COVERINGS OF SKEW-PRODUCTS AND CROSSED PRODUCTS
BY COACTIONS

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ABSTRACT. Consider a projective limit $G$ of finite groups $G_n$. Fix a compatible family $\delta^n$ of coactions of the $G_n$ on a $C^*$-algebra $A$. From this data we obtain a coaction $\delta$ of $G$ on $A$. We show that the coaction crossed product of $A$ by $\delta$ is isomorphic to a direct limit of the coaction crossed products of $A$ by the $\delta^n$.

If $A = C^*(\Lambda)$ for some $k$-graph $\Lambda$, and if the coactions $\delta^n$ correspond to skew-products of $\Lambda$, then we can say more. We prove that the coaction crossed-product of $C^*(\Lambda)$ by $\delta$ may be realised as a full corner of the $C^*$-algebra of a $(k+1)$-graph. We then explore connections with Yeend’s topological higher-rank graphs and their $C^*$-algebras.

1. Introduction

In this article we investigate how certain coactions of discrete groups on $k$-graph $C^*$-algebras behave under inductive limits. This leads to interesting new connections between $k$-graph $C^*$-algebras, nonabelian duality, and Yeend’s topological higher-rank graph $C^*$-algebras.

We consider a particularly tractable class of coactions of finite groups on $k$-graph $C^*$-algebras. A functor $c$ from a $k$-graph $\Lambda$ to a discrete group $G$ gives rise to two natural constructions. At the level of $k$-graphs, one may construct the skew-product $k$-graph $\Lambda \times_c G$; and at the level of $C^*$-algebras, $c$ induces a coaction $\delta$ of $G$ on $C^*(\Lambda)$. It is a theorem of [15] that these two constructions are compatible in the sense that the $k$-graph algebra $C^*(\Lambda \times_c G)$ is canonically isomorphic to the coaction crossed-product $C^*$-algebra $C^*(\Lambda) \times_\delta G$.

The skew-product construction is also related to discrete topology: given a regular covering map from a $k$-graph $\Gamma$ to a connected $k$-graph $\Lambda$, one obtains an isomorphism of $\Gamma$ with a skew-product of $\Lambda$ by a discrete group $G$ [15, Theorem 6.11]. Further results of [15] then show how to realise the $C^*$-algebra of $\Gamma$ as a coaction crossed product of the $C^*$-algebra of $\Lambda$.

The results of [12] investigate the relationship between $C^*(\Lambda)$ and $C^*(\Gamma)$ from a different point of view. Specifically, they show how a covering $p$ of a $k$-graph $\Lambda$ by a $k$-graph $\Gamma$ induces an inclusion of $C^*(\Lambda)$ into $C^*(\Gamma)$. A sequence of compatible coverings therefore gives rise to an inductive limit of $C^*$-algebras. The main results

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of [12] show how to realise this inductive limit as a full corner in the $C^*$-algebra of a $(k+1)$-graph.

We can combine the ideas discussed in the preceding three paragraphs as follows. Fix a $k$-graph $\Lambda$, a projective sequence of finite groups $G_n$, and a sequence of functors $c_n : \Lambda \to G_n$ which are compatible with the projective structure. We obtain from this data a sequence of skew-products $\Lambda \times c_n G_n$ which form a sequence of compatible coverings of $\Lambda$. By results of [12], we therefore obtain an inductive system of $k$-graph $C^*$-algebras $C^*(\Lambda \times c_n G_n)$. The results of [15] show that each $C^*(\Lambda \times c_n G_n)$ is isomorphic to a coaction crossed product $C^*(\Lambda) \times_{\delta_n} G_n$. It is therefore natural to ask whether the direct limit $C^*$-algebra $\varinjlim(C^*(\Lambda \times c_n G_n))$ is isomorphic to a coaction crossed product of $C^*(\Lambda)$ by the projective limit group $\varinjlim G_n$.

After summarising in Section 2 the background needed for our results, we answer this question in the affirmative and in greater generality in Theorem 3.1. Given a $C^*$-algebra $A$, a projective limit of finite groups $G_n$ and a compatible system of coactions of the $G_n$ on $A$, we show that there is an associated coaction $\delta$ of $\varinjlim G_n$ on $A$, such that $A \times_{\delta} (\varinjlim G_n) \cong \varinjlim (A \times_{\delta_n} G_n)$.

In Section 4 we consider the consequences of Theorem 3.1 in the original motivating context of $k$-graph $C^*$-algebras. We consider a $k$-graph $\Lambda$ together with functors $c_n : \Lambda \to G_n$ which are consistent with the projective limit structure on the $G_n$. In Theorem 4.3 we use Theorem 3.1 to deduce that $C^*(\Lambda) \times_{\delta} G$ is isomorphic to $\varinjlim(C^*(\Lambda) \times_{\delta_n} G_n)$. Using results of [12], we realise $C^*(\Lambda) \times_{\delta} G$ as a full corner in a $(k+1)$-graph algebra (Corollary 4.5). We digress in Section 5 to investigate simplicity of $C^*(\Lambda) \times_{\delta} G$ via the results of [18].

We conclude in Section 6 with an investigation of the connection between our results and Yeend’s notion of a topological $k$-graph [21, 20]. We construct from an infinite sequence of coverings $p_n : \Lambda_{n+1} \to \Lambda_n$ of $k$-graphs a projective limit $\Lambda$ which is a topological $k$-graph. We show that the $C^*$-algebra $C^*(\Lambda)$ of this topological $k$-graph coincides with the direct limit of the $C^*(\Lambda_n)$ under the inclusions induced by the $p_n$. In particular, the system of cocycles $c_n : \Lambda \to G_n$ discussed in the preceding paragraph yields a cocycle $c : \Lambda \to G := \varinjlim(G_n, g_n)$, the skew-product $\Lambda \times_c G$ is a topological $k$-graph, and the $C^*$-algebras $C^*(\Lambda \times_c G)$ and $C^*(\Lambda) \times_{\delta} G$ are isomorphic, generalising the corresponding result [15, Theorem 7.1(ii)] for discrete groups.

2. Preliminaries

Throughout this paper, we regard $\mathbb{N}^k$ as a semigroup under addition with identity element 0. We denote the canonical generators of $\mathbb{N}^k$ by $e_1, \ldots, e_k$. For $n \in \mathbb{N}^k$, we denote its coordinates by $n_1, \ldots, n_k \in \mathbb{N}$ so that $n = \sum_{i=1}^k n_ie_i$. For $m, n \in \mathbb{N}^k$, we write $m \leq n$ if $m_i \leq n_i$ for all $i \in \{1, \ldots, k\}$.

We will at times need to identify $\mathbb{N}^k$ with the subsemigroup of $\mathbb{N}^{k+1}$ consisting of elements $n$ whose last coordinate is equal to zero. For $n \in \mathbb{N}^k$, we write $(n, 0)$ for the corresponding element of $\mathbb{N}^{k+1}$. When convenient, we regard $\mathbb{N}^k$ as (the morphisms of) a category with a single object in which the composition map is the usual addition operation in $\mathbb{N}^k$. 
2.1. $k$-graphs. Higher-rank graphs are defined in terms of categories. In this paper, given a category $\mathcal{C}$, we will identify the objects with the identity morphisms, and think of $\mathcal{C}$ as the collection of morphisms only. We will write composition in our categories by juxtaposition.

