Section 4. Set Theory

4.1. Definitions

A set may be viewed as any well defined collection of objects, called elements or members of the set.

Sets are usually denoted with upper case letters, $A, B, X, Y, \ldots$ while lower case letters are used to denote elements $a, b, x, y, \ldots$ of a set.

Membership in a set is denoted as follows:

• $a \in S$ denotes that $a$ is a member or element of a set $S$. Similarly $a, b \in S$ denotes that $a$ and $b$ are both elements of a set $S$.

• $a \not\in S$ denotes that $a$ is not an element of a set $S$. Similarly $a, b \not\in S$ denotes that neither of $a$ and $b$ are elements of a set $S$.

In Set Theory, we work within a Universe, $U$, and consider sets containing elements from $U$. 
A set may be specified in essentially two ways:

1. The elements of the set are listed within braces, \{ \}, and separated by commas.

Technically, the listing of elements can be done only for finite sets. However, if an infinite set is defined by a “simple” rule, we sometimes write a few elements and then use “…” to mean roughly “and so on” or “by the same rule”.

Examples:

- \( A = \{1,3,5,7,9\} \). The set \( A \) is the finite collection of odd integers, 1 to 9 inclusive

- \( B = \{\ldots, -4, -2, 0, 2, 4, \ldots\} \). The set \( B \) is the infinite collection of even integers.

Exercises:

- List a finite set, \( C \), containing even integers between 10 and 20 inclusive.

- List an infinite set, \( D \), containing natural numbers that are divisible by 3
2. A statement defining the properties which characterise the elements in the set is written within braces

**Examples:**

- \( A = \{ z \in \mathbb{Z} : (z \text{ is odd } \land 1 \leq z \leq 9) \} \). The set \( A \) is the finite collection of odd integers, 1 to 9 inclusive.
- \( B = \{ z \in \mathbb{Z} : \exists k \in \mathbb{Z}, z = 2k \} \). The set \( B \) is the infinite collection of even integers.

**Exercises:**

- Define a finite set, \( C \), containing even integers between 10 and 20 inclusive.
- Define an infinite set, \( D \), containing natural numbers that are divisible by 3.
4.1.2. **Axiom of Specification.**

Given a Universe $U$ and any statement $P(x)$ involving $x \in U$, then there exists a set $A$ such that
$\forall x \in U, (x \in A \iff P(x))$. Further, we write
$A = \{x \in U : P(x)\}$.

In other words, the Axiom of Specification says that we can pick a set and a property and build a new set. This is why the notation for $A$ is sometimes referred to as set-builder notation.

**Example:**

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $P(x)$ be the statement “$x$ is odd”.

$\therefore$ by the Axiom of Specification, $A = \{x \in U : x \text{ is odd}\}$.

**Notes:**

1. We know an element $x$ belongs to the set $A = \{x \in U : P(x)\}$ if $x$ satisfies the condition $P(x)$.

2. This notation is more simply written $\{x \in D : P(x)\}$.

This is called set builder notation. In using this notation, the elements of the domain, $D$, must belong to the Universe, $U$,
and $P(x)$ can be any predicate involving $x$. $D$ could be all of $U$.

**Examples:**

- The interval $[0, 1]$ can be written in set builder notation as:
  \[
  \{x : x \in \mathbb{R} \land 0 \leq x \leq 1\} = \{x \in \mathbb{R} : 0 \leq x \leq 1\}
  \]

- The set of all rational numbers, $\mathbb{Q}$ can be written as:
  \[
  \mathbb{Q} = \left\{ \frac{a}{b} : a, b \in \mathbb{Z} \land b \neq 0 \right\} = \left\{ x : x = \frac{a}{b} : a, b \in \mathbb{Z} \land b \neq 0 \right\}
  \]

- $\{x \in \mathbb{R} : x^3 = x\} = \{0, 1, -1\}.$

**Exercises:**

Write down the following sets by listing their elements:

- $\{x \in \mathbb{N} : x^3 = x\}$
- $\{x \in \mathbb{R} : x^2 = 9\}$
- $\{x \in \mathbb{Z} : x^2 = 7\}$
4.2. Venn Diagrams

Venn diagrams are a pictorial method of demonstrating the relationship between set. The universal set, $U$, is represented by a rectangle and sets within the universe are depicted with circles.

While a Venn diagram may be used to demonstrate the relationship between sets, it does not provide a method of proving those relationships.
4.3. Special Sets

4.3.1. The Singleton Set

Sets having a single element are frequently called singleton sets.

Example:

• \{1\} is read “singleton 1”.

