5.3.2. One-to-one

Let $F$ be a function from $A$ to $B$. $F$ is one-to-one if and only if
\[
\forall x_1, x_2 \in A, ((x_1, y) = (x_2, y) \Rightarrow x_1 = x_2).
\]

For one-to-one functions, any given element from the Range is related to only one element from the Domain. That is each element in both the domain and the range is related to just one element.

Notes:

* Only functions can be one-to-one.

* It is often the case that if a function $F$ is one-to-one, it satisfies a horizontal line test.

* To establish if a relation is one-to-one show if the relation is, in fact, a function. Then determine if it is one-to-one.

* To show a function is one-to-one, show each element in the range occurs once in an ordered pair.

* To show a function is \textit{not} one-to-one, give a \textit{counterexample}, that is, find an element of the range that is related to two elements in the domain.
Examples:

- Consider the relation $F_1$ on $\mathbb{R}$ given by

  $$F_1 = \{(x, y) : y = x^2\}.$$  Is $F_1$ a one-to-one function?

  ![Graph of $y = x^2$](image)

  Dom $F_1 = \mathbb{R}$, vertical line test holds, thus $F_1$ is a function. Horizontal line test fails: $(-1, 1) \in F_1 \land (1, 1) \in F$, therefore $F_1$ is not a one-to-one function.
• Consider the relation $F_2$ on $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$ given by $F_2 = \{(x, y) : y = x^2\}$. Is $F_2$ a one-to-one function?

![Graph of $F_2$](image)

$\text{Dom } F_2 = \mathbb{R}^+$, vertical line test holds, thus $F_2$ is a function. Horizontal line test holds, therefore $F_2$ is a one-to-one function.

• Let $X = \{0, 1, 2, 3\}$.

Consider the function $F_3$ from $\mathcal{P}(X)$ to $\mathbb{N}$ given by $F_3 = \{(A, n) : n \text{ is the number of elements in the set } A\}$.

Is $F_3$ a one-to-one function?

Consider $A = \{0, 1\} \in \mathcal{P}(X)$ and $B = \{1, 2\} \in \mathcal{P}(X)$.

Then $(A, 2) \in F_3$ and $(B, 2) \in F_3$, that is, $2 \in \mathbb{N}$ appears twice.

Thus, $F_3$ is not a one-to-one function.
Exercises:

Which of the following relations are one-to-one functions?

- $F_1$ on $A = \{1, 2, 3\}$, $F_1 = \{(1, 2), (2, 3), (3, 1)\}$.

- $F_2$ on $A = \{1, 2, 3\}$, $F_2 = \{(1, 2), (2, 1), (3, 1)\}$.
\[ F_3 \text{ on } \mathbb{Z}, \quad F_3 = \{(x, y): y = 2x\} . \]

\[ F_4 \text{ from } \mathbb{Z} - \{0\} \text{ to } \mathbb{R}, \quad F_4 = \{(x, y): y = \sqrt{x^2 - 1}\} . \]
5.3.3. Onto

Let $F$ be a function from $A$ to $B$. $F$ is onto if and only if

$\text{Range } F = B$, that is,

$$\forall y \in B, \exists x \in A, (x, y) \in F.$$ 

For a function to be onto, every given element from the
range must be related to at least one element from the
domain.

Notes:

* Only functions can be onto.

* To establish if a relation is onto show if the relation
  is, in fact, a function. Then determine if it is onto.

* To show a function $F$ from $A$ to $B$ is onto, show that
  $\text{Range } F = B$, that is every element in the range occurs at
  least once in an ordered pair.

* To show a function is not onto, give a
  \textit{counterexample}, that is, find an element of the range that is
  not related to an element in the domain.
Example:

• Consider the relation $F_1$ from

$A = \{x \in \mathbb{R} : -1 \leq x \leq 1\}$ to $\mathbb{R}$ given by

$F_1 = \{(x, y) : y = \sqrt{1-x^2}\}$. Is $F_1$ an onto function?

By defining the function to $F_2$ from

$A = \{x \in \mathbb{R} : -1 \leq x \leq 1\}$ to $B = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$ given by

$F_2 = \{(x, y) : y = \sqrt{1-x^2}\}$.

Now Range $F_2 = \{y \in \mathbb{R} : 0 \leq y \leq 1\} = B$, thus the function $F_2$ is an onto function.
Exercises:

Which of the following functions are onto?

