundamental domain. Then the isomorphism $\varinjlim(\alpha^t)_*$ of $C^*(E)_\infty$ onto $C^*(E_\infty)$ satisfies*

$$(\alpha_*)^\infty_s := (\varinjlim(\alpha^t)_*)^{-1} \circ (\alpha^\infty_s)_* \circ \varinjlim(\alpha^t)_*. \tag{3.2}$$

Proof. Since Ss is cofinal and both sides of (3.2) are continuous, it is enough to check (3.2) on elements of the form $(\alpha_*)^r(b)$ for $b \in C^*(E)_r = C^*(E)$ and $r \in Ss$. On the one hand, we have

$$(\alpha^\infty_s)_* \circ \varinjlim(\alpha^t)_*((\alpha_*)^r(b)) = (\alpha^\infty_s)_*((\alpha^r)_*(b)) = (\alpha^\infty_s \circ \alpha^r)_*(b) = (\alpha^{rs^{-1}})_*(b),$$

using Proposition 2.1(1); on the other hand,

$$\varinjlim(\alpha^t)_* \circ (\alpha_*)^\infty_s((\alpha_*)^r(b)) = \varinjlim(\alpha^t)_*((\alpha_*)^{rs^{-1}}(b)) = (\alpha^{rs^{-1}})_*(b),$$

using (3.1). □

Applying this proposition to the action τ by left translation on a skew product, and composing the resulting isomorphism $\varinjlim(\tau^t)_*$ with the isomorphism ψ_* induced by the isomorphism $\psi : (E \times_{c,1} S)_\infty \to E \times_{c,1} \Gamma$ of Proposition 2.4, we obtain:

Corollary 3.9. *Suppose S is an Ore semigroup, E a row-finite directed graph and $c : E^1 \to S$ a function. Then*

$$(C^*(E \times_{c,1} S)_\infty, \Gamma, (\tau_*)^\infty) \cong (C^*(E \times_{c,1} \Gamma), \Gamma, \sigma_*),$$

where σ is the action by left translation on $E \times_{c,1} \Gamma$.

4. Semigroup Dynamical Systems and Crossed Products

A homomorphism ϕ from a C^*-algebra A to a multiplier algebra $M(B)$ is *extendible* if there is an approximate identity $\{a_j\}$ for A and a projection p_ϕ in $M(B)$ such that $\phi(a_j) \to p_\phi$ strictly in $M(B)$. We say an action $\alpha : S \to \operatorname{End} A$ of a semigroup on a C^*-algebra is extendible if each α_t is extendible. The following basic proposition is proved in [1, Proposition 3.1.1] and [7, Proposition 1.1.13].

Proposition 4.1. *A homomorphism $\phi : A \to M(B)$ is extendible if and only if there is a strictly continuous homomorphism $\bar{\phi}$ of $M(A)$ into $M(B)$ such that $\bar{\phi}|_A = \phi$.*

Lemma 4.2. *Let $\phi : A \to M(B)$ be a homomorphism. Suppose there is a dense *-subalgebra B_0 of B and an approximate identity $\{a_j\}_{j\in\Lambda}$ in A such that $\{\phi(a_j)b\}_{j\in\Lambda}$ is Cauchy for every $b \in B_0$. Then ϕ is extendible.*

Proof. An $\varepsilon/3$ argument shows that $\{\phi(a_j)b\}$ is Cauchy for all $b \in B$. We define maps $L_p, R_p : B \to B$ by $L_p(b) = \lim_j \phi(a_j)b$ and $R_p(b) = \lim_j b\phi(a_j)$ (which exists because $\lim_j \phi(a_j)b^* = \lim_j(b\phi(a_j))^*$ exists). For $b, c \in B$ we have

$$R_p(b)c = \lim_j b\phi(a_j)c = b \lim_j \phi(a_j)c = bL_p(c),$$

so (L_p, R_p) is a double centraliser of B. Thus there is a multiplier $p \in M(B)$ such that $pb = L_p(b)$ and $bp = R_p(b)$. By definition $\phi(a_j) \to p$ strictly, and since each a_j is self-adjoint so is p. For any $a \in A$ we have

$$\phi(a)pb = \lim_j \phi(aa_j)b = \phi(\lim_j aa_j)b = \phi(a)b$$

so

$$\|p^2b - pb\| \leq \|p(pb) - \phi(a_j)(pb)\| + \|\phi(a_j)pb - pb\| \to 0,$$

and $p^2 = p$. Thus p is a projection. □

Lemma 4.3. *Let $(\{A_x : x \in X\}, \{\pi_y^x\})$ be a direct system of C^*-algebras. If all the linking maps are extendible, then each $\pi^x : A_x \to \varinjlim A_x$ is extendible.*

