WUCT121

Discrete Mathematics

Logic

1. Logic
2. Predicate Logic
3. Proofs
4. Set Theory
5. Relations and Functions
Section 1. Logic

1.1. Introduction.

In developing a mathematical theory, assertions or statements are made. These statements are made in the form of sentences using words and mathematical symbols. When proving a theory, a mathematician uses a system of logic. This is also the case when developing an algorithm for a program or system of programs in computer science. The system of logic is applied to decide if a statement follows from, or is a logical consequence of, one or more other statements.

You are familiar with using numbers in arithmetic and symbols in algebra. You are also familiar with the ‘rules’ of arithmetic and algebra.

Examples:

\[(3 + 4) + 6 = 3 + (4 + 6)\] (Associativity)
\[= 3 + 10\]
\[= 13\]

\[3x - 5x = (3 - 5)x\] (Distributivity)
\[= -2x\]
In a similar way, Logic deals with statements or sentences by defining symbols and establishing ‘rules’.

Roughly speaking, in arithmetic an operation is a rule for producing new numbers from a pair of given numbers, like addition (+) or multiplication (×).

In logic, we form new statements by combining short statements using connectives, like the words *and*, *or*.

**Examples:**

- This room is hot and I am tired.
- \[ x < 1 \text{ or } x > 7. \]

**1.2. Statements**

**1.2.1. Definition**

**Definition: Statement.** A *statement* or *proposition* is an assertion or declarative sentence which is true or false, but not both.

The truth value of a mathematical statement can be determined by application of known rules, axioms and laws of mathematics.
A statement which is **true** requires a *proof*.

**Examples:**

- Is the following statement True or False?

  For a real number $x$, if $x^2 = 1$, then $x = 1$ or $x = -1$.

  The statement is TRUE. Therefore, we must prove it.

  Consider $x^2 = 1$.

  Adding $-1$ to both sides gives $x^2 - 1 = 0$.

  Factorising this equation, we have $(x - 1)(x + 1) = 0$.

  Therefore, $x - 1 = 0$ or $x + 1 = 0$.

  **Case 1:** $x - 1 = 0$.

  Add 1 to both sides and we have $x = 1$.

  **Case 2:** $x + 1 = 0$.

  Add $-1$ to both sides and we have $x = -1$. 
A statement which is **false** requires a *demonstration*.

**Example:**

- Is the following statement True or False?

\[ 5 - (3 - 2) = (5 - 3) - 2 \]

The statement is **FALSE**. Therefore, we must demonstrate it.

\[
5 - (3 - 2) = 5 - 1 \\
\quad = 4 \\
(5 - 3) - 2 = 2 - 2 \\
\quad = 0 \\
\therefore 5 - (3 - 2) \neq (5 - 3) - 2
\]
Exercise:

Determine which of the following sentences are statements. For those which are statements, determine their truth value.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>$2 + 3 = 5$</td>
</tr>
<tr>
<td>(ii)</td>
<td>It is hot and sunny outside.</td>
</tr>
<tr>
<td>(iii)</td>
<td>$2 + 3 = 6$</td>
</tr>
<tr>
<td>(iv)</td>
<td>Is it raining?</td>
</tr>
<tr>
<td>(v)</td>
<td>Go away!</td>
</tr>
<tr>
<td>(vi)</td>
<td>There exists an even prime number.</td>
</tr>
<tr>
<td>(vii)</td>
<td>There are six people in this room.</td>
</tr>
<tr>
<td>(viii)</td>
<td>For some real number $x$, $x &lt; 2$</td>
</tr>
<tr>
<td>(ix)</td>
<td>$x &lt; 2$</td>
</tr>
<tr>
<td>(x)</td>
<td>$x + y = y + x$</td>
</tr>
</tbody>
</table>
Strictly speaking, as we don’t know what $x$ or $y$ are, in parts (ix) and (x), these should not be statements. In Mathematics, $x$ and $y$ usually represent real numbers and we will assume this is the case here.

Therefore, (ix) is either true or false (even if we don’t know which) and (x) is always true, so we will allow both.

### 1.2.2. Simple Statements

**Definition: Simple Statement.** A *simple* or *primitive* statement is a statement which cannot be broken down into anything simpler.