Fix an integer $k \geq 1$. A $k$-graph is a pair $(\Lambda, d)$ where $\Lambda$ is a countable category and $d : \Lambda \to \mathbb{N}^k$ is a functor satisfying the factorisation property: whenever $\lambda \in \Lambda$ and $m,n \in \mathbb{N}^k$ satisfy $d(\lambda) = m + n$, there are unique $\mu, \nu \in \Lambda$ with $d(\mu) = m$, $d(\nu) = n$, and $\lambda = \mu \nu$. If $p \leq q \leq d(\lambda)$, we denote by $\lambda(p,q)$ the unique path in $\Lambda^{q-p}$ such that $\lambda = \lambda'(p,q)\lambda''$ for some $\lambda' \in \Lambda^p$ and $\lambda'' \in \Lambda^{d(\lambda)-q}$.

For $n \in \mathbb{N}^k$, we write $\Lambda^n$ for $d^{-1}(n)$. Applying the factorisation property with $m = 0$, $n = d(\lambda)$ and with $m = d(\lambda)$, $n = 0$, one shows that $\Lambda^0$ is precisely the set of identity morphisms in $\Lambda$. The codomain and domain maps in $\Lambda$ therefore determine maps $r,s : \Lambda \to \Lambda^0$. We think of $\Lambda^0$ as the vertices — and $\Lambda$ as the paths — in a “$k$-dimensional directed graph.”

Given $F \subset \Lambda$ and $v \in \Lambda^0$ we write $vF$ for $F \cap r^{-1}(v)$ and $Fv$ for $F \cap s^{-1}(v)$. We say that $\Lambda$ is row-finite if $v\Lambda^n$ is a finite set for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$, and we say that $\Lambda$ has no sources if $v\Lambda^n$ is always nonempty.

We denote by $\Omega_k$ the $k$-graph $\Omega_k := \{(p,q) \in \mathbb{N}^k \times \mathbb{N}^k : p \leq q\}$ with $r(p,q) := (p,p)$, $s(p,q) := (q,q)$ and $d(p,q) := q-p$. As a notational convenience, we will henceforth denote $(p,p) \in \Omega_k^n$ by $p$. An infinite path in a $k$-graph $\Lambda$ is a degree-preserving functor (otherwise known as a $k$-graph morphism) $x : \Omega_k \to \Lambda$. The collection of all infinite paths is denoted $\Lambda^{\infty}$. We write $r(x)$ for $x(0)$, and think of this as the range of $x$.

For $\lambda \in \Lambda$ and $x \in s(\lambda)\Lambda^{\infty}$, there is a unique infinite path $\lambda x \in r(\lambda)\Lambda^{\infty}$ satisfying $(\lambda x)(0,p) := \lambda x(0,p - d(\lambda))$ for all $p \geq d(\lambda)$. In particular, $r(x)x = x$ for all $x \in \Lambda^{\infty}$, so we denote $\{x \in \Lambda^{\infty} : r(x) = v\}$ by $v\Lambda^{\infty}$. If $\Lambda$ has no sources, then $v\Lambda^{\infty}$ is nonempty for all $v \in \Lambda^0$.

The factorisation property also guarantees that for $x \in \Lambda^{\infty}$ and $n \in \mathbb{N}^k$ there is a unique infinite path $\sigma^n(x) \in x(n)\Lambda^{\infty}$ such that $\sigma^n(x)(p,q) = x(p+n,q+n)$. We somewhat imprecisely refer to $\sigma$ as the shift map. Note that $\sigma^{d(\lambda)}(\lambda x) = x$ for all $\lambda \in \Lambda$, $x \in s(\lambda)\Lambda^{\infty}$, and $x = x(0,n)\sigma^n(x)$ for all $x \in \Lambda^{\infty}$ and $n \in \mathbb{N}^k$.

We say a row-finite $k$-graph $\Lambda$ with no sources is cofinal if, for every $v \in \Lambda^0$ and every $x \in \Lambda^{\infty}$ there exists $n \in \mathbb{N}^k$ such that $v\Lambda x(n) \neq \emptyset$. Given $m \neq n \in \mathbb{N}^k$ and $v \in \Lambda^0$, we say that $\Lambda$ has local periodicity $m,n$ at $v$ if $\sigma^m(x) = \sigma^n(x)$ for all $x \in v\Lambda^{\infty}$. We say that $\Lambda$ has no local periodicity if, for every $m,n \in \mathbb{N}^k$ and every $v \in \Lambda^0$, we have $\sigma^m(x) \neq \sigma^n(x)$ for some $x \in v\Lambda^{\infty}$.

2.2. Skew-products. Let $\Lambda$ be a $k$-graph, and let $G$ be a group. A cocycle $c : \Lambda \to G$ is a functor from $\Lambda$ to $G$ where the latter is regarded as a category with one object. That is, $c : \Lambda \to G$ satisfies $c(\mu \nu) = c(\mu)c(\nu)$ whenever $\mu, \nu$ can be composed in $\Lambda$. It follows that $c(v) = e$ for all $v \in \Lambda^0$, where $e \in G$ is the identity element.

Given a cocycle $c : \Lambda \to G$, we can form the skew-product $k$-graph $\Lambda \times_c G$. We follow the conventions of [15, Section 6]. Note that these are different to those of [9]
Section 5. The paths in $\Lambda \times_c G$ are

$$(\Lambda \times_c G)^n := \Lambda^n \times G$$

for each $n \in \mathbb{N}$. The range and source maps $r, s : \Lambda \times_c G \to (\Lambda \times_c G)^0$ are given by $r(\lambda, g) := (r(\lambda), c(\lambda)g)$ and $s(\lambda, g) := (s(\lambda), g)$. Composition is determined by $(\mu, c(\nu)g)(v, g) = (\mu\nu, g)$. It is shown in [15, Section 6] that $\Lambda \times_c G$ is a k-graph.

2.3. Coverings and $(k+1)$-graphs. We recall here some definitions and results from [12] regarding coverings of k-graphs. Given k-graphs $\Lambda$ and $\Gamma$, a k-graph morphism $\phi : \Lambda \to \Gamma$ is a functor which respects the degree maps. A covering of k-graphs is a triple $(\Lambda, \Gamma, p)$ where $\Lambda$ and $\Gamma$ are k-graphs, and $p : \Gamma \to \Lambda$ is a k-graph morphism which is surjective and is locally bijective in the sense that for each $v \in \Gamma^0$, the restrictions $p|_{\delta v} : v\Gamma \to p(v)\Lambda$ and $p|_{\Gamma v} : \Gamma v \to \Lambda p(v)$ are bijective.

Remark 2.1. What we have called a covering of k-graphs is a special case of what was called a “covering system of k-graphs” in [12]. In general, a covering system consists of a covering of k-graphs together with some extra combinatorial data. We do not need the extra generality, so we have dropped the word “system.”

A covering $(\Lambda, \Gamma, p)$ is row-finite if $\Lambda$ (equivalently $\Gamma$) is row-finite, and $|p^{-1}(v)| < \infty$ for all $v \in \Lambda^0$. Proposition 2.6 of [12] shows that we can associate to a row-finite covering $p : \Gamma \to \Lambda$ of k-graphs a row-finite $(k+1)$-graph $\Lambda^2 \Gamma$ containing disjoint copies $i(\Lambda)$ and $j(\Gamma)$ of $\Lambda$ and $\Gamma$ with an edge of degree $e_{k+1}$ connecting each vertex $j(v) \in j(\Gamma^0)$ to its image $i(p(v)) \in i(\Lambda^0)$.