• If \(a \in U\), then \(\{x \in U : x = a\} = \{a\}\)

Note: The singleton set \(\{a\}\) is NOT the same as the element \(a\).

4.3.2. The Empty Set

The empty set or null set is a set which contains no elements.

It is denoted by the symbol \(\emptyset\) or by empty braces \(\{\}\).

Using set builder notation, one way of defining the empty set is: \(\emptyset = \{x \in \mathbb{N} : x \neq x\}\)
4.4. Subsets

4.4.1. Definition: Subset.

If $A$ and $B$ are sets, then $A$ is called a subset of $B$, written $A \subseteq B$, if and only if, every element in $A$ is also in $B$.

Examples:

- $\{1, 2\} \subseteq \{1, 2, 3\}$
- The Venn diagram demonstrating $A \subseteq B$ is:

![Venn diagram](image)

Exercises:

- Write the definition of subset using logic notation.
- Is $\{\text{cat, dog}\} \subseteq \{\text{bird, fish, cat, dog}\}$?
4.4.2. Definition: Proper Subset.

If $A$ and $B$ are sets, then $A$ is called a proper subset of $B$, written $A \subset B$, if and only if, every element in $A$ is also in $B$ but there is at least one element of $B$ that is not in $A$.

$A$ is a proper subset of $B$ if $A \subseteq B$ but $A \neq B$.

Examples:

- $\{1, 2\} \subset \{1, 2, 3\}$

Exercises:

- Draw a Venn diagram demonstrating $A \subset B$, where $A = \{1, 2\}$ and $B = \{1, 2, 3, 4, 5\}$

- Is $\{a, b, c\} \subset \{c, b, a\}$?
Notes.

1. If $A \subseteq B$, then each element of $A$ belongs to $B$, or for each $x \in A$, it is true that $x \in B$.

2. If $A$ is a subset of $B$, then $B$ is sometimes called a superset of $A$.

3. If $A$ and $B$ are sets, then to prove $A \subseteq B$, we need to prove $\forall x, x \in A \Rightarrow x \in B$.

4. If $A$ is a proper subset of $B$, there must be at least one element in $B$ that is not in $A$.

5. If $A$ and $B$ are sets, to prove $A$ is not a subset of $B$, denoted $A \not\subseteq B$, we need to prove $\sim (A \subseteq B)$:

   \[
   \sim (\forall x, x \in A \Rightarrow x \in B) \equiv \exists x, \sim (x \in A \Rightarrow x \in B) \\
   \equiv \exists x, \sim (x \in A \lor x \in B) \\
   \equiv \exists x, (x \in A \land x \notin B)
   \]

6. The following relationships hold in the number system:

   $\emptyset \subseteq \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$
4.4.3. The null set as a subset.

For any set $A$ in a Universe $U$, $\emptyset \subseteq A$

Proof:
Suppose $\neg (\emptyset \subseteq A)$. Then, there exists $x \in \emptyset$ such that $x \notin A$. This, therefore, means that $\emptyset$ is not empty, which is a contradiction. Therefore, $\emptyset \subseteq A$.

4.4.4. Distinction between elements and subsets

Examples:

- $2 \in \{1,2,3\}$, $2 \notin \{1,2,3\}$
- $\{2\} \notin \{1,2,3\}$, $\{2\} \subseteq \{1,2,3\}$
- $1 \in \{x \in \mathbb{N} : x^2 = 1\}$, $\{1\} \subseteq \{x \in \mathbb{N} : x^2 = 1\}$

Exercises:

Let $S$ be a set in a Universe $U$. Determine whether the following are true or false.

- $S \in S$
- $S \in \{S\}$
- $\emptyset \subseteq \{S\}$
- $\emptyset \in \{S\}$
- $\emptyset \subseteq \{S\}$
- $\{\emptyset\} \subseteq \{S\}$
4.5. Set Equality

4.5.1. Definition: Set Equality.

If $A$ and $B$ are sets, then $A$ equals $B$, written $A = B$, if and only if, every element in $A$ is also in $B$ and every element in $B$ is also in $A$.

Equivalently, $A = B$ if, and only if $A \subseteq B$ and $B \subseteq A$.

Note: To prove that two sets are equal two things must be shown:: $A \subseteq B$ and $B \subseteq A$.

Examples:

- The Venn diagram demonstrating $A = B$ is:
Exercises:

- Write the definition of set equality using logic notation.

4.5.2. Axiom of Extent.

If $A$ and $B$ are sets then $A = B \iff (\forall x \in U, x \in A \iff x \in B)$.

The Axiom of Extent says that a set is completely determined by its elements, the order in which the elements are listed is irrelevant, as is the fact that some members may be listed more than once.