A simple statement is denoted by use of letters $p, q, r...$

**Examples:**

- $p$: There are seven days in a week
  
  $p$ is a simple statement

- $p : 2 + 3 = 6$

  $p$ is a simple statement
1.2.3.  Compound Statements

Definition: Compound Statement. A compound or composite statement is a statement which is comprised of simple statements and logical operations.

A compound statement is denoted by use of letters $P, Q, R...$

Examples:

- $P$: There are seven days in a week and twelve months in a year.
  Is a compound statement.
  $p$: There are seven days in a week
  $q$: There twelve months in a year
  Operation: $\text{and}$

- $P$: $2 + 3 = 6$ or $5 - (3 - 2) = (5 - 3) - 2$.
  Is a compound statement.
  $p$: $2 + 3 = 6$
  $q$: $5 - (3 - 2) = (5 - 3) - 2$
  Operation: $\text{or}$
• \( P \): If it is not raining then I will go outside and eat my lunch.

Is a compound statement

\( p \): It is raining

\( q \): I will go outside

\( r \): I will eat my lunch

Negation of \( p \)

Operations: \textit{If … then, and}
*Exercises:*

Determine which of the following are simple statements, and which are not. For those which are not, identify the simple statement(s) used.

<table>
<thead>
<tr>
<th></th>
<th>Simple Statement</th>
<th>Operation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>(2 + 3 = 5)</td>
<td></td>
</tr>
<tr>
<td>(ii)</td>
<td>It is hot and sunny outside.</td>
<td></td>
</tr>
<tr>
<td>(iii)</td>
<td>(2 + 3 \neq 6)</td>
<td></td>
</tr>
<tr>
<td>(iv)</td>
<td>(x \leq 2)</td>
<td></td>
</tr>
<tr>
<td>(v)</td>
<td>(-5 &lt; x &lt; 2)</td>
<td></td>
</tr>
<tr>
<td>(vi)</td>
<td>If I study hard then I will pass my exam</td>
<td></td>
</tr>
</tbody>
</table>
1.3. Truth Tables

A statement $P$ can hold one of two truth values, true or false. These are denoted “T” and “F” respectively.

Note: Some books may use “1” for true and “0” for false.

When determining the truth value of a compound statement all possible combinations of the truth values of the statements comprising it must be considered.

This is done systematically by the use of truth tables. Each connective is defined by its own unique truth table.

There are five fundamental truth tables which will be covered in the following sections.

1.3.1. Truth Table Construction

To construct a truth table assign each statement a column.

The number of rows in the table is determined by the number of statements. For $n$ statements, $2^n$ rows will be required.

Systematically assign truth values to each of the statements, beginning in the first column.
Once all possible truth values for the simple statements are inserted, determine the truth values of the compound statements following the rules for the operations.

**Example:**

- Given three statements $P, Q, R$. The table setup is:

<table>
<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$R$</th>
<th>Compound Statement</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td></td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td></td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td></td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td></td>
</tr>
</tbody>
</table>
1.4. Logical Operations

There are five main operations which when applied to a statement will return a statement.

If $P$ and $Q$ are statements, the five primary operations used are:

not $P$, the **negation** of $P$.

$P$ or $Q$, the **disjunction** of $P$ and $Q$.

$P$ and $Q$, the **conjunction** of $P$ and $Q$.

$P$ implies $Q$, the **conditional** of $P$ and $Q$.

$P$ if and only if $Q$, the **biconditional** of $P$ and $Q$.

1.4.1. Negation, “not”

**Definition: Statement Negation.**

If $P$ is a statement, the *negation* of $P$ is “not $P$” or “it is not the case that $P$” and is denoted $\sim P$. 
Examples:

- There are not seven days in a week
  \( p \): There are seven days in a week

- \( P \): It is raining outside.
  \( \sim P \): \( \sim \) (It is raining outside.)
  It is not raining outside.

- \( Q \): \( x > 2 \) or \( x < 2 \)
  \( \sim Q \): \( \sim (x > 2 \) or \( x < 2 ) \)
  Simplified: \( x = 2 \).

Exercises:

For each statement \( P \), write down \( \sim P \).

- \( P \): Discrete Maths is interesting.

