4.8. Set Operations

There are five main set theoretic operations, one corresponding to each of the logical connectives.

Set Operation	Name	Logical Connective	Name
$\overline{A}$	Complement	$\sim P$	Negation
$A \cup B \bigcup$	Union	$P \vee Q \bigvee$	Disjunction
$A \cap B \bigcap$	Intersection	$P \wedge Q$	Conjunction
$A \subseteq B \subseteq$	Subset	$P \Rightarrow Q$	Conditional
$A = B =$	Equality	$P \Leftrightarrow Q$ $P \equiv Q$	Biconditional Equivalence

The set operations can be defined in terms of the corresponding logical operations. This means that each of the tautologies proved by truth tables for the logical connectives will have a corresponding theorem in set theory.

We have seen how the logical conditional operator, $P \Rightarrow Q$ is related to subset, $A \subseteq B$ and how the logical biconditional operator, $P \Leftrightarrow Q$ (or equivalence, $P \equiv Q$) is related to set equality, $A = B$.

The following sections will cover the three remaining set operations: complement, union and intersection.

In our discussion of set theory, we will let U be a fixed set and all other sets, whether denoted A, B, C, etc, will be subsets of U. In other words, $A, B, C \in \mathcal{P}(U)$. Thus, each result should start with a statement similar to "Let A, B, C be subsets of a universal set U" or "Let $A, B, C \in \mathcal{P}(U)$".

4.8.1. Definition: Compliment

Let U be a universal set, and let $A \subseteq U$. Then the **complement of *A***, denoted by $\overline{A}$, is given by $\overline{A} = \{x \in U : \sim (x \in A)\} = \{x \in U : x \notin A\}$.

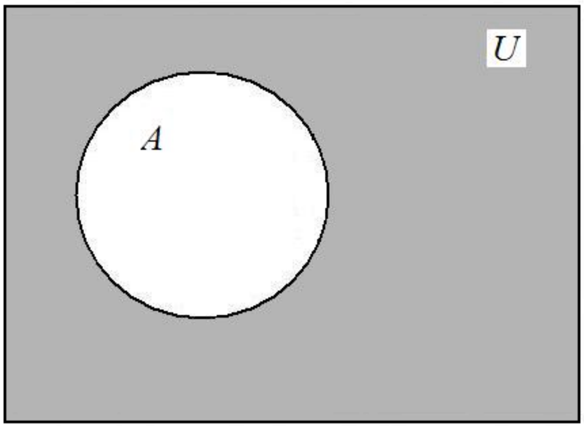
Notes.

1. $U \setminus A$, A' and A^c are also used for $\overline{A}$ in some books.

2. If the set U is fixed in a discussion, then $\overline{A}$ is sometimes written as $\overline{A} = \{x : x \notin A\}$

Example:

- The shaded area in the following Venn diagram depicts $\overline{A}$:



Exercises:

Let $U = \mathbb{Z}$. Write down $\overline{A}$ for the following sets:

- $A = \{1, 2, 3\}$

- $A = \{x \in \mathbb{Z} : x \text{ is even}\}$

- $A = \{x \in \mathbb{Z} : x > 0 \vee x < 0\}$

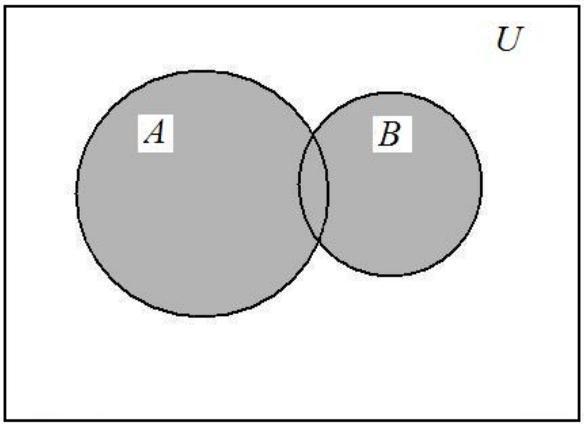
4.8.2. Definition: Union

Let A and B be subsets of a universe U. Then the **union of *A* and *B***, denoted by $A \cup B$, is given by

$A \cup B = \{x \in U : x \in A \vee x \in B\}$.

Example:

- The shaded area in the following Venn diagram depicts $A \cup B$:



Exercises:

- Let $U = \mathbb{R}$. Write down $A \cup B$ for the following sets:

o $A = \{1\}$ and $B = \{2\}$.

o A is the set of all even integers, B is the set of all odd integers.

o $A = \{x \in \mathbb{R} : 0 \le x \le 2\}$ and $B = \{x \in \mathbb{R} : 1 \le x \le 3\}$

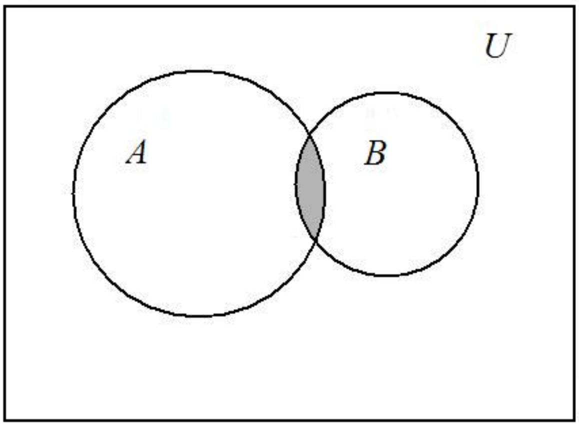
- If $A \subseteq U$ and $B \subseteq U$, is it true that $A \cup B \subseteq U$?

