Diploma in Information Technology

Final Examination
Spring Session 2008

WUCT121
Discrete Mathematics

This exam represents 60% of the total subject marks

Reading Time: 5 minutes
Time allowed: 3 Hours

DIRECTIONS TO CANDIDATES
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DIRECTIONS TO CANDIDATES

1. Total number of questions: 6.
2. All questions are to be attempted.
3. All answers must be written in the Examination Answer Booklet supplied.
4. Questions are of equal value.
5. All working must be clearly shown.
6. Non-programmable calculators may be used.
7. No other examination aids are permitted.
8. Electronic dictionaries are not allowed.
9. Non-English speaking background students may use an approved English to Foreign language translation dictionary
10. The examination paper must not be removed from the examination room and must be submitted with the examination answer booklet.
11. During the 5 minutes reading time, you may make notes on the examination paper only. Answer Booklets must remain closed.
Question 1 (50 marks)

(a) Using the ‘quick method’, prove that the following statement is a tautology:

\[(\neg p \land \neg q) \Rightarrow (p \lor q)\]

[6 marks]

(b) Using full truth tables, determine whether the following statement is a tautology, a contradiction, or is contingent:

\[\sim ((q \lor \sim p) \land (p \lor \sim q)) \iff (\sim p \iff q)\]

[12 marks]

(c) Write the following statement in predicate calculus notation using quantifiers and variables:

Some students cannot correctly answer all questions in this exam.

[8 marks]

(d) Write in simple English without using quantifiers or variables:

\[\forall x \in \mathbb{R}^+, \exists y \in \mathbb{R}^+, (y < x), \text{ where } \mathbb{R}^+ \text{ is the set of positive Real numbers.}\]

[4 marks]

(e) Write down, then simplify, the negations of each of the statements in parts (c) and (d) using:

(i) Predicate calculus notation using quantifiers and variables.

[6 marks]

(ii) Simple English without using quantifiers and variables.

[6 marks]

(f) For the following sentence:

‘The statement that all statements are sentences
but not all sentences are statements is false.’

(i) Determine if it is a statement; and explain why or why not.

[4 marks]

(ii) If it is a statement, determine whether it is true or false, giving reasons for your answer.

[4 marks]
**Question 2**

(a) Using a direct proof, prove the following statement:

“For any real number \( x \), \( 20(x - 1) \leq 4x^2 + 5 \)”

(b) Briefly explain how the following tautologies may be used in the method of proof by contradiction:

(i) \((\sim p \Rightarrow (q \land \sim q)) \Rightarrow p\)

(ii) \((p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)\)

(c) Using proof by contradiction or otherwise, prove the following statements:

(i) “For all integers \( n \), if \( n^3 \) is even then \( n \) is even”.

(ii) “There is no largest even integer”.

(d) Consider the following statements:

\[
\begin{align*}
P : & \quad 2x + 1 = 3 \\
Q : & \quad 2x = 2 \\
R : & \quad x = 1
\end{align*}
\]

If we assume that \( P \) is a true statement, explain how these statements, together with Modus Ponens and the Law of Syllogism, can be used to prove that \( R \) is also a true statement.

(e) Complete the following tautology, which is used in the method of proof by cases:

\[ \ldots \Leftrightarrow ((p \Rightarrow r) \ldots (q \Rightarrow r)) \]

(f) Using proof by cases or otherwise, prove the following statement:

“For all integers \( n \), \( 2n^2 - 1 \) is odd.”
Question 3

(a) Use Mathematical Induction to prove that $5^n - 1$ is divisible by 4 for all $n \in \mathbb{N}$. [6 marks]

(b) 

(i) Use the Euclidean Algorithm to find $\gcd(126, -39)$. [4 marks]

(ii) Find $m, n \in \mathbb{Z}$ such that $126m - 39n = \gcd(126, -39)$. [6 marks]

(c) 

(i) **Showing all steps clearly**, use the Sieve of Eratosthenes to find all the prime numbers between 2 and 52. [8 marks]

(ii) Write down all the twin primes between 2 and 52. [2 marks]

(d) Use the Generalized Pigeonhole Principle to justify your answer to the following question:

In a group of 2000 students, must at least five have the same day of the year as their birthday? [4 marks]

(e) Find a value of $x$ such that $x = 5^7 \pmod{17}$ and $0 \leq x < 17$. [4 marks]

(f) Write down the complete set of residue classes modulo 11, $\mathbb{Z}_{11}$. [3 marks]

(g) Express each of the elements $[-3], [7], [44] \in \mathbb{Z}_{11}$ as $[x] \in \mathbb{Z}_{11}$, $0 \leq x < 11$. [3 marks]

(h) Let $S \subseteq \mathbb{Z}_{11}$ be given by $S = \{[1], [4], [7]\}$.