Proof. Given $x \in X$ let $\{a_j\}_{j\in\Lambda}$ be an approximate identity in A_x. By Lemma 4.2 it suffices to show that $\{\pi^x(a_j)b\}_{j\in\Lambda}$ is Cauchy for each $b \in B_0 := \bigcup_{x\in X} \pi^x(A_x)$. We have $b = \pi^y(c)$ for some $y \in X$ and $c \in A_y$, and we can choose $z \in X$ such that $x \leq z$ and $y \leq z$. Then

$$\pi^x(a_j)b = (\pi^z \circ \pi_z^x(a_j))(\pi^z \circ \pi_z^y(c)) = \pi^z(\pi_z^x(a_j)\pi_z^y(c)),$$

which converges because π_z^x is extendible. □

Proposition 4.4. *Suppose $\pi : E \to F$ is an injective graph morphism between row-finite graphs such that $\pi(E)$ is saturated in F. Then $\pi_* : C^*(E) \to C^*(F)$ is extendible. In particular, if α is a free action of an Ore semigroup on E and α has a fundamental domain, then α_* is an extendible action.*

Proof. List $E^0 = \{v_1, v_2, \dots\}$ and let $p_n = \sum_{i=1}^n p_{v_i}$. Then $\{p_n\}$ is an approximate identity for $C^*(E)$. For any spanning element $s_\mu s_\nu^*$ for $C^*(F)$, we have

$$\pi_*(p_n) s_\mu s_\nu^* = \sum_{i=1}^n p_{\pi(v_i)} s_\mu s_\nu^* = \begin{cases} s_\mu s_\nu^* & \text{if } s(\mu) = \pi(v_i) \text{ for some } i \le n, \\ 0 & \text{otherwise.} \end{cases}$$

Now a standard argument (see [2, Lemma 1.1]) shows that $\pi_*(p_n)$ converges strictly to a projection $p \in M(C^*(F))$ such that

$$p s_\mu s_\nu^* = \begin{cases} s_\mu s_\nu^* & \text{if } s(\mu) \in \pi(E^0), \\ 0 & \text{otherwise.} \end{cases}$$

Thus π_* is extendible. The last statement follows from Lemma 3.6. □

Suppose α is an action of a semigroup S by endomorphisms of a C^*-algebra A. A *covariant representation* of (A, S, α) on a Hilbert space $\mathcal{H}$ is a pair (π, V) consisting of a nondegenerate representation $\pi : A \to \mathcal{B}(\mathcal{H})$ and an isometric representation $V : S \to \operatorname{Isom}(\mathcal{H})$ satisfying $\pi(\alpha_t(a)) = V_t \pi(a) V_t^*$ for $a \in A, t \in S$. A *crossed product* for (A, S, α) is a triple (B, i_A, i_S) where $i_A : A \to B$ is a nondegenerate homomorphism and $i_S : S \to \operatorname{Isom}(M(B))$ is a semigroup homomorphism, satisfying:

(1) $i_A(\alpha_t(a)) = i_S(t) i_A(a) i_S(t)^*$ for $a \in A$ and $t \in S$;
(2) for every covariant representation (π, V) of (A, S, α) there is a nondegenerate representation $\pi \times V$ of B with $(\pi \times V) \circ i_A = \pi$ and $\overline{(\pi \times V)} \circ i_S = V$; and
(3) B is generated as a C^*-algebra by $\{i_A(a) i_S(t) : a \in A, t \in S\}$.

If S is Ore and α is extendible, and if there is a nonzero covariant representation, then [13, Proposition 1.4] says there is a crossed product $(A \times_\alpha S, i_A, i_S)$, and that it is unique up to isomorphism.

The following mild generalisation of [12, Theorem 2.2.1] will be used in the proof of our main theorem.