More generally, given a sequence $(\Lambda_n, \Lambda_{n+1}, p_n)$ of row-finite coverings of k-graphs, Corollary 2.10 of [12] shows how to build a $(k+1)$-graph $\lim(\Lambda_n; p_n)$, which we sometimes refer to as a tower graph, containing a copy $i_n(\Lambda_n)$ of each individual k-graph in the sequence, and an edge of degree $e_{k+1}$ connecting each $i_{n+1}(v) \in i_{n+1}(\Lambda_{n+1}^0)$ to its image $i_n(p_n(v)) \in i_n(\Lambda_n^0)$. The $(k+1)$-graph $\lim(\Lambda_n; p_n)$ has no sources if the $\Lambda_n$ all have no sources.

Given a covering $(\Lambda, \Gamma, p)$, [12, Proposition 3.2 and Theorem 3.8] show that the covering map $p : \Gamma \to \Lambda$ induces an inclusion $i_p : C^*(\Lambda) \to C^*(\Gamma)$. If $(\Lambda_n, \Lambda_{n+1}, p_n)_{n=1}^\infty$ is a sequence of coverings, the $(k+1)$-graph algebra $C^*(\lim(\Lambda_n; p_n))$ is Morita equivalent to the direct limit $\lim_i(C^*(\Lambda_n), i_p)$.

2.4. Coactions and coaction crossed products. Here we give some background on group coactions on $C^*$-algebras and coaction crossed products. For a detailed treatment of coactions and coaction crossed-products, see [1, Appendix A].

Given a locally compact group $G$, we write $C^*(G)$ for the full group $C^*$-algebra of $G$. We prefer to identify $G$ with its canonical image in $M(C^*(G))$, but when confusion is likely we use $s \mapsto u(s)$ for the canonical inclusion of $G$ in $M(C^*(G))$. If $A$ and $B$ are $C^*$-algebras, then $A \otimes B$ denotes the spatial tensor product. For a group $G$, we write $\delta_G$ for the natural comultiplication $\delta_G : C^*(G) \to M(C^*(G) \otimes C^*(G))$ given by the integrated form of the strictly continuous map which takes $s \in G$ to $s \otimes s \in UM(C^*(G) \otimes C^*(G))$.

As in [1, Definition A.21], a coaction of a group $G$ on a $C^*$-algebra $A$ is an injective homomorphism $\delta : A \to M(A \otimes C^*(G))$ satisfying
(1) the coaction identity $(\delta \otimes 1_G) \circ \delta = (1_A \otimes \delta_G) \circ \delta$ (as maps from $A$ to $M(A \otimes C^*(G) \otimes C^*(G))$); and

(2) the nondegeneracy condition $\delta(A)(1_A \otimes C^*(G)) = M(A \otimes C^*(G))$.

As in [7,8], the nondegeneracy condition (2) — rather than the weaker condition that $\delta$ be a nondegenerate homomorphism — is part of our definition of a coaction (cf. Definition A.21 and Remark A.22(3) of [4]). Since we will be dealing only with coactions of compact (and hence amenable) groups, the two conditions are equivalent in our setting in any case (see [14, Lemma 3.8]).

Let $\delta : A \to M(A \otimes C^*(G))$ be a coaction of $G$ on $A$. We regard the map which takes $s \in G$ to $u(s) \in M(C^*(C))$ as an element $w_G$ of $UM(C_0(G) \otimes C^*(G))$. Given a $C^*$-algebra $D$, a covariant homomorphism of $(A, G, \delta)$ into $M(D)$ is a pair $(\pi, \mu)$ of homomorphisms $\pi : A \to M(D)$ and $\mu : C_0(G) \to M(D)$ satisfying the covariance condition:

$$(\pi \otimes \text{id}_D) \circ \delta(a) = (\mu \otimes \text{id}_G)(w_G)(\pi(a) \otimes 1)(\mu \otimes \text{id}_G)(w_G)^*$$

for all $a \in A$.

The coaction crossed-product $A \rtimes_\delta G$ is the universal $C^*$-algebra generated by the image of a universal covariant representation $(j_A, j_G)$ of $(A, G, \delta)$ (see [4, Theorem A.41]).

3. Continuity of coaction crossed-products

In this section, we prove a general result regarding the continuity of the coaction crossed-product construction. Specifically, consider a projective system of finite groups $G_n$ and a system of compatible coactions $\delta^n$ of the $G_n$ on a fixed $C^*$-algebra $A$. We show that this determines a coaction $\delta$ of the projective limit $\varprojlim G_n$ on $A$, and that the coaction crossed product of $A$ by $\delta$ is isomorphic to a direct limit of the coaction crossed products of $A$ by the $\delta^n$.

The application we have in mind is when $A = C^*(\Lambda)$ is a $k$-graph algebra, and the $\delta^n$ arise from a system of skew-products of $\Lambda$ by the $G_n$. We consider this situation in Section 4.

**Theorem 3.1.** Let $A$ be a $C^*$-algebra, and let

$$\cdots \xrightarrow{q_{n+1}} G_{n+1} \xrightarrow{q_n} G_n \xrightarrow{q_1} G_1$$

be surjective homomorphisms of finite groups. For each $n$ let $\delta^n$ be a coaction of $G_n$ on $A$. Suppose that the diagram

$$(1) \quad \xymatrix{ A \ar[r]^{\delta^{n+1}} \ar[rd]_{\delta^n} & M(A \otimes C^*(G_{n+1})) \ar[d]^{\text{id} \otimes q_n} \cr & M(A \otimes C^*(G_n)) }$$

commutes for each $n$.

For each $n$, write $Q_n$ for the canonical surjective homomorphism of $\varprojlim(G_m, q_m)$ onto $G_n$; write $q_n^* : C(G_n) \to C(G_{n+1})$ for the induced map $q_n^*(f) := f \circ q_n$; and write $J_n$ for the homomorphism $J_n := j^A_{\delta^{n+1}} \times (j_{G_{n+1}} \circ q_n^*)$ from $A \times_{\delta^n} G_n$ to $A \times_{\delta^{n+1}} G_{n+1}$. 


Then there is a unique coaction $\delta$ of $\varprojlim (G_n, q_n)$ on $A$ such that:

(i) the diagrams

$$
\begin{array}{ccc}
A & \xrightarrow{\delta} & M(A \otimes C^*(\varprojlim G_n)) \\
\downarrow{\delta^n} & & \downarrow{id \otimes Q_n} \\
M(A \otimes C^*(G_n)) & \end{array}
$$

commute; and

(ii) $A \times_\delta \varprojlim (G_n, q_n) \cong \varprojlim (A \times_{\delta^n} G_n, J_n)$.

**Remark 3.2.** In equation (1) we could replace $M(A \otimes C^*(G_n))$ with $A \otimes C^*(G_n)$ and $M(A \otimes C^*(G_{n+1}))$ with $A \otimes C^*(G_{n+1})$ because $G_n, G_{n+1}$ are discrete.

**Proof of Theorem 3.1.** Put

$$
\begin{align*}
G &= \varprojlim G_n \\
B_n &= A \times_{\delta^n} G_n \\
J_n &= j_{\delta^{n+1}} \times (j_{G_{n+1}} \circ q_n^*) : B_n \to B_{n+1} \\
B &= \varinjlim (B_n, J_n) \\
K_n &= \text{the canonical embedding } B_n \to B.
\end{align*}
$$

We aim to apply Landstad duality [17]: we will show that $B$ is of the form $C \times_\delta G$ for some coaction $(C, G, \delta)$, and then we will show that we can take $C = A$. To apply [17] we need:

- an action $\alpha$ of $G$ on $B$, and
- a nondegenerate homomorphism $\mu : C(G) \to M(B)$ which is $rt - \alpha$ equivariant, where $rt$ is the action of $G$ on $C(G)$ by right translation.