4.8.3. Definition: Intersection

Let *A* and *B* be subsets of a universe *U*. Then the **intersection of *A* and *B***, denoted by $A \cap B$, is given by $A \cap B = \{x \in U : x \in A \wedge x \in B\}$.

Example:

- The shaded area in the following Venn diagram depicts $A \cap B$:



Exercises:

- Let $U = \mathbb{R}$. Write down $A \cap B$ for the following sets:

  - $A = \{1, 2, 3, 5\}$ and $B = \{1, 4, 5, 6\}$.

  - *A* is the set of all even integers, *B* is the set of all odd integers.

  - $A = \{x \in \mathbb{R} : 0 \le x \le 2\}$ and $B = \{x \in \mathbb{R} : 1 \le x \le 3\}$

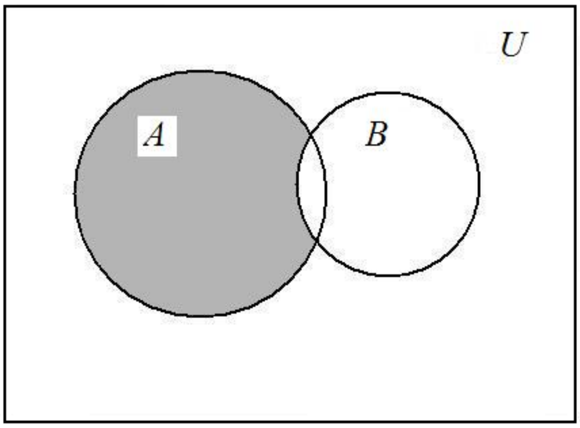
- If $A \subseteq U$ and $B \subseteq U$, is it true that $A \cap B \subseteq U$?

4.8.4. Definition: Difference

Let *A* and *B* be subsets of a universe *U*. Then the **difference of *A* and *B***, denoted by $A - B$, is given by $A - B = \{x \in U : x \in A \wedge x \notin B\}$.

Example:

- The shaded area in the following Venn diagram depicts $A - B$:



Notes.

1. The difference of $A - B$ is sometimes called the **relative complement of *B* in *A***.

2. If we let $A = U$, then we have

$$\begin{aligned} U - B &= \{x \in U : x \in U \wedge x \notin B\} \\ &= \{x \in U : x \in \overline{B}\} \\ &= \overline{B} \end{aligned}$$

3. Using Definitions for complement and intersection, we can simplify the definition of difference as follows:

$$\begin{aligned} A - B &= \{x \in U : x \in A \wedge x \notin B\} \\ &= \{x \in U : x \in A \wedge x \in \overline{B}\} \\ &= A \cap \overline{B} \end{aligned}$$

Exercises:

- Let $U = \mathbb{R}$. Write down $A - B$ for the following sets:

o $A = \{1, 2, 3, 5\}$ and $B = \{1, 4, 5, 6\}$.

.

o *A* is the set of all even integers, *B* is the set of all odd integers.

o $A = \{x \in \mathbb{R} : 0 \le x \le 2\}$ and $B = \{x \in \mathbb{R} : 1 \le x \le 3\}$

- If $A \subseteq U$ and $B \subseteq U$, is it true that $A - B \subseteq U$?

- Let $U = \mathbb{R}$, $A = \{1, 2, 3\}$, $B = \{2\}$, $C = \{2, 3, 4\}$ and $D = [0, 1] = \{x \in \mathbb{R} : 0 \le x \le 1\}$. Write down:

o $A - C$

o $B - C$

o $D - B$

o $D - A$

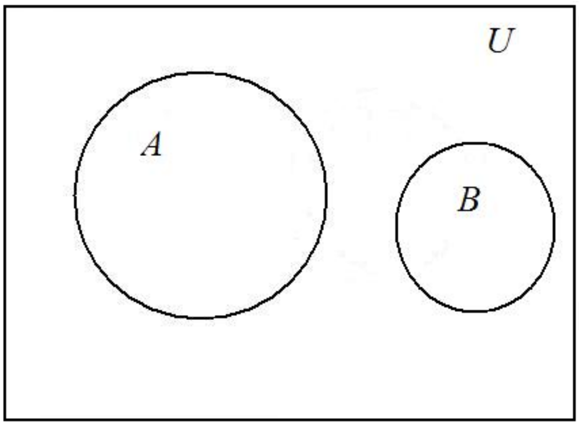
o $A - D$

4.8.5. Definition: Disjoint sets

Let A and B be subsets of a universe U. Then A and B are said to be **disjoint** if $A \cap B = \emptyset$.