Theorem 4.5. *Suppose (A, S, α) is an semigroup dynamical system with S Ore and α extendible. Then $A \times_\alpha S$ is canonically isomorphic to $\overline{i_{A_\infty} \circ \alpha^1}(1)(A_\infty \times_{\alpha^\infty} \Gamma)\overline{i_{A_\infty} \circ \alpha^1}(1)$.*

Proof. The proof follows that of [12, Theorem 2.2.1] with one exception. Laca requires that the C^*-algebra A be unital, and then the projection p is $i_{A_\infty} \circ \alpha^1(1)$. Since A does not have an identity, we need to extend i_{A_∞} and α^1 to multiplier algebras before defining p, but this can be done because both homomorphisms are extendible: i_{A_∞} because it is nondegenerate, and α^1 by Lemma 4.3. We can then replace Laca's projection with $\overline{i_{A_\infty} \circ \alpha^1}(1)$ throughout his proof without problem: the main tools, namely Theorem 2.1.1 and Lemma 2.1.3 of [12], hold for nonunital A. □

Remark 4.6. Since the endomorphisms α_t are injective, so is α^1. Thus the embedding of (A, S, α) in $\overline{i_{A_\infty} \circ \alpha^1}(1)(A_\infty \times_{\alpha^\infty} \Gamma)\overline{i_{A_\infty} \circ \alpha^1}(1)$ gives a nonzero covariant representation (i, v) of (A, S, α) with i faithful. Indeed, the above argument proves that the system has a crossed product, namely $\overline{i_{A_\infty} \circ \alpha^1}(1)(A_\infty \times_{\alpha^\infty} \Gamma)\overline{i_{A_\infty} \circ \alpha^1}(1)$, thus improving [13, Proposition 1.4].

5. Proof of the Main Theorem

We are now ready to prove our main theorem. So suppose α is a free action of an Ore semigroup S on a row-finite directed graph E, and that α has a fundamental domain. The next lemma follows from an $\varepsilon/3$ argument:

Lemma 5.1. *Let A be a C^*-algebra and $\{m_j\}$ be a norm-bounded sequence in $M(A)$. Suppose there is a dense *-subalgebra B such that $\{m_j b\}$ and $\{bm_j\}$ are Cauchy for each $b \in B$. Then $\{m_j\}$ converges strictly in $M(A)$.*

The proof of the theorem involves a string of isomorphisms. First we use Theorem 1.8 to get

$$C^*(E) \times_{\alpha_*} S \cong C^*(E/S \times_{c,1} S) \times_{\tau_*} S.$$

Proposition 4.4 implies that τ is extendible, so Theorem 4.5 gives

$$C^*(E/S \times_{c,1} S) \times_{\tau_*} S \cong p_1(C^*(E/S \times_{c,1} S)_\infty \times_{(\tau_*)^\infty} \Gamma)p_1,$$

where $p_1 = \overline{i_{C^*(E/S\times S)_\infty} \circ (\tau_*)^1}(1)$.

We know from Corollary 3.9 that there is an isomorphism ψ_* of $(C^*(E/S \times_{c,1} S)_\infty, \Gamma, (\tau_*)^\infty)$ onto $(C^*(E/S \times_{c,1} \Gamma), \Gamma, \sigma_*)$, so we have an isomorphism ϕ between the corresponding corners. Moreover,

$$\begin{aligned}\overline{\phi \circ i_{C^*(E/S\times S)_\infty} \circ (\tau_*)^1}(1) &= \overline{i_{C^*(E/S\times\Gamma)} \circ \psi_* \circ (\tau_*)^1}(1)\\ &= \overline{i_{C^*(E/S\times\Gamma)} \circ (\tau^1)_*}(1),\end{aligned}$$

and hence

$$p_1(C^*(E/S \times_{c,1} S)_\infty \times_{(\tau_*)^\infty} \Gamma)p_1 \cong p_2(C^*(E/S \times_{c,1} \Gamma) \times_{\sigma_*} \Gamma)p_2,$$

where $p_2 = \overline{i_{C^*(E/S\times\Gamma)} \circ (\tau^1)_*}(1)$.

We are now going to show that [9, Theorem 3.1] extends to give an isomorphism Θ of $C^*(E/S \times_{c,1} \Gamma) \times_{\sigma_*} \Gamma$ onto $C^*(E/S) \otimes \mathcal{K}(\ell^2(\Gamma))$. As our action σ_* differs from the action in Theorem 3.1 of [9], we construct a different Cuntz-Krieger $E/S \times_{c,1} \Gamma$-family $\{t_{(e,g)}, q_{(v,g)}\}$ in $C^*(E/S) \otimes \mathcal{K}(\ell^2(\Gamma))$ and group homomorphism $V : \Gamma \to UM(C^*(E/S) \otimes \mathcal{K}(\ell^2(\Gamma)))$ by setting