(i) Construct the multiplication table for $S$. [6 marks]

(ii) Using this table, or otherwise, determine the identity for multiplication and the multiplicative inverses of all the elements in $S$ which have them. [4 marks]
Question 4 (50 marks)

(a) Let $\mathbb{U} = \{d, i, s, r, e, t, e, m, a, t, h, e, m, a, t, i, c, s\}$ be the Universal set. Let $S = \{x \in \mathbb{U} \mid x \in \{s, e, c, r, e, t, s\}\}$, $T = \{x \in \mathbb{U} \mid x \in \{t, h, e, m, e, s\}\}$ and $C = \{x \in \mathbb{U} \mid x \in \{t, a, c, t, i, c, s\}\}$ be subsets of the Universal set.

(i) Draw a Venn diagram showing $S$, $T$, $C$ and $\mathbb{U}$. [6 marks]

(ii) Write down the following sets:

   a) $S \cup T$
   c) $S - C$

   b) $T - S$
   d) $C \cap T$ [8 marks]

(iii) Which of the following are false? **Give reasons.**

   a) $c \subseteq C$
   c) $\emptyset \in \mathcal{P}(S)$

   b) $\{c\} \subseteq C$
   d) $\emptyset \subseteq \mathcal{P}(S)$ [8 marks]

(b) Briefly explain what is meant by the term Singleton set. [2 marks]

(c) Let $A = \{1\}$ and $B = \{4, 3, 2\}$. Write down the following sets:

   (i) $(A \times B)$ [4 marks]

   (ii) $\mathcal{P}(A \times B)$ [6 marks]

(d) State the Axiom of Specification and the Axiom of Extent. [4 marks]

(e) Using (d), and the definitions of Intersection, Difference, and Complement, prove the following statement using a **typical element argument**:

   $X - Y = X \cap \overline{Y}$ [12 marks]
Question 5  (50 marks)

(a) Explain what is meant by an ordered pair. [3 marks]

(b) Explain what is meant by a binary relation. [3 marks]

(c) Explain what is meant by a function from $A$ to $B$. [3 marks]

(d) Explain what is meant by a permutation. [3 marks]

(e) Let $T$ be a relation on $\{1, 2, 3, 4, 5\}$ defined as follows:

$$T = \{(x, y) | x > y \land x + y \text{ is odd and prime}\}.$$

(i) Write down the domain and range of $T$. [6 marks]

(ii) Write down the elements and sketch the graph of $T$. [6 marks]

(iii) Write down the elements and sketch the graph of $T^{-1}$. [6 marks]

(iv) Are either $T$ or $T^{-1}$ functions? Give a brief explanation why or why not. [6 marks]

(v) Are either $T$ or $T^{-1}$ onto $\{1, 2, 3, 4, 5\}$? Give a brief explanation why or why not. [6 marks]

(f) Simplify the following permutations:

(i) $(1\ 2) \cdot (4\ 3) \cdot (2\ 5)$ [4 marks]

(ii) $((1\ 3\ 5)^{-1} \cdot (4\ 3\ 5))^{-1}$ [4 marks]
**Question 6** (50 marks)

(a) Explain what is meant by the following terms:

(i) A graph [2 marks]

(ii) A simple graph [2 marks]

(iii) A simple path [2 marks]

(b) 

(i) Draw a graph that is simple and connected. [2 marks]

(ii) Draw a graph that is connected but not simple. [3 marks]

(c) Draw a graph with the specified properties or explain why no such graph exists:

(i) Graph with three vertices of degrees 1, 2, and 3 respectively. [4 marks]

(ii) Graph with four vertices of degrees 1, 2, 3, and 4 respectively. [4 marks]

(iii) Graph with five vertices of degrees 0, 1, 1, 1, and 5 respectively. [4 marks]

(d) Let $G = \{V, E\}$ be the graph given below:

![Graph](image)

(i) Draw the subgraph $H_1$ or explain why no such subgraph exists.

Subgraph $H_1 = \{\{v_1, v_2, v_4\}, \{e_3, e_5, e_6\}\}$. [3 marks]

(ii) Starting at $v_1$, write down any circuit of graph $G$. [3 marks]

(iii) Is graph $G$ an Eulerian graph? Explain. [3 marks]

(iv) Does an Eulerian path exist in $G$? Explain. [3 marks]
(e) For the graph $G$ given in question (d) above, either write down a path with the specified properties, or explain why no such path exists.

(i) A simple path of length 3 from $v_1$ to $v_2$. [2 marks]

(ii) A path of length 4 from $v_1$ to $v_3$. [2 marks]

(f) Is the graph $G$ given in question (d) above connected? If so, what is the least number of edges you must remove to make the graph disconnected? Give an example. [3 marks]

(g) If the edges in graph $G$ have the following weightings, use Kruskal’s Algorithm to find a minimum spanning tree for $G$:

$e_1 = 4, e_2 = 7, e_3 = 9, e_4 = 2, e_5 = 6, e_6 = 3, e_7 = 1$. [4 marks]

(h) Label the vertices and edges of the following graphs in any convenient way and show whether or not they are isomorphic.

[4 marks]

END OF EXAM