WUCT121

Discrete Mathematics

Numbers

1. Natural Numbers
2. Integers and Real Numbers
3. The Principle of Mathematical Induction
4. Elementary Number Theory
5. Congruence Arithmetic
Section 1. Natural Numbers

1.1. Introduction.

The Natural Numbers (also called counting numbers) consist of all positive “whole” numbers. It is given by the set \{1, 2, 3, \ldots\}, and is denoted by \(\mathbb{N}\).

In some areas of Mathematics, 0 is included giving the set \(\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}\). Here the use of the subscript “0” indicates the inclusion of 0 in the set.

The former is generally used in number theory, while the latter is preferred in mathematical logic, set theory, and computer science.

In this course we will use \(\mathbb{N}\) unless otherwise specified.

Natural numbers have two main purposes: they can be used for counting, as in "there are 10 students in this room", and they can be used for ordering as in "this is the 3rd largest city in the country".

The natural numbers are represented as points on a straight line, called the number line.
Discussion:

• Is 0 an element of \( \mathbb{N} \)? (notation: \( 0 \in \mathbb{N} \))

• Is \( 10^{10^{10}} \in \mathbb{N} \)?

• Is there a largest element of \( \mathbb{N} \)?
  If there is, what is it? If there isn’t, can you prove it?

• What is the smallest element of \( \mathbb{N} \)?

• Let \( E = \{2, 4, 6, 8, \ldots\} \), that is, the even natural numbers.
  
  * What is the largest element of \( E \)?

  * What is the smallest element of \( E \)?

• What are the four main operations we can do with the elements of \( \mathbb{N} \)?
1.2. Operations

1.2.1. Definition.

An *operation* is a rule for combining two elements of a set. The main operations on numbers are:

- Addition
- Subtraction
- Multiplication
- Division

1.2.2. Closed Operation

A *closed operation* is a rule for combining any two elements of a set which always produces another element in the same set.

We say the set is *closed* under the operation
Exercise:

Which of these operations are closed operations on \( \mathbb{N} \)?

Why, or why not?

- Addition
- Subtraction
- Multiplication
- Division

In this course we will consider only closed operations.

1.2.3. Addition

Addition is a closed operation on \( \mathbb{N} \).

This means for any \( a, b \in \mathbb{N}, a + b \in \mathbb{N} \).

1.2.4. Multiplication

Multiplication is a closed operation on \( \mathbb{N} \).

This means for any \( a, b \in \mathbb{N}, a \times b \in \mathbb{N} \).
1.3. **Identities**

An identity, \( i \), is an element of a set which under an operation with any member, say \( a \), of a set will return that member, \( a \).

1.3.1. **Additive Identity.**

There is no identity under addition in \( \mathbb{N} \).

We require \( i + a = a \).

This would mean \( i = 0 \), however 0 is not an element of \( \mathbb{N} \).

Thus there is no additive identity for \( \mathbb{N} \).

1.3.2. **Multiplicative Identity.**

The identity under multiplication in \( \mathbb{N} \) is 1.

We require \( i \times a = a \).

This means \( i = 1 \), and 1 is an element of \( \mathbb{N} \).

Thus the multiplicative identity for \( \mathbb{N} \) is 1.
1.4. **Inverses**

An inverse of an element, say \( a \), is an element of a set, say \( b \), which under an operation with \( a \), will return the identity.

1.4.1. **Additive Inverses.**

Since there is no additive identity for \( \mathbb{N} \), there are no elements which will have an additive inverse.

1.4.2. **Multiplicative Inverses.**

For multiplicative inverses in \( \mathbb{N} \) we require \( a \times b = 1 \), where \( a,b \in \mathbb{N} \). Thus \( b = \frac{1}{a} \).

For \( b \in \mathbb{N} \), \( b = 1 \) when \( a = 1 \).

Thus the inverse for 1 is 1. No other elements of \( \mathbb{N} \) have multiplicative inverses.
Exercise:

Carefully write down every small step you take when simplifying the following expressions without the aid of a calculator.

Can you give reasons for each step?

- \(3x + 4y + 2x + y\)

- \((157 + 25) + 75\)

- \((x + 3)(x + 9)\)
1.5. Properties of Addition and Multiplication

1.5.1. Commutativity.

Let $a, b \in \mathbb{N}$. Then:

- $a + b = b + a$
- $a \times b = b \times a$

We say $\mathbb{N}$ is *commutative* under addition and multiplication. This property allows us to change the order or rearrange elements.