- $F_1$ from $A = \{1, 2, 3, 4, 5\}$ to $B = \{a, b, c, d\}$, 
  $F_1 = \{(1, a), (2, c), (3, c), (4, d), (5, d)\}$

- $F_2$ from $A = \{1, 2, 3, 4, 5\}$ to $B = \{a, b, c, d\}$, 
  $F_2 = \{(1, a), (2, b), (3, c), (4, d), (5, a)\}$.

- $F_3$ on $\mathbb{R}$, $F_3 = \{(x, y) : y = 4x - 1\}$.

- $F_4$ on $\mathbb{Z}$, $F_4 = \{(x, y) : y = 4x - 1\}$. 
5.3.4. Inverse

Every relation has an inverse and this holds for functions also.

For any function, there is an inverse relation; however, this inverse relation is not always a function.

The inverse of a function $F$ will also be a function when $F$ is one-to-one and onto.

**Example:**

Consider the relation $F$ on the interval $[-1,1] = \{x \in \mathbb{R} : -1 \leq x \leq 1\}$, given by

$$F = \{(x,y) : y = \sqrt{1-x^2}\}.$$

- Sketch $F$. Is $F$ a one-to-one and onto function?
Dom $F = [-1,1]$, and the vertical line test holds. Thus $F$ is a function. Horizontal line test fails, thus $F$ is not a one-to-one function. Range $F = [0,1] \neq [-1,1]$, thus $F$ is not an onto function.

- Sketch $F^{-1}$. Is $F^{-1}$ a function?

Since $F$ is function, and thus a relation, there is an inverse relation $F^{-1}$ on $[-1,1]$ given by

$$F^{-1} = \{(x, y) : x = \sqrt{1 - y^2}\}.$$

\[\text{Dom } F^{-1} = [0,1] \neq [-1,1], \text{ and the vertical line test fails.}
\]

Thus $F^{-1}$ is not a function.
Exercises:

Consider the relation $F$ on $A = \{x \in \mathbb{R} : x \geq 0\}$ given by

$$F = \{(x, y) : y = x^2\}.$$ 

Sketch $F$. Is $F$ a one-to-one and onto function?

- Sketch $F^{-1}$. Is $F^{-1}$ a function?
5.4. Permutations

5.4.1. Definition

Let \( A \) be a set and let \( F \) be a function on \( A \). Then \( F \) is a permutation of \( A \) if \( F \) is one-to-one and onto.

Example:

Let \( A = \{0, 1, 2, 3\} \). Define \( F = \{(0,1), (1,2), (2,3), (3,0)\} \).

\( F \) is a one-to-one and onto function on \( A \) and thus is a permutation of the elements of \( A \).

Using conventional function notation each ordered pair in \( F \) can be written as:
\[
F(0) = 1, \quad F(1) = 2, \quad F(2) = 3, \quad F(3) = 0
\]

“Matrix” representation can also be used for permutations. The function \( F \) can be written as
\[
F = \begin{pmatrix}
0 & 1 & 2 & 3 \\
1 & 2 & 3 & 0
\end{pmatrix}
\]

\( F \) is one possible permutation of the set \( A \).

Other permutations are:
\[
I = \begin{pmatrix}
0 & 1 & 2 & 3 \\
0 & 1 & 2 & 3
\end{pmatrix}, G = \begin{pmatrix}
0 & 1 & 2 & 3 \\
1 & 0 & 3 & 2
\end{pmatrix}, H = \begin{pmatrix}
0 & 1 & 2 & 3 \\
1 & 3 & 2 & 0
\end{pmatrix}
\]
There will be \(4! = 4 \times 3 \times 2 \times 1\) total different permutations of the set \(A\).

\(I\) is known as the identity permutation, where each element in \(A\) is mapped to itself.

Notes:

* In general, if \(A\) is a set with \(n\) elements, there are \(n!\) different permutations of \(A\).

* The set of all permutations on a set \(A\) with \(n\) elements is often denoted by \(S_n\).

**Exercises:**

Let \(A = \{0, 1, 2, 3\}\) and let \(G = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \end{pmatrix}\) and \(H = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 3 & 2 & 0 \end{pmatrix}\) be permutation on \(A\).

Write down the following.

- \(G(1)\)
- \(G(3)\)
- \(H(0)\)
- \(H(1)\)
- \(G(H(0))\)
- \(G(H(1))\)
5.4.2. Cycle notation

Obviously, the matrix notation for permutations can be confusing when we start to combine permutations.

This notation can be mistaken for “normal” matrix multiplication. Therefore, we introduce what is called cycle notation for permutations.