Then [17] will provide a coaction $(C, G, \delta)$ and an isomorphism

$$
\theta : B \xrightarrow{\cong} C \times_\delta G
$$

such that

$$
\theta \circ \mu = j_G \quad \text{and} \quad \theta(B^\alpha) = j_C(C).
$$

This is simpler than the general construction of [17], because our group $G$ is compact (and then we are really using Landstad’s unpublished characterisation [13] of crossed products by coactions of compact groups).

We begin by constructing the action $\alpha$: for each $s \in G$ the diagrams

$$
\begin{array}{ccc}
B_{n+1} & \xrightarrow{\delta^n Q_{n+1}(s)} & B_{n+1} \\
J_n \downarrow{} & & \downarrow{J_n} \\
B_n & \xrightarrow{\delta^n Q_n(s)} & B_n
\end{array}
$$
commute because
\[
\tilde{\delta}^{n+1} Q_{n+1(s)} \circ J_n \circ j_A^n = \tilde{\delta}^{n+1} Q_{n+1(s)} \circ j_A^{n+1} \\
= j_A^{n+1} \\
= J_n \circ j_A^n \\
= J_n \circ \tilde{\delta}^n Q_n(s) \circ j_A^n
\]
and
\[
\tilde{\delta}^{n+1} Q_{n+1(s)} \circ J_n \circ j_G^n = \tilde{\delta}^{n+1} Q_{n+1(s)} \circ j_G^{n+1} \circ q_s^n \\
= j_G^{n+1} \circ rt Q_{n+1(s)} \circ q_s^n \\
= j_G^{n+1} \circ q_s^n \circ rt q_n Q_{n+1(s)} \\
= J_n \circ j_G^n \circ rt Q_n(s) \\
= J_n \circ \tilde{\delta}^n Q_n(s) \circ j_G^n.
\]
Thus, because the \(\tilde{\delta}_n Q_n(s)\) are automorphisms, by universality there is a unique automorphism \(\alpha_s\) such that the diagrams
\[
\begin{array}{ccc}
B & \overset{\alpha_s}{\rightarrow} & B \\
\downarrow K_n & & \downarrow K_n \\
B_n & \overset{\tilde{\delta}_n Q_n(s)}{\rightarrow} & B_n
\end{array}
\]
commute. It is easy to check that this gives a homomorphism \(\alpha : G \rightarrow \text{Aut } B\). We verify continuity: each function \(s \mapsto \alpha_s(b)\) for \(b \in B\) is a uniform limit of functions of the form \(s \mapsto \alpha_s \circ K_n(b)\) for \(b \in B_n\). But we have
\[
\alpha_s \circ K_n(b) = K_n \circ \tilde{\delta}_n Q_n(s)(b),
\]
which is continuous since \(K_n\), \(Q_n\), and \(t \mapsto \tilde{\delta}_n(t) : G_n \rightarrow B_n\) are.

We turn to the construction of the nondegenerate homomorphism \(\mu\): first note that the increasing union \(\bigcup Q_n^\ast(C(G_n))\) is dense in \(C(G)\) by the Stone-Weierstrass Theorem, and it follows that there is an isomorphism
\[
C(G) \cong \lim_{\rightarrow} (C(G_n), q_n^\ast)
\]
taking \(Q_n\) to the canonical embedding. We have a compatible sequence of nondegenerate homomorphisms
\[
\begin{array}{ccc}
C(G_{n+1}) & \overset{j_G^{n+1}}{\rightarrow} & M(B_{n+1}) \\
\downarrow q_n^\ast & & \downarrow J_n \\
C(G_n) & \overset{j_G^n}{\rightarrow} & M(B_n)
\end{array}
\]
so by universality there is a unique homomorphism $\mu$ making the diagrams

$$
\begin{align*}
C(G) & \xrightarrow{\mu} M(B) \\
Q_n^* & \uparrow K_n \\
C(G_n) & \xrightarrow{j_G} M(B_n)
\end{align*}
$$

commute. Moreover, $\mu$ is nondegenerate since $K_n$ and $j_{G_n}$ are.

We now have $\alpha$ and $\mu$, and the equivariance

$$
\alpha_s \circ \mu = \mu \circ \text{rt}_s
$$

follows from

$$
\alpha_s \circ \mu \circ Q_n^* = \alpha_s \circ K_n \circ j_{G_n} = K_n \circ \hat{\delta_n} \circ j_{G_n} = K_n \circ j_{G_n} \circ \text{rt}_{Q_n(s)} = \mu \circ Q_n^* \circ \text{rt}_{Q_n(s)} = \mu \circ \text{rt}_s \circ Q_n^*.
$$

Thus we can apply [17] to obtain a coaction $(C, G, \delta)$ and an isomorphism

$$
\theta : B \xrightarrow{\cong} C \times_{\delta} G
$$

such that

$$
\theta \circ \mu = j_G \quad \text{and} \quad \theta(B^\alpha) = j_C(C).
$$

We want to take $C = A$. Note that we have a compatible sequence of nondegenerate homomorphisms

$$
\begin{align*}
A & \xrightarrow{j_A^{n+1}} B_{n+1} \\
j_A^n & \downarrow J_n \\
A & \xrightarrow{j_A^n} B_n
\end{align*}
$$

so by universality there is a unique homomorphism $j$ making the diagrams

$$
\begin{align*}
A & \xrightarrow{j} B \\
j_A^n & \downarrow K_n \\
A & \xrightarrow{j_A^n} B_n
\end{align*}
$$

commute. Moreover, $j$ is injective and nondegenerate since $K_n$ and $j_A^n$ are. Because $j, j_C, \text{and} \theta$ are faithful, to show that we can take $C = A$ it suffices to show that

$$
j(A) = B^\alpha.
$$

We have

$$
j(A) \subset B^\alpha.
$$
because

\[ \alpha_s \circ j = \alpha_s \circ K_n \circ j_A^\delta n \]
\[ = K_n \circ \hat{\delta}^n Q_n(s) \circ j_A^\delta n \]
\[ = K_n \circ j_A^\delta n \]
\[ = j. \]

For the opposite containment, let \( b \in B^n \). There is a sequence \( b_n \in B_n \) such that \( K_n(b_n) \to b \). The functions \( s \mapsto \alpha_s \circ K_n(b) \) converge uniformly to the function \( s \mapsto \alpha_s(b) \), so

\[ \int_G \alpha_s \circ K_n(b_n) \, ds \to \int_G \alpha_s(b) \, ds = b. \]

We have

\[ \int_G \alpha_s \circ K_n(b_n) \, ds = \int_G K_n \circ \hat{\delta}^n Q_n(s)(b_n) \, ds = K_n \left( \int_G \hat{\delta}^n Q_n(s)(b_n) \, ds \right). \]

Since

\[ \int_G \hat{\delta}^n Q_n(s)(b_n) \, ds \in B_n^{\hat{\delta} n} = j_A^\delta n(A), \]

we conclude that

\[ b \in K_n \circ j_A^\delta n(A) = j(A). \]

Therefore we can take \( C = A \), so that we have a coaction \( (A, G, \delta) \) and an isomorphism

\[ \theta : B \xrightarrow{\cong} A \times_\delta G \]

such that

\[ \theta \circ \mu = j_G. \]