**Example:** $2 + 4 = 4 + 2 \text{ and } 2 \times 4 = 4 \times 2$

1.5.2. Associativity.

Let $a, b, c \in \mathbb{N}$. Then:

- $(a + b) + c = a + (b + c)$
- $(a \times b) \times c = a \times (b \times c)$

We say $\mathbb{N}$ is *associative* under addition and multiplication. This property allows us to omit parenthesis in some expressions.

**Example:** $(3 + 4) + 2 = 3 + 4 + 2$
1.5.3. Distributivity.

Let $a, b, c \in \mathbb{N}$. Then:

- $a \times (b + c) = (a \times b) + (a \times c)$
- $(a + b) \times c = (a \times c) + (b \times c)$

We say multiplication *distributes* over addition in $\mathbb{N}$. This property allows us to expand expressions.

**Example:** $2 \times (3 + 4) = (2 \times 3) + (2 \times 4)$

**Note:** addition does not distribute over multiplication. That is for any $a, b, c \in \mathbb{N}$, $a + (b \times c) \neq (a + b) \times (a + c)$

**Exercise:**

Let $x, y \in \mathbb{N}$. Simplify the following expression giving reasons for each step.

$[8(x + y)] + 2x$
1.6. **Order Properties.**

*Discussion:*

- Given two natural numbers $a, b$, what are the three possible orderings of the two numbers?

- What does “$a < b$” mean on a number line?

- Write down a definition for “$a < b$”.

- Use your definition to prove the following:
  
  * $5 < 20$
* If $x, y, s, t \in \mathbb{N}$ with $x < y$ and $s < t$, then $x + s < y + t$

* If $x, y, c \in \mathbb{N}$ with $x < y$, then $cx < cy$
1.6.1. Law of Trichotomy

The Natural Numbers, \( \mathbb{N} \), is a set of numbers with order properties. These are more formally expressed as the Law of Trichotomy.

**Property: The Law of Trichotomy**

If \( a, b \in \mathbb{N} \), then one and only one of the following relationships hold:

- \( a < b \)
- \( a = b \)
- \( a > b \)

1.6.2. Transitivity

If \( a, b, c \in \mathbb{N} \). Then:

- If \( a < b \) and \( b < c \) then \( a < c \)
- If \( a = b \) and \( b = c \) then \( a = c \)
- If \( a > b \) and \( b > c \) then \( a > c \)
Discussion:

- Let’s turn our attention for a moment to the real numbers, represented by \( \mathbb{R} \).

Consider the following sets of real numbers:

* Write down the smallest and largest elements in each set.

- \( A = \left\{ 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2} \right\} \)

- \( B = \{ x \in \mathbb{R} : 0 < x \leq 1 \} \)

- \( C = \left\{ \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots \right\} \)

- \( D = \{ \ldots, -10, -6, -2, 2, 6, 10, \ldots \} \)

- Consider these sets of natural numbers.

* What is the smallest element in each set?

- \( E = \{ 2, 4, 6, 8, \ldots \} \)

- \( O = \{ 1, 3, 5, 7, 9, \ldots \} \)

- \( P = \{ 2, 3, 5, 7, 11, \ldots \} \)
• Write down any non-empty subset of \( \mathbb{N} \).

• Does it have a smallest element? 

• Can you give a subset of \( \mathbb{N} \) that does not have a smallest element?

The last question above demonstrates an important order property of the natural numbers.

1.6.3. Well-Ordering Property

Property: The Well-Ordering Property for \( \mathbb{N} \).

If \( A \) is any non-empty subset of \( \mathbb{N} \), then \( A \) has a least element.

We say that \( \mathbb{N} \) is a well-ordered set.

Definition: Well-Ordered.

We say a set is well-ordered if every non-empty subset has a least element.
Exercise:

Are the following sets well-ordered?

- \([0, 1) = \{x \in \mathbb{R} : 0 \leq x < 1\}\), the interval of real numbers between 0 and 1, including 0.

- \(\mathbb{R}\), the set of all real numbers.

- \(E = \{2, 4, 6, 8, \ldots\}\), the set of even natural numbers.

The well-ordering property is an important statement about \(\mathbb{N}\). It can sometimes be used to prove an infinite number of claims as we shall see in later sections.