Example:

Let \( A = \{1, 2, 3, 4, 5\} \) and let \( F \) be a permutation on \( A \) given by \( F = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix} \)

we note that:

1 “goes to” 2
2 “goes to” 3
3 “goes to” 4
4 “goes to” 5
5 “goes to” 1.

This can be written as a cycle: \((1 \ 2 \ 3 \ 4 \ 5)\).

Diagrammatically, this can be represented as

\[
(1 \ 2 \ 3 \ 4 \ 5)
\]
If an element is mapped onto itself, then it is left out of the cycle.

**Examples:**

Write the following permutations using cycle notation.

- Let \( A = \{0, 1, 2, 3\} \)
  \[ F = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 1 & 0 & 3 \end{pmatrix} = (0 \ 2) \]

- \( A = \{1, 2, 3, 4, 5\} \)
  \[ G = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 5 & 4 \end{pmatrix} = (1 \ 2)(4 \ 5) \]

- \( A = \{1, 2, 3\} \)
  \[ I = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = (1) \text{ or } (2) \text{ or } (3) \text{ or } (1)(2)(3) \]
Exercises:

Write down the following permutations on $A = \{0, 1, 2, 3\}$, using cycle notation.

- $F = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 3 & 2 & 1 \end{pmatrix}$
- $G = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 3 & 0 & 2 \end{pmatrix}$
- $H = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \end{pmatrix}$
5.4.3. Composition

In traditional Calculus, composition of functions is defined to be \((g \circ f)(x) = g(f(x))\).

The same idea is used when considering composition of permutations.

**Examples:**

Let \(A = \{1, 2, 3, 4\}\) and let \(F = (1 \quad 2 \quad 3 \quad 4)\), \(G = (1 \quad 2)(3 \quad 4)\) be permutations on \(A\).

Write down the following:

- \(G(F(1)) = G(2) = 1\)
- \(G(F(2)) = G(3) = 4\)
- \(G(F(3)) = G(4) = 3\)
- \(G(F(4)) = G(1) = 2\)

What is \(G \circ F\) written using cyclic notation?

\(G \circ F = (2 \quad 4)\)
This could be calculated by writing each function in cyclic notation in the appropriate order, then determining the resultant permutation.

\[ G \circ F = FG = (1 \ 2 \ 3 \ 4)(1 \ 2)(3 \ 4) \]

1 “goes to” 2 in the first cycle, then 2 “goes to” 1 in the second. Thus, 1 “goes to” 1 overall.

2 “goes to” 3 in the first cycle, then 3 “goes to” 4 in the third. Thus, 2 “goes to” 4 overall.

3 “goes to” 4 in the first cycle, then 4 “goes to” 3 in the third. Thus, 3 “goes to” 3 overall.

4 “goes to” 1 in the first cycle, then 1 “goes to” 2 in the second. Thus, 4 “goes to” 2 overall.

These calculations give \( G \circ F = FG = (2 \ 4) \).

\[(1 \ 2 \ 3 \ 4)(1 \ 2)(3 \ 4) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} = (2 \ 4)\]
Exercises:

Calculate the following compositions of permutations on $A = \{0,1,2,3\}$.

- $(1\ 2)(1\ 0\ 2)$

- $(0\ 1)(2\ 3)(0\ 1\ 2\ 3)$

- $(1\ 2\ 3)(3\ 2)$

5.4.4. Inverse Permutations

Permutations are one-to-one and onto functions, thus their inverses are also functions which are one-to-one and onto. Thus, the inverse of a permutation is also a permutation.

Recall that to find the inverse of a relation or function, we simply reverse the ordered pairs. For permutations, the process is identical.
Examples:

Let $A = \{1, 2, 3, 4\}$ and let $F = (1 \ 2 \ 4 \ 3)$.

In $F$:

1 “goes to” 2. Thus, in $F^{-1}$, 2 “goes to” 1.

2 “goes to” 4. Thus, in $F^{-1}$, 4 “goes to” 2.

3 “goes to” 1. Thus, in $F^{-1}$, 1 “goes to” 3.

4 “goes to” 3. Thus, in $F^{-1}$, 3 “goes to” 4.

Putting all these calculations together, we have

$$F^{-1} = (1 \ 2 \ 4 \ 3)^{-1}$$

$$= (1 \ 3 \ 4 \ 2)$$

$$= (3 \ 4 \ 2 \ 1)$$

Note that $F^{-1}$ is just $F$ written in the reverse order.

Exercises:

Let $A = \{0, 1, 2, 3\}$ Write down the following.

- $(1 \ 2 \ 3)^{-1}$
- $(0 \ 3 \ 1)^{-1}$