We have proved (ii). For (i), we calculate:

\[ (j_A^\delta \otimes \delta) \circ (\text{id} \otimes q_n) \circ \delta = (\text{id} \otimes q_n) \circ (j_A^\delta \otimes \text{id}) \circ \delta \]
\[ = (\text{id} \otimes q_n) \circ \text{Ad}(j_G \otimes \text{id})(w_G) \circ (j_A^\delta \otimes 1) \]
\[ = \text{Ad}(\text{id} \otimes q_n)((j_G \otimes \text{id})(w_G)) \circ (\text{id} \otimes q_n) \circ (j_A^\delta \otimes 1) \]
\[ = \text{Ad}(j_G \otimes \text{id})((\text{id} \otimes q_n)(w_G)) \circ (j_A^\delta \otimes 1) \]
\[ = \text{Ad}(j_G \otimes Q_n^* \otimes \text{id})(w_{G_n}) \circ (j_A^\delta \otimes 1) \]
\[ = \text{Ad}(K_n \circ j_G \otimes \text{id})(w_{G_n}) \circ (K_n \circ j_A^\delta \otimes 1) \]
\[ = (K_n \otimes \text{id}) \circ \text{Ad}(j_G \otimes \text{id})(w_{G_n}) \circ (j_A^\delta \otimes 1) \]
\[ = (K_n \otimes \text{id}) \circ (j_A^\delta_n \otimes \text{id}) \circ \delta^n \]
\[ = (K_n \circ j_A^\delta_n \otimes \text{id}) \circ \delta^n \]
\[ = (j_A^\delta \otimes \text{id}) \circ \delta^n. \]

Since \( j_A^\delta \) is faithful, we therefore have \( \delta \circ (\text{id} \otimes q_n) = \delta^n \). \( \square \)
The following application of Theorem 3.1 motivates the work of the following sections.

**Example 3.3.** Let $A = C(\mathbb{T}) = C^*(\mathbb{Z})$, and let $z$ denote the canonical generating unitary function $z \mapsto z$. For $n \in \mathbb{N}$, let $G_n := \mathbb{Z}/2^{n-1}\mathbb{Z}$ be the cyclic group of order $2^{n-1}$. We write $1$ for the canonical generator of $G_n$ and $0$ for the identity element. Let $g \mapsto u_n(g)$ denote the canonical embedding of $G_n$ into $C^*(G_n)$. Define $q_n : G_{n+1} \to G_n$ by $q_n(m) := m \text{ (mod } 2^{n-1})$, and write $q_n$ also for the homomorphism $q_n : C^*(G_{n+1}) \to C^*(G_n)$ satisfying $q_n(u_{n+1}(g)) = u_n(q_n(g))$. For each $n$, let $\delta^n$ be the coaction of $G_n$ on $A$ determined by $\delta^n(z) := z \otimes u_n(1)$.

Let $g \mapsto u(g)$ denote the canonical embedding of $\lim G_n$ as unitaries in the multiplier algebra of $C^*(\lim G_n)$. The coaction $\delta$ of $\lim G_n$ on $A$ described in Theorem 3.1 is the one determined by $\delta(z) := z \otimes u(1, 1, \ldots)$; the corresponding coaction crossed-product is known to be isomorphic to the Bunce-Deddens algebra of type $2^\infty$ (see, for example, [6, 8.4.4]).

4. Coverings of skew-products

In this section and the next, we adopt the following notation and assumptions.

**Notation 4.1.** Let $\Lambda$ be a connected row-finite $k$-graph with no sources. Fix a vertex $v \in A^0$, and denote by $\pi \Lambda$ the fundamental group $\pi_1(\Lambda, v)$ of $\Lambda$ with respect to $v$. Fix a cocycle $c : \Lambda \to \pi \Lambda$ such that the skew product $\Lambda \rtimes_c \pi \Lambda$ is isomorphic to the universal covering $\Omega\Lambda$ of $\Lambda$ (such a cocycle exists by [15, Corollary 6.5]).

Fix a descending chain of finite-index normal subgroups

\[ \cdots < H_{n+1} < H_n < \cdots < H_1 := \pi \Lambda. \]

For each $n$, let $G_n := \pi \Lambda/H_n$, and let $q_n : G_{n+1} \to G_n$ be the induced homomorphism

\[ q_n(g_{n+1}) := gH_n. \]

Then

\[ \cdots \xrightarrow{q_n^{-1}} G_{n+1} \xrightarrow{q_n} G_n \xrightarrow{q_{n-1}} \cdots \xrightarrow{q_1} G_1 := \{e\} \]

is a chain of surjective homomorphisms of finite groups. Let $G$ denote the projective limit group $\lim (G_n, q_n)$.

For each $n$, let $c_n : \Lambda \to G_n$ be the induced cocycle $c_n(\lambda) = c(\lambda)H_n$, and let

\[ \Lambda_n := \Lambda \rtimes c_n G_n \]

be the skew-product $k$-graph. Define covering maps $p_n : \Lambda_{n+1} \to \Lambda_n$ by $p_n(\lambda, g) := (\lambda, q_n(g))$.

As in [15, Theorem 7.1(1)], for each $n$ there is a coaction $\delta^n : C^*(\Lambda) \to C^*(\Lambda) \otimes C^*(G_n)$ determined by $\delta^n(s_\lambda) := s_\lambda \otimes c_n(\lambda)$. Denote by $J_n$ the inclusion

\[ J_n := j_{\Lambda_{n+1}}^\delta \times (j_{G_{n+1}} \circ q_n^*) : C^*(\Lambda) \times \delta^n G_n \to C^*(\Lambda) \times \delta^n G_{n+1} \]

described in Theorem 3.1(ii).

As in [15, Theorem 7.1(ii)], for each $n$ there is an isomorphism $\phi_n$ of $C^*(\Lambda_n) = C^*(\Lambda \rtimes c_n G_n)$ onto $C^*(\Lambda) \times \delta^n (G_n)$ which satisfies $\phi_n(s(\lambda, g)) := (s_\lambda, g)$. 

Example 4.2 (Example 3.3 Continued). Let \( \Lambda \) be the path category of the directed graph \( B_1 \) consisting of a single vertex \( v \) and a single edge \( f \) with \( r(f) = s(f) = v \). Note that as a category, \( \Lambda \) is isomorphic to \( \mathbb{N} \), and the degree functor is then the identity function from \( \mathbb{N} \) to itself.

Then \( \pi \Lambda \) is the free abelian group generated by the homotopy class of \( f \), and so is isomorphic to \( \mathbb{Z} \). We define a functor \( c : \Lambda \to \mathbb{Z} \) by \( c(f) = 1 \).

For each \( n \in \mathbb{N} \), let \( H_n := 2^{n-1}\mathbb{Z} \subset \mathbb{Z} \), so that \( \cdots \subset H_{n+1} \subset H_n \subset \cdots \subset H_1 := \pi \Lambda \) is a descending chain of finite-index normal subgroups. For each \( n \), \( G_n := \mathbb{Z}/H_n \) is the cyclic group of order \( 2^n - 1 \), and \( q_n : G_{n+1} \to G_n \) is the quotient map described in Example 3.3. The induced cocycle \( c_n : \Lambda \to G_n \) obtained from \( c \) is determined by \( c_n(f) = 1 \in \mathbb{Z}/2^{n-1}\mathbb{Z} \).

For \( p \in \mathbb{N} \), let \( C_p \) denote the simple cycle graph with \( p \) vertices: \( C_p^0 := \{ v^p_j : j \in \mathbb{Z}/p\mathbb{Z} \} \) and \( C_p^1 := \{ \varepsilon^p_j : j \in \mathbb{Z}/p\mathbb{Z} \} \), where \( r(\varepsilon^p_i) = v^p_i \) and \( s(\varepsilon^p_i) = v^p_{i+1 \mod p} \). For each \( n \), the skew-product graph \( \Lambda_n := \Lambda \times_{c_n} G_n \) is isomorphic to the path-category of \( C_{2^{n-1}} \). The associated covering map \( p_n : \Lambda_{n+1} \to \Lambda_n \) corresponds to the double-covering of \( C_{2^n} \) by \( C_{2^n} \) satisfying \( v^p_{1 \mod 2^n} \mapsto v^p_{1 \mod 2^{n-1}} \) and \( \varepsilon^p_i \mapsto \varepsilon^p_{i \mod 2^{n-1}} \).

