Discrete Mathematics

Graphs

Tutorial Exercises

Solutions
**Graphs**

**Question 1** Let $G = \{V, E\}$ be a graph. Draw a graph with the following specified properties, or explain why no such graph exists.

(a) A graph having $V = \{v_1, v_2, v_3, v_4, v_5\}$; $E = \{e_1, e_2, e_3, e_4, e_5\}$, where 
   
   $e_1 = (v_1, v_3)$, $e_2 = (v_2, v_4)$, $e_3 = (v_1, v_4)$, $e_4 = (v_3, v_3)$ and $e_5 = (v_3, v_5)$.

   How many components does $G$ have?

   ![Graph with 5 vertices and 5 edges](attachment:graph.png)

   $G$ has one component.

(b) A connected graph having $V = \{v_1, v_2, v_3, v_4, v_5\}$, and $\forall v_i \in V$, $\delta(v_i) = 2$.

   How many edges does $G$ have?

   ![Graph with 5 vertices and 5 edges](attachment:graph.png)

   $G$ has 5 edges
Question 2  Either draw a graph with the following specified properties, or explain why no such graph exists:

(a) A graph with four vertices having the degrees of its vertices 1, 2, 3 and 4.

(b) A simple graph with five vertices with degrees 2, 3, 3, 3, and 5.

It is impossible to draw this graph. A simple graph has no parallel edges nor any loops. There are only 5 vertices, so each vertex can only be joined to at most four other vertices, so the maximum degree of any vertex would be 4. Hence, you can’t have a vertex of degree 5.

(c) A simple graph in which each vertex has degree 3 and which has exactly 6 edges.
(d) A graph with four vertices having the degrees of its vertices 1, 1, 2 and 2.

(e) A graph with four vertices having the degrees of its vertices 1, 1, 2 and 6.

(f) A graph with five edges having the degrees of its vertices 1, 1, 3 and 3.
Not possible, number of edges is 5, thus the sum of the degrees of the vertices
must be $2 \times 5 = 10 \sum_{i=1}^{4} \delta(v_i) = 1 + 1 + 3 + 3 = 8 \neq 2 \times 5$
Question 3  Let $G = \{V, E\}$ be the graph drawn below.

![Graph Image]

(a) Draw a sub-graph $H = \{V_H, E_H\}$ of $G$ for which $\sum_{v \in V_H} \delta(v) = 4$.

2 possibilities are:

1. ![Sub-graph 1 Image]

There are other possibilities.

(b) Draw a sub-graph $H = \{V_H, E_H\}$ of $G$ having 2 components, one vertex of even degree and one pair of vertices of odd degree.

2 possibilities are:

1. ![Sub-graph 2 Image]
There are other possibilities.

(c) Either write down a path with the specified properties, or explain why no such path exists.

(i) A path of length 5 from $v_1$ to $v_2$.
   One possible: $v_1, e_1, v_1, e_2, v_1, e_1, v_1, e_5, v_2$

(ii) A path of length 3 from $v_1$ to $v_3$.
    One possible: $v_1, e_2, v_1, e_5, v_2, e_3, v_3$

**Question 4**

(a) Draw all spanning trees for $K_3$.  
(b) Draw 8 different spanning trees for $K_4$. It can be shown that there are $n^{n-2}$ spanning trees for $K_n$. Can you write down 16 spanning trees for $K_4$?

![Diagram of $K_4$ and 8 spanning trees]
(c) Use Kruskal’s algorithm to find the minimum spanning tree for the following weighted graph. Write down the number of steps you need to complete the algorithm.

Minimum spanning tree is

The algorithm can be completed in 3 steps.