Examples:

- $\{1, 2\} = \{1, 2\}$
- $\{a, b, c\} = \{c, b, a\}$

Exercises:

- Is $\{a, b, c, d\} = \{b, d, a, c\}$
- Is $\{\text{Ann, Bob, Cal}\} = \{\text{Bob, Cal, Ann, Cal}\}$
4.5.3. **Theorem: Equality by Specification**

Let $U$ be a universe and let $P(x)$ be a statement.

If $\forall x \in U, (P(x) \iff Q(x))$, that is $\forall x \in U, (P(x) \equiv Q(x))$
then $\{x \in U : P(x)\} = \{x \in U : Q(x)\}$

The Theorem states that subsets of the same universe $U$
which are defined by equivalent statements are equal sets.

This theorem allows the use of tautologies of logic to prove
set theoretic statements, as will be outlined later.

**Example:**

- We know that $x^2 = 1 \iff (x = 1 \lor x = -1)$.

Therefore
\[
\{x \in \mathbb{R} : x^2 = 1\} = \{(x = 1 \lor x = -1)\} = \{1, -1\}
\]

- If $a_1, a_2, a_3, \ldots a_n \in U$, then we can write
  \[
  A = \{x \in U : x = a_1 \lor x = a_2 \lor \ldots \lor x = a_n\}
  \]
  \[
  = \{a_1, a_2, a_3, \ldots a_n\}
  \]

In other words, if we know the elements of a set, we know
the set.

- $A = \{x \in \mathbb{N} : x = 1 \lor x = 2 \lor x = 3\} = \{1, 2, 3\}$
Exercise:

• Are the following sets equal? Using logic, can you prove your answer?
  \{1,3,1,2\}, \{3,2,1\}, \{1,2,3\}

• Are the following two sets equal? Give reasons.
  \[E = \{n \in \mathbb{N} : n \text{ is even}\} \text{ and } T = \{n \in \mathbb{N} : n^2 \text{ is even}\}.\]
4.6. Power Sets

4.6.1. Definition: Power Set

If $X$ is any set, then $\{A : A \subseteq X\}$ is the \textbf{power set} of $X$.

The power set of $X$ is often written as $\mathcal{P}(X)$.

So $\mathcal{P}(X) = \{A : A \subseteq X\}$.

A power set is a set whose elements are sets.

If the elements of $X$ are in a universe $U$, those of $\mathcal{P}(X)$ are in a universe $\mathcal{P}(U)$.

Examples:

- Let $X = \{1\}$ and let $S$ be the set of all subsets of $X$.
  Write down the set $S$ by listing its elements.
  $S = \{A : A \subseteq X\}$.
  $\emptyset \subseteq \{1\}$ and $\{1\} \subseteq \{1\}$.
  Thus $S = \{\emptyset, \{1\}\}$

- Let $X = \{1, 2\}$ and $S = \{A : A \subseteq X\}$. Write down the set $S$ by listing its elements.
  $\emptyset \subseteq \{1, 2\}$, $\{1\} \subseteq \{1, 2\}$, $\{2\} \subseteq \{1, 2\}$, and $\{1, 2\} \subseteq \{1, 2\}$.
  Thus $S = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$
Exercises:

- Let \( X = \{1, 2, 3\} \).
  - Write down the set \( \mathcal{P}(X) \) by listing its elements.
  - How many elements are there in \( \mathcal{P}(X) \)?
  - Is \( \emptyset \in \mathcal{P}(X) \)?
  - Is \( \emptyset \subseteq \mathcal{P}(X) \)?
  - Is \( 1 \in \mathcal{P}(X) \)?
  - Is \( \{1\} \in \mathcal{P}(X) \)?
  - Is \( \{2\} \subseteq \mathcal{P}(X) \)?
  - Is \( \{\{1, 2\}\} \subseteq \mathcal{P}(X) \)?
4.7. **Hasse Diagrams**

The elements of $\mathcal{P}(X)$ can be represented by diagrams using the following procedure:

1. An upward directed line between two sets indicates that the “lower” set is a subset of the “upper” set.

2. $\emptyset$ is at the bottom and $X$ is at the top.

3. Each pair of sets is joined by an upward directed line to the “smallest” set which contains each as a subset.

4. Each pair of sets is joined by a downward directed line to the “largest” set which is a subset of each.

**Example:**

Let $X = \{1, 2\}$, thus $\mathcal{P}(X) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ and the Hasse diagram is given by:

```
   {1, 2}
  /     \
{1}    {2}
|     |
\emptyset
```