Modulo a relabelling of the generators of \( \mathbb{N}^2 \), the 2-graph \( \lim(G_n, p_n) \) obtained from this data as in [12] (see Section 2.3) is isomorphic to the 2-graph of [16] Example 6.7. Combining this with the final observation of Example 3.3 we obtain a new proof that the \( C^* \)-algebra of this 2-graph is Morita equivalent to the Bunce-Deddens algebra of type \( 2^\infty \).

**Theorem 4.3.** Adopt the notation and assumptions [11]. Taking \( A := C^*(\Lambda) \), the coactions \( \delta^n \) and the quotient maps \( q_n \) make the diagrams (11) commute. Let \( \delta \) denote the coaction of \( G := \lim(G_n, q_n) \) on \( C^*(\Lambda) \) obtained from Theorem 3.3. Let \( P_0 \) denote the projection \( \sum_{v \in A_0} s_v \) in the multiplier algebra of \( C^*(\lim(\Lambda_n, p_n)) \). Then \( P_0 \) is full and

\[
P_0 C^*(\lim(\Lambda_n, p_n)) P_0 \cong C^*(\Lambda) \times \delta G.
\]

To prove this theorem, we first show that in the setting described above, the inclusions of \( k \)-graph algebras induced from the coverings \( p_n : \Lambda_{n+1} \to \Lambda_n \) as in [12] are compatible with the inclusions of coaction crossed products induced from the quotient maps \( q_n : G_{n+1} \to G_n \).

**Lemma 4.4.** With the notation and assumptions [11] fix \( n \in \mathbb{N} \), and let \( \iota_{p_n} \) be the inclusion of \( C^*(\Lambda_n) \) into \( C^*(\Lambda_{n+1}) \) obtained from [12] Proposition 3.3(iv)]. Then the inclusion \( \iota_n \) and the isomorphisms \( \phi_n, \phi_{n+1} \) of Notation 4.1 make the following diagram commute.

\[
\begin{array}{ccc}
C^*(\Lambda_n) & \xrightarrow{\iota_{p_n}} & C^*(\Lambda_{n+1}) \\
\downarrow \phi_n & & \downarrow \phi_{n+1} \\
C^*(\Lambda) \times \delta^n G_n & \xrightarrow{\iota_n} & C^*(\Lambda) \times \delta^{n+1} G_{n+1}
\end{array}
\]

**Proof.** By definition, we have

\[
\iota_{p_n}(s_{(\lambda, gH_n)}) = \sum_{p(\lambda', g'H_{n+1}) = (\lambda, gH_n)} s_{(\lambda', g'H_{n+1})}.
\]
By definition of \( p_n \), this becomes
\[
\ell_p(s_{(\lambda,gH_n)}) = \sum_{\{g'H_{n+1} \in G_{n+1} : g'H_n = gH_n\}} s_{(\lambda,g'H_{n+1})}.
\]
Hence
\[
\varphi_{n+1} \circ \ell_{P_n}(s_{(\lambda,gH_n)}) = \sum_{\{g'H_{n+1} \in G_{n+1} : g'H_n = gH_n\}} (s_{\lambda,g'H_{n+1}}).
\]
But this is precisely \( \ell(\varphi_n(s_{(\lambda,gH_n)})) \) by definition of \( \ell \) and \( \varphi_n \).

**Corollary 4.5.** With the notation and assumptions \([4.1]\), let \( P_0 \) denote the projection \( \sum_{\nu \in \Lambda_0} s_{\nu} \) in the multiplier algebra of \( C^*(\lim(\Lambda_n,p_n)) \). Then \( P_0 \) is full, and
\[
P_0 C^*(\lim(\Lambda_n,p_n)) P_0 \cong \lim(C^*(\Lambda) \times_{g_n} G_n, t_n).
\]

**Proof.** Equation (3.2) of \([12]\) implies that \( P_0 C^*(\lim(\Lambda_n,p_n)) P_0 \) is isomorphic to \( \lim(C^*(\Lambda_n), t_{p_n}) \). The latter is isomorphic to \( \lim(\tilde{C}^*(\Lambda) \times_{g_n} G_n, t_n) \) by Lemma 4.4 and the universal property of the direct limit.

**Proof of Theorem 4.3** It is immediate from the definitions of the maps involved that the maps \( \delta_n \) and \( q_n \) make the diagram (1) commute. The rest of the statement then follows from Corollary 4.5 and Theorem 3.1(ii).

5. Simplicity

In this section we frequently embed \( \mathbb{N}^k \) into \( \mathbb{N}^{k+1} \) as the subset consisting of elements whose \((k+1)\)st coordinate is equal to zero. For \( n \in \mathbb{N}^k \), we write \((n,0)\) for the corresponding element of \( \mathbb{N}^{k+1} \).

**Theorem 5.1.** Adopt the notation and assumptions \([4.1]\). Then \( C^*(\lim(\Lambda_n,p_n)) \) is simple if and only if the following two conditions are satisfied:

(i) each \( \Lambda_n \) is cofinal, and

(ii) whenever \( v \in \Lambda_0 \), \( p \neq q \in \mathbb{N}^k \) satisfy \( \sigma^p(x) = \sigma^q(x) \) for all \( x \in v\Lambda_0 \), there exists \( x \in v\Lambda^\infty \), \( l \in \mathbb{N}^k \) and \( N \in \mathbb{N} \) such that \( c_N(x(p,p+l)) \neq c_N(x(q,q+l)) \).

The idea is to prove the theorem by appealing to \([18\text{, Theorem 3.1}]\). To do this, we will first describe the infinite paths in \( \lim(\Lambda_n,p_n) \). We identify \( \lim(G_n,q_n) \) with the set of sequences \( g = (g_n)_{n=1}^\infty \) such that \( g_n(g_{n+1}) = g_n \) for all \( n \).

**Lemma 5.2.** Adopt the notation and assumptions \([4.1]\). Fix \( x \in \Lambda^\infty \) and \( g = (g_n)_{n=1}^\infty \in \lim(G_n,q_n) \). For each \( n \in \mathbb{N} \) there is a unique infinite path \( (x,g_n) \in \Lambda_n^\infty \) determined by \((x,g_n)(0,m) = (x(0,m),c_n(x(0,m))^{-1}g_n)\) for all \( m \in \mathbb{N}^k \). There is a unique infinite path \( x^g \in (\lim(\Lambda_n,p_n)) \) such that \( x^g(x^0,m,0) = x^0(m,0) \) for all \( m \in \mathbb{N}^k \) and \( x^g(nc_{k+1}) = (x(0),g_n) \) for all \( n \in \mathbb{N} \); moreover \( \sigma^{nc_{k+1}}(x^g)(0,m,0) = (x,g_n)(0,m) \) for all \( m \in \mathbb{N}^k \). Finally, every infinite path \( y \in (\lim(\Lambda_n,p_n))_0 \) is of the form \( \sigma^{nc_{k+1}}(x^g) \) for some \( n \in \mathbb{N} \), \( x \in \Lambda^\infty \) and \( g \in \lim(G_n,q_n) \).