Example:

- The following Venn diagram depicts disjoint sets A and B:



Note. Disjoint sets have no elements in common.

Exercises:

- Let $U = \mathbb{R}$, $A = \{1, 2, 3\}$, $B = \{2\}$, $C = \{2, 3, 4\}$ and $D = [0, 1] = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$. Which pairs of sets from A, B, C, D are disjoint?

4.9. Order of Operations for Set Operators.

The order of operation for set operators is as follows:

1. Evaluate complement first

2. Evaluate $\cup$ and $\cap$ second. When both are present, parenthesis may be needed, otherwise work left to right.

3. Evaluate $\subseteq$ and $=$ third. When both are present, parenthesis may be needed, otherwise work left to right.

Note: Use of parenthesis will determine order of operations which over ride the above order.

Examples:

Indicate the order of operations in the following:

- $\underbrace{\overline{A}}_{1}\underbrace{\cap}_{2}B$
- $\underbrace{\overline{(A\underbrace{\cap}_{1}B)}}_{2}$
- $\underbrace{\overline{A}}_{1}\underbrace{\cap}_{3}(B\underbrace{\cup}_{2}C)$
- $\underbrace{\overline{A}}_{1}\underbrace{\subseteq}_{3}B\underbrace{\cap}_{2}C$

Exercises:

Indicate the order of operations in the following:

- $(\underbrace{\overline{A}}\underbrace{\subseteq}B)\underbrace{\cap}C$
- $\underbrace{\overline{A}}\underbrace{\subseteq}B\underbrace{\cup}C$
- $\underbrace{\overline{(A\underbrace{\cup}B)}}$
- $\underbrace{\overline{A}}\underbrace{=}B\underbrace{\cap}C$

Notes.

1. $\cup$ and $\cap$ are operations on sets, thus $\cup$ and $\cap$ can only be put between two sets.

2. $\vee$ and $\wedge$ are operations on statements, thus $\vee$ and $\wedge$ can only be placed between statements.

Example:

- If *A*, *B*, and *C* are sets then $(A \subseteq B \wedge B \subseteq C) \Rightarrow A \subseteq C$ is interpreted as $((A \subseteq B) \wedge (B \subseteq C)) \Rightarrow (A \subseteq C)$
- $(A \subseteq B \wedge B \subseteq C) \not\equiv (A \subseteq (B \wedge B) \subseteq C)$

4.10. Set Laws

Let *A*, *B*, and *C* be subsets of a universal set *U*. That is $A, B, C \in \mathcal{P}(U)$. Then for all sets *A*, *B*, and *C* following set laws hold:

1. Commutative Laws:

- $(A \cup B) = (B \cup A)$
- $(A \cap B) = (B \cap A)$
- $(A = B) = (B = A)$

2. Associative Laws:

- $((A \cup B) \cup C) = (A \cup (B \cup C))$
- $((A \cap B) \cap C) = (A \cap (B \cap C))$
- $((A = B) = C) = (A = (B = C))$

3. Distributive Laws:

- $(A \cup (B \cap C)) = ((A \cup B) \cap (A \cup C))$
- $(A \cap (B \cup C)) = ((A \cap B) \cup (A \cap C))$

4. Double Complement (Involution) Law:

- $\overline{(\overline{A})} = A$

5. De Morgan's Laws:

- $\overline{(A \cup B)} = (\overline{A} \cap \overline{B})$
- $\overline{(A \cap B)} = (\overline{A} \cup \overline{B})$

6. Identity Laws:

- $(A \cup \emptyset) = A$
- $(A \cap U) = A$

7. Negation (Complement) Laws:

- $(A \cup \overline{A}) = U$
- $(A \cap \overline{A}) = \emptyset$
- $\overline{U} = \emptyset$
- $\overline{\emptyset} = U$

8. Dominance Laws:

- $(A \cup U) = U$
- $(A \cap \emptyset) = \emptyset$

9. Idempotent Laws:

- $(A \cup A) = A$
- $(A \cap A) = A$

10. Absorption Laws:

- $A \cap (A \cup B) = A$
- $A \cup (A \cap B) = A$

11. Set Difference

- $A - B = A \cap \overline{B}$

12. Subset properties of $\cup$ and $\cap$

- $\left(A \subseteq (B \cap C)\right) \Leftrightarrow \left((A \subseteq B) \wedge (A \subseteq C)\right)$
- $\left((A \cup B) \subseteq C\right) \Leftrightarrow \left((A \subseteq C) \wedge (B \subseteq C)\right)$

13. Subset property inclusion of intersection

- $A \cap B \subseteq A$
- $A \cap B \subseteq B$

14. Subset property inclusion in union

- $A \subseteq A \cup B$
- $B \subseteq A \cup B$

15. Transitive Property.

- $((A \subseteq B) \wedge (B \subseteq C)) \Rightarrow (A \subseteq C)$
- $((A = B) \wedge (B = C)) \Rightarrow (A = C)$