Section 5. Relations and Functions

5.1. Cartesian Product

5.1.1. Definition: Ordered Pair

Let $A$ and $B$ be sets and let $a \in A$ and $b \in B$.

An ordered pair $(a, b)$ is a pair of elements with the property that:

$$(a, b) = (c, d) \iff (a = c) \land (b = d).$$

Notes:

* A pair set $\{a, b\}$ is NOT an ordered pair, since $\{a, b\} = \{b, a\}$.

* It should be clear from the context when $(a, b)$ is an ordered pair, and when $(a, b) = \{x \in \mathbb{R} : a < x < b\}$ is an open interval of real numbers.
Examples:

- Points in the plane $\mathbb{R}^2$ are represented as ordered pairs.

From the graph it can be seen $(1, 2) \neq (2, 1)$ and $(-1, -2) \neq (-2, -1)$.

- Complex numbers $a + ib$ where $i = \sqrt{-1}$ and $a, b \in \mathbb{R}$, are ordered pairs in the sense that,
  
  $$a + ib = c + id \iff (a = c) \land (b = d).$$
5.1.2. **Definition: Cartesian Product**

Let $A$ and $B$ be sets, then the Cartesian Product of $A$ and $B$, denoted $A \times B$, is defined by

$$A \times B = \{(a, b) : a \in A \land b \in B\}.$$ 

**Example:**

- $\mathbb{R} \times \mathbb{R} = \{(x, y) : x \in \mathbb{R} \land y \in \mathbb{R}\}$.

Sketch a graph of $\mathbb{R} \times \mathbb{R}$, otherwise known as $\mathbb{R}^2$. $\mathbb{R}^2$ is the usual Cartesian plane with the usual graph.
Exercises:

• Let $A = \{3\}$ and $B = \{2, 3\}$. Write down $A \times B$. Sketch a graph of $A \times B$ in $\mathbb{R}^2$.

• Let $C = \{x \in \mathbb{R} : -1 \leq x \leq 1\}$ and $D = \{1, 2\}$. Write down $C \times D$. Sketch a graph of $C \times D$ in $\mathbb{R}^2$. 
5.2. Relations

5.2.1. Definition: Binary Relation

Let $A$ and $B$ be sets. We say that $R$ is a (binary) relation from $A$ to $B$ if $R \subseteq A \times B$.

Notes:

* If $R \subseteq A \times A$, we say that $R$ is a relation on $A$.

* If $(a, b) \in R$, we will frequently write $aRb$ and say that “$a$ is in the relation $R$ to $b$”.

* Every relation is a subset of a Cartesian product

Examples:

- Let $A$ be the set of all male human beings and let $B$ be the set of all human beings. The relation $T$ from $A$ to $B$ is given by $T = \{(x, y) : x$ is the father of $y\}$.

- $W = \{(1, 2), (2, 1), (5, \pi)\}$.

Note: $W$ cannot be defined by a “rule”. Sometimes relations are simply defined by a listing of elements.
Let \( R \) be the relation on \( \mathbb{R} \), defined by
\[
R = \{(x, y) : x^2 + y^2 = 1\}.
\]
Sketch the graph of \( R \) in \( \mathbb{R}^2 \)

\[\text{Exercise:}\]

Let \( S \) be the relation on \( \mathbb{R} \), defined by
\[
S = \{(x, y) : 4x + y = 4\}.
\]
Sketch the graph of \( S \) in \( \mathbb{R}^2 \)
Example:
Consider the relation $R$ on $\mathbb{R}$ given by \( R = \{(x, y): x = y\} \)

- Sketch the graph of $R$ in $\mathbb{R}^2$

- Are the following true or false?
  - $1R1$ True
  - $1R2.2$ False
  - $(-3, 3) \in R$ False
  - If $aR100$, what is the value of $a$? 100

Note. The relation $R$ in this example is called the identity relation on $\mathbb{R}$ and is usually written $R = \{(x, x): x \in \mathbb{R}\}$. 
Exercises:

- Let \( X = \{0, 1, 2, 3\} \), and let the relation \( R \) on \( X \) be given by \( R = \{(x, y) : \exists z \in \mathbb{N}, x + z = y\} \).
  
  o  What is an easier way of expressing the relation \( R \)?

  o  List all the elements of \( R \).

  o  Sketch \( X \times X \), and circle the elements of \( R \).

- Let \( S \) be the relation on \( \mathbb{Z} - \{0\} \) given by
  \[ S = \{(x, y) : \exists z \in \mathbb{Z}, xz = y\} \]

  o  Describe the relation \( S \).
Are the following true or false?

- $(2, -4) \in S$
- $-3 S 0$
- $(3, 5) \in S$

Let $R$ be the relation on $\mathbb{N}$ given by

$$R = \{(x, y) : y = x^2\}$$

and let $S$ be the relation on $\mathbb{R}$ given by $S = \{(x, y) : y = x^2\}$.