**Proof.** That the formula given determines unique infinite paths \( (x,g_n), n \in \mathbb{N} \) follows from \([9\text{, Remarks 2.2}]\). That there is a unique infinite path \( x^g \) such that
\(x^g(0, (m, 0)) = x(0, m)\) for all \(m \in \mathbb{N}^k\) and \(x^g(ne_{k+1}) = (x(0), g_n)\) for all \(n \in \mathbb{N}\) follows from the observation that for each \(n \in \mathbb{N}\) there is a unique path
\[
\alpha = \alpha_{g,n} := e(x(0), g_1)e(x(0), g_2) \ldots e(x(0), g_n)
\]
with \(d(\alpha_{g,n}) = ne_{k+1}, r(\alpha) = x(0) \in \Lambda^0\) and \(s(\alpha) = (x(0), g_n) \in \Lambda^0_n\), and that for each \(m \in \mathbb{N}^k\),
\[
\alpha(x, g_n)(0, m) = x(0, m)e(x(m), c_1(x(0, m))^{-1}g_1) \ldots e(x(m), c_n(x(0, m))^{-1}g_n)
\]
is the unique minimal common extension of \(x(0, m)\) and \(\alpha\). This also establishes the assertion that \(\sigma^{ne_{k+1}}(x^g)(0, (m, 0)) = (x, g_n)(0, m)\) for all \(m \in \mathbb{N}^k\).

For the final assertion, fix \(y \in (\lim(\Lambda_n, p_n))^\infty\). We must have \(y(0) = (v, g_n)\) for some \(v \in \Lambda^0, g_n \in G_n = \pi\Lambda/H_n\) and \(n \in \mathbb{N}\). Let \(x \in \Lambda_n^\infty\) be the infinite path determined by \(x(0, m) := y(0, (m, 0))\) for all \(m \in \mathbb{N}^k\). By definition of \(\Lambda_n = \Lambda \times_{c_\rho} G_n\), we have \(x(0, m) := (\alpha_m, c_n(\alpha_m)^{-1}g_n)\) where each \(\alpha_m \in \nu\Lambda^m\) and \(g\) is the element of \(\pi\Lambda\) such that \(y(0) = v(g_n)\) as above. There is then an infinite path in \(x' \in \Lambda^\infty\) determined by \(x'(0, m) = \alpha_m\) for all \(m \in \mathbb{N}^k\). For \(n > i \geq 1\), inductively define \(g_i := g_i(g_{i+1})\), and for \(n < i\) let \(g_i\) be the unique element of \(G_i\) such that \(y((i-n)e_{k+1}) = (v, g_i)\); that such \(g_i\) exist follows from the definition of \(\lim(\Lambda_n, p_n)\).

Then \(g := (g_i)_{i \in H_\infty}^\infty\) is an element of \(\lim(G_n, q_n)\) by definition, and routine calculations using the definitions of the \(\Lambda_n\) show that \(x = \sigma^{ne_{k+1}}(x')^g\). \(\square\)

**Lemma 5.3.** Adopt the notation and assumptions \([4.1]\). Then the \((k + 1)\)-graph \(\lim(\Lambda_n, p_n)\) is cofinal if and only if each \(\Lambda_n\) is cofinal.

**Proof.** First suppose that each \(\Lambda_n\) is cofinal. Fix \(y \in \lim(\Lambda_n, p_n)\) and \(w \in \lim(\Lambda^0)\).

By Lemma 5.2 we have \(y = \sigma^{i_0e_{k+1}}(x^g)\) for some \(g = (g_n)_{n=1}^\infty \in \lim(G_n, q_n)\), some \(i_0 \in \mathbb{N}\) and some \(x \in \Lambda^\infty\). We must show that \(w(\lim(\Lambda_n, p_n))y(q) \neq \emptyset\) for some \(q\).

We have \(w \in \Lambda^0_m\) for some \(m \in \mathbb{N}\), so \(w = (w', h)\) for some \(h \in G_m\). If \(m < i_0\), fix any \(h' \in \pi\Lambda\) such that \(h'H_{i_0} = h\), and note that \(w(\lim(\Lambda_n, p_n))(w', h'H_{i_0})\) is nonempty, so that it suffices to show that \((w', h'H_{i_0})\lim(\Lambda_n, p_n))y(q) \neq \emptyset\) for some \(q\). That is to say, we may assume without loss of generality that \(m \geq i_0\). But now \(w \in \Lambda^0_m\) and \(\sigma^{0, \ldots, 0, m-i_0}(y) \in (\lim(\Lambda_n, p_n))^\infty\) with \(r(y) = \Lambda^0_{i_0}\). Since \(\Lambda_n\) is cofinal, we have \(w(\Lambda_{i_0})x(g_m)(q) \neq \emptyset\) for some \(q \in \mathbb{N}^k\) (recall that \(x, (g_i)_{i \in H_\infty}^\infty\) are such that \(y = \sigma^{i_0e_{k+1}}(x^g)\)). By definition, \((x, g_m)(q) = y(q_1, \ldots, q_k, m-i_0)\) and this shows that \(w(\lim(\Lambda_n, p_n))y(q) \neq \emptyset\) for \(q = (q_1, \ldots, q_k, m-n)\).

Now suppose that \(\lim(\Lambda_n, p_n)\) is cofinal. Fix \(n \in \mathbb{N}\) and a vertex \(w\) and an infinite path \(x\) in \(\Lambda_n\). Then \(x(0) = (v, gH_n)\) for some \(v \in \Lambda^0, g \in \pi\Lambda\). There are paths \(\alpha_m \in \Lambda^m, m \in \mathbb{N}^k\) determined by \(x(0, m) = (\alpha_m, c_n(\alpha_m)^{-1}gH_n)\); there is then an infinite path \(x' \in \Lambda^\infty\) such that \(x'(0, m) = \alpha_m\) for all \(m \in \mathbb{N}^k\). Let \(g_i := gH_i\) for all \(i \in \mathbb{N}\). By abuse of notation we denote by \(g\) the element \((gH_i)_{i=1}^\infty\) of \(\lim(G_n, q_n)\). Let \(y = \sigma^n(x')^g\) be the infinite path of \(\lim(\Lambda_n, p_n)\) provided by Lemma 5.2. As \(\lim(\Lambda_n, p_n)\) is cofinal, we may fix a path \(\lambda \in \lim(\Lambda_n, p_n)\) such that \(r(\lambda) = w\) and \(s(\lambda)\) lies on \(y\). By definition of \(y\), there exist \(n' \geq n\) and \(m \in \mathbb{N}^k\) such that \(s(\lambda) = (x'(m), c_{n'}(\alpha_m)^{-1}g_{n'})\). We then have \(d(\lambda)_{k+1} = n' - n,\) and we may factorise \(\lambda = \lambda'\lambda''\) where \(d(\lambda') = d(\lambda) - (n' - n)e_{k+1}\) and \(d(\lambda'') = (n' - n)e_{k+1}\). By construction
Lemma 5.4. Adopt the notation and assumptions 4.1. Then the \((k + 1)\)-graph 

\(\lim (\Lambda_n, p_n)\) has no local periodicity if and only if it satisfies condition 2 of Theorem 5.1.

Proof. First suppose that condition 2 of Theorem 5.1 holds. Fix a vertex \(v \in (\lim (\Lambda_n, p_n))^0\) and \(p \neq q \in \mathbb{N}^{k+1}\). So \(v \in \Lambda_n^0\) for some \(n\), and \(v\) therefore has the form \(v = (w, gH_n)\) for some \(w \in \Lambda_0^0\) and \(g \in \pi \Lambda\). We must show that there exists \(x \in \nu(\lim (\Lambda_n, p_n))^\infty\) such that \(\sigma^p(x) \neq \sigma^q(x)\).