- \( P \): \( x^2 - 1 = 0 \)
1.4.1.1 Truth Table for Negation

The negation of $P$ has the opposite truth value from $P$, 
$\sim P$ is false when $P$ is true; $\sim P$ is true when $P$ is false.

<table>
<thead>
<tr>
<th>$P$</th>
<th>$\sim P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

All possible truth values for $P$

All possible truth values for $\sim P$ depending on the value of $P$.

Example:

Write down the truth value of the following statements.

<table>
<thead>
<tr>
<th>$P$</th>
<th>$\sim P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bullet$ $2 + 5 = 7$</td>
<td>$2 + 5 \neq 7$</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>$\bullet$ This room is empty</td>
<td>This room is not empty</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>
Exercise:

Write down the truth value of the following statements.

<table>
<thead>
<tr>
<th>$P$</th>
<th>$\sim P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 \in \mathbb{N}$</td>
<td>$1 \notin \mathbb{N}$</td>
</tr>
<tr>
<td>Division is a closed operation on $\mathbb{N}$</td>
<td>Division is not a closed operation on $\mathbb{N}$</td>
</tr>
</tbody>
</table>

Note:
- The truth table for negation tells us that for any statement $P$, exactly one of $P$ or $\sim P$ is true. So, to prove $P$ is true, we have two methods:
  - Direct: Start with some facts and end up proving $P$ in a direct step-by-step manner.
  - Indirect: Don’t prove $P$ is true directly, but prove that $\sim P$ is false.
- Generally, brackets are left out around ‘~ $P$’.
  Thus, $\sim P \lor Q$ means $(\sim P) \lor Q$, and not $\sim (P \lor Q)$.
  This is similar to arithmetic where $-x + y$ means $(x) + y$ and not $-(x + y)$.
1.4.2. Disjunction, “or”

Definition: Disjunction.

If $P$ and $Q$ are statements the *disjunction* of $P$ and $Q$ is “$P$ or $Q$”, denoted $P \lor Q$.

Examples:

- Given $P : 2 + 3 = 5$, $Q : 2 + 3 = 6$, write down $P \lor Q$.
  
  $P \lor Q : 2 + 3 = 5 \text{ or } 2 + 3 = 6$
  
  alternatively: $(2 + 3 = 5) \lor (2 + 3 = 6)$
  
  simplified: $2 + 3 = 5 \text{ or } 6$

- Write $P : x \leq 5$ using “$\lor$”.
  
  $(x < 5) \lor (x = 5)$

Exercises:

- Write the following statements using “$\lor$”
  
  * I am catching the bus or train home.

  * A month has 30 or 31 days.
• For the statements $P$ and $Q$, write down $P \lor Q$.

$\ast$ $P: \quad x > 0 \quad Q: \quad x = 0$

$\ast$ $P: \ x$ is the square of an integer, $Q: \ x$ is prime

### 1.4.2.1 Truth Table for Disjunction

The disjunction of $P$ and $Q$ is true when either $P$ is true, or $Q$ is true, or both $P$ and $Q$ are true; it is false only when both $P$ and $Q$ are false.

<table>
<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$P \lor Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>
Example:

Write down the truth value of the following statements.

<table>
<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$P \lor Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2 + 3 = 5$</td>
<td>$2 + 3 = 6$</td>
<td>$T$</td>
</tr>
<tr>
<td>$T$</td>
<td>$F$</td>
<td>$T$</td>
</tr>
<tr>
<td>$1 \notin \mathbb{N}$</td>
<td>$0 \in \mathbb{N}$</td>
<td>$F$</td>
</tr>
<tr>
<td>$F$</td>
<td>$F$</td>
<td>$F$</td>
</tr>
</tbody>
</table>

Exercise:

Write down the truth value of the following statements.

<table>
<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$P \lor Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2 &gt; 1$</td>
<td>$(x+1)^2 = x^2 + 2x + 1$</td>
<td></td>
</tr>
<tr>
<td>$2$ is odd</td>
<td>$5$ is odd</td>
<td></td>
</tr>
<tr>
<td>$2 &lt; 1$</td>
<td>This room is empty</td>
<td></td>
</tr>
</tbody>
</table>