Sketch each relation. What difference does the “input” set make to the elements in each relation.
Note. Care must be taken when writing relations. As can be seen from this example, it must be very clear the sets a relation is from and to.

- Let $A = \{0, 1\}$ and $B = \{-1, 0, 1\}$. Let two relations from $A$ to $B$ be given by $R_1 = \{(0,-1), (1,-1), (1, 0)\}$, and $R_2 = \{(0,0), (1,1), (1, -1)\}$.

Determine:

- $R_1 \cap R_2$.
- $R_1 \cup R_2$
Let $R_3$ and $R_4$ be relations on $\mathbb{R}$ defined by $R_3 = \{(x, y): x = y\}$, and $R_4 = \{(x, y): x = -y\}$. Determine:

- $R_3 \cup R_4$
- $R_3 \cap R_4$
5.2.2. **Definition: Domain**

Let $R$ be a relation from $A$ to $B$.

Then the domain of $R$, denoted $\text{Dom} \ R$, is given by

$\text{Dom} \ R = \{x : \exists y, \ xRy\}$.

Notes:

* Let $R$ be a relation from $A$ to $B$, then $\text{Dom} \ R \subseteq A$.

* $\text{Dom} \ R$ is the set of all first elements in the ordered pairs that belong to $R$.

5.2.3. **Definition: Range**

Let $R$ be a relation from $A$ to $B$.

Then the range of $R$, denoted $\text{Range} \ R$, is given by

$\text{Range} \ R = \{y : \exists x, \ xRy\}$.

Notes:

* Let $R$ be a relation from $A$ to $B$, then $\text{Range} \ R \subseteq B$.

* $\text{Range} \ R$ is the set of all second elements in the ordered pairs that belong to $R$. 
Examples:

- Let $A = \{0, 1, 2, 3\}$ and let $R_1$ be the relation on $A$ given by $R_1 = \{(0, 0), (0, 1), (0, 2), (3, 0)\}$.

Determine:

  - $\text{Dom } R_1 = \{0, 3\}$
  - $\text{Range } R_1 = \{0, 1, 2\}$

Exercises:

- Let $R_2$ be the relation on $\mathbb{Z}$ given by $R_2 = \{(x, y) : xy \neq 0\}$.

Determine:

  - $\text{Dom } R_2$
  - $\text{Range } R_2$

- Let $R_3$ be the relation from $\mathbb{Z}$ to $\mathbb{Q}$ given by $R_3 = \{(x, y) : x \neq 0 \land y = \frac{1}{x}\}$.

Determine:

  - $\text{Dom } R_3$
  - $\text{Range } R_3$
5.2.4. Definition: Inverse Relations

Let $R$ be a relation from $A$ to $B$. The inverse relation, denoted $R^{-1}$, from $B$ to $A$ is defined as

$$R^{-1} = \{(y, x) : (x, y) \in R\}.$$ 

Notes:

* For a relation $R$ from $A$ to $B$, the inverse relation $R^{-1}$ can be defined by interchanging the elements of all the ordered pairs of $R$. This turns out to be easier for a finite (listed) relation than an infinite (given by formula) relation.

* $\text{Dom } R^{-1} = \text{Range } R \subseteq B$ and $\text{Range } R^{-1} = \text{Dom } R \subseteq A$.

Examples:

- Define a relation $R$ on $\mathbb{N}$ as $R = \{(x, y) : y = 2x\}$.
  
  o Write down 3 elements of $R$.
    
    (1, 2), (2, 4), (3, 6)

  o Write down 3 elements of $R^{-1}$
    
    (2, 1), (4, 2), (6, 3)
Sketch a graph of $R$ and $R^{-1}$ on coordinate axis, circle elements of $R^{-1}$.

Write down a simple definition for $R^{-1}$.

$$R^{-1} = \{(y, x) : y = 2x\}$$
$$= \{(x, y) : x = 2y\}$$
$$= \{(x, y) : y = \frac{1}{2}x\}$$

Exercise:

Let $S$ be the identity relation on the set of reals. What is $S^{-1}$?

$$S = \{(x, x) : x \in \mathbb{R}\}$$
5.2.5. Directed Graph of a Relation

When a relation $R$ is defined on a set $A$, we can represent it with a directed graph. This is a graph in which an arrow is drawn from each point in $A$ to each related point.

$\forall x, y \in A$, there is an arrow from $x$ to $y \Leftrightarrow xRy$,
$\Leftrightarrow (x, y) \in R$

If a point is related to itself, a loop is drawn that extends out from the point and goes back to it.

Example:

- Let $A = \{0, 1, 2, 3\}$ and let $R_1$ be the relation on $A$ given by $R_1 = \{(0, 0), (0, 1), (0, 2), (3, 0)\}$. Draw the directed graph of $R_1$. 

![Directed Graph of a Relation](image-url)
Exercise:

- Let $A = \{0, 1, 2, 3\}$ and let $R_2$ be the relation on $A$ given by $R_2 = \{(0, 0), (1, 2), (2, 2)\}$.

Draw the directed graph of $R_2$. 