$$t_{(e,g)} = s_e \otimes \chi_g \rho_{c(e)}, \quad q_{(v,g)} = p_v \otimes \chi_g \quad \text{and} \quad V_g = 1 \otimes \lambda_g,$$

where $\{s_e, p_v\}$ denote the canonical generators of $C^*(E/S)$ and λ and ρ are the left and right regular representations, respectively. The proof of [9, Theorem 3.1] then gives an isomorphism Θ such that

$$\Theta(s_{(e,g)}) = t_{(e,g)}, \quad \Theta(p_{(v,g)}) = q_{(v,g)} \quad \text{and} \quad \Theta(U_g) = V_g$$

where $\{s_{(e,g)}, p_{(v,g)}\}$ is the canonical Cuntz-Krieger $(E/S \times_{c,1} \Gamma)$-family and U is the canonical homomorphism of Γ into $UM(C^*(E/S \times_{c,1} \Gamma) \times_{\sigma_*} \Gamma)$ satisfying

$$U_h s_{(e,g)} = s_{(e,hg)} U_h \quad \text{and} \quad U_h p_{(v,g)} = p_{(v,hg)} U_h \quad \text{for} \quad g \in \Gamma.$$

We claim that $\bar{\Theta}(p_2) = 1_{M(C^*(E/S))} \otimes \chi_S$. Since p_2 is the image of $1 \in M(C^*(E/S \times_{c,1} S))$, we compute $\bar{\Theta}(p_2)$ by looking at the image of the approximate identity in $C^*(E/S \times_{c,1} S)$ arising from the partial sums of $\sum_{(v,t)\in(E/S\times S)^0} p_{(v,t)}$. Extendibility of $(\tau^1)_*$ and nondegeneracy of $i_{C^*(E/S\times\Gamma)}$ and Θ imply that

$$\sum_{v\in(E/S)^0,\, t\in S} p_v \otimes \chi_t$$

converges strictly to $\bar{\Theta}(p_2)$, so it suffices to show that this sum converges strictly to the element $1_{M(C^*(E/S))} \otimes \chi_S$. By Lemma 5.1, and because each $p_v \otimes \chi_t$ is self-adjoint, it suffices to show that

$$\Big(\sum_{v\in(E/S)^0,\, t\in S} p_v \otimes \chi_t\Big) b$$

converges to $(1_{M(C^*(E/S))} \otimes \chi_S)b$ for all b in a dense *-subalgebra B of $C^*(E/S) \otimes \mathcal{K}(\ell^2(\Gamma))$. Operators of the form $s_\mu s_\nu^* \otimes \chi_g \rho_h$ for μ, ν paths in (E/S) and $g, h \in \Gamma$, span a dense *-subalgebra B of $C^*(E/S) \otimes \mathcal{K}(\ell^2(\Gamma))$, and for such a generator

$$p_v s_\mu s_\nu^* = \begin{cases} s_\mu s_\nu^* & \text{if } s(\mu) = v \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\chi_t \chi_g = \begin{cases} \chi_g & \text{if } t = g \\ 0 & \text{otherwise.} \end{cases}$$

Thus if $g \in S$, then for any finite subset H of S containing g and any finite subset F of $(E/S)^0$ containing $s(\mu)$, we have

$$\Big(\sum_{v \in F,\, t \in H} p_v \otimes \chi_t \Big)(s_\mu s_\nu^* \otimes \chi_g \rho_h) = s_\mu s_\nu^* \otimes \chi_g \rho_h,$$

and if $g \notin S$, then we always get 0. Hence it follows from Lemma 5.1 that

$$\sum_{v \in (E/S)^0,\, t \in S} p_v \otimes \chi_t$$

converges strictly to the multiplier $1_{M(C^*(E/S))} \otimes \chi_S$, as claimed.

We now have

$$p_2(C^*(E/S \times_{c,1} \Gamma) \times_{\sigma_*} \Gamma)p_2 \cong p_3(C^*(E/S) \otimes \mathcal{K}(\ell^2(\Gamma)))p_3$$

where $p_3 = 1_{M(C^*(E/S))} \otimes \chi_S$. But $\chi_S(\ell^2(\Gamma)) = \ell^2(S)$, so

$$p_3(C^*(E/S) \otimes \mathcal{K}(\ell^2(\Gamma)))p_3 \cong C^*(E/S) \otimes \mathcal{K}(\ell^2(S)).$$

This completes the proof of the Theorem.

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