We first consider the case where \(p_{k+1} \neq q_{k+1}\). By construction of the tower graph \(\lim (\Lambda_n, p_n)\), this forces the vertices \(x(p)\) and \(x(q)\) to lie in distinct \(\Lambda_n\) for any \(x \in \nu(\lim (\Lambda_n, p_n))^\infty\); in particular they cannot be equal.

Now suppose that \(p_{k+1} = q_{k+1}\). If \(\sigma^p(x) = \sigma^q(x)\) for every \(x \in \nu(\lim (\Lambda_n, p_n))^\infty\), then for any \(\alpha \in \nu(\lim (\Lambda_n, p_n))p^\infty_{k+1}\) and any \(y \in s(\alpha)(\lim (\Lambda_n, p_n))^\infty\), we have \(\sigma^p(\alpha y) = \sigma^q(\alpha y)\); that is,

\[
\sigma^{p-p_{k+1}e_{k+1}}(y) = \sigma^{q-q_{k+1}e_{k+1}}(y) \quad \text{for all } y \in s(\alpha)(\lim (\Lambda_n, p_n))^\infty.
\]

So we may assume without loss of generality that \(p_{k+1} = q_{k+1} = 0\). Write \(p'\) and \(q'\) for the elements of \(\mathbb{N}^k\) whose entries are the first \(k\) entries of \(p\) and \(q\).

We have \(v \in \Lambda_n\) for some \(n\), so there exists \(w \in \Lambda_0^0\) and \(g \in \pi \Lambda\) such that \(v = (w, gH_n)\). Suppose first that there exists \(x \in \nu \Lambda^\infty\) such that \(\sigma^{p'}(x) \neq \sigma^{q'}(x)\), then the infinite path \((x, gH_n) \in \nu \Lambda^\infty\) such that

\[
(x, gH_n)(0, m) := (x(0, m), c_n(x(0, m))^{-1}gH_n)
\]

also satisfies \(\sigma^{p'}((x, gH_n)) \neq \sigma^{q'}((x, gH_n))\). By Lemma 5.2, we may choose an infinite path \(y\) such that \(y|_{\mathbb{N}^\times\{0\}} = (x, gH_n)\), and then \(y \in \nu(\lim (\Lambda_n, p_n))^\infty\) satisfies \(\sigma^p(y) \neq \sigma^q(y)\).

Now suppose that every path \(x \in \nu \Lambda^\infty\) satisfies \(\sigma^{p'}(x) = \sigma^{q'}(x)\). Then by condition 2 of Theorem 5.1, we may fix \(x \in \nu \Lambda^\infty\) and \(N \in \mathbb{N}\) such that \(c_N(x(0, p')) \neq c_N(x(0, q'))\). It then follows from the definition of the \(c_j\) that \(c_j(x(0, p')) \neq c_j(x(0, q'))\) whenever \(j \geq N\). So with \(j := \max\{N, n\}\), we have

\[
(x, gH_j)((j - n)e_{k+1} + p) \neq (x, gH_j)((j - n)e_{k+1} + q),
\]

and therefore \(x(g)\) satisfies \(\sigma^p(x^g) \neq \sigma^q(x^g)\) as required. Hence condition 2 of Theorem 5.1 implies that \(\lim (\Lambda_n, p_n)\) has no local periodicity.

To show that if \(\lim (\Lambda_n, p_n)\) has no local periodicity then condition 2 of Theorem 5.1 holds, we prove the contrapositive statement. Suppose that condition 2 of Theorem 5.1 does not hold. Fix \(v \in \Lambda^0\) and \(p, q \in \mathbb{N}^k\) such that \(\sigma^p(x) = \sigma^q(x)\) for
all \( x \in v\Lambda^\infty \) and \( c_n(x(p, p + l)) = c_n(x(q, q + l)) \) for all \( n \in \mathbb{N} \), \( l \in \mathbb{N}^k \). Then for each \( x \in v\Lambda^\infty \) and each \( g = (g_n)^\infty_{n=1} \in \lim(G_n, p_n) \), we have \( \sigma^p(x, g_n)(0, l) = \sigma^n(x, g_n)(0, l) \) for all \( n \in \mathbb{N} \) and \( l \in \mathbb{N}^k \). Hence Lemma 5.2 implies that every \( y \in v(\lim(\Lambda_n, p_n))^\infty \) satisfies \( \sigma^{(p, 0)}(y) = \sigma^{(q, 0)}(y) \).

\[ \square \]

6. Projective limit \( k \)-graphs

Let \( (\Lambda_n, \Lambda_{n+1}, p_n)_{n=1}^\infty \) be a sequence of row-finite coverings of \( k \)-graphs with no sources as in Section 2.3. We aim to show that the sets \( (\lim(\Lambda_i))^m := \lim(\Lambda_i^m, p_i) \) under the projective limit topology with the natural (coordinate-wise) range and source maps specify a topological \( k \)-graph (in the sense of Yeend). Moreover, we show that the associated topological \( k \)-graph \( C^* \)-algebra is isomorphic to the full corner \( P_0 C^*(\lim(\Lambda_n; p_n)) P_0 \) determined by \( P_0 := \sum_{n \in \mathbb{N}^0} s_n \). In particular, when the \( \Lambda_n \) and \( p_n \) are as in 4.1 the \( C^* \)-algebra of the projective limit topological \( k \)-graph is isomorphic to the crossed product of \( C^*(\Lambda) \) by the coaction of the projective limit of the groups \( G_i \) obtained from Theorem 3.1.

Let \( (\Lambda_n, \Lambda_{n+1}, p_n)_{n=1}^\infty \) be a sequence of row-finite coverings of \( k \)-graphs with no sources. Let \( \lim(\Lambda_i, p_i) \) be the projective limit category, equipped with the projective limit topology. That is, \( \lim(\Lambda_i, p_i) \) consists of all sequences \( (\lambda_i)_{i=1}^\infty \) such that each \( \lambda_i \in \Lambda_i \) and \( p_i(\lambda_{i+1}) = \lambda_i \); the structure maps \( \tilde{r}, \tilde{s}, \tilde{d} \) and \( \tilde{id} \) on \( \lim(\Lambda_i, p_i) \) are obtained by pointwise application of the corresponding structure maps for \( \Lambda \). The cylinder sets \( Z(\lambda_1, \ldots, \lambda_j) := \{(\mu_i)_{i=1}^\infty \in \lim(\Lambda_i, p_i) : \mu_i = \lambda_i \text{ for } 1 \leq i \leq j\} \), form a basis of compact open sets for a locally compact Hausdorff topology.

Define \( \tilde{d} : \lim(\Lambda_i, p_i) \to \mathbb{N}^k \) by \( \tilde{d}(\lambda_i)_{i=1}^\infty := d(\lambda_i) \). Since the \( p_i \) are degree-preserving, we have

\[ \tilde{d}(\lambda_i)_{i=1}^\infty = d(\lambda_i) \quad \text{for all } i \geq 1. \]

For fixed \( \lambda = (\lambda_i)_{i=1}^\infty \in \lim(\Lambda_i, p_i)^{m+n} \), the unique factorisation property for each \( \lambda_i \) produces unique elements \( \lambda(0, m) := (\lambda_i(0, m))_{i=1}^\infty \in \lim(\Lambda_i, p_i)^m \) and \( \lambda(m, n) := (\lambda_i(m, n))_{i=1}^\infty \in \lim(\Lambda_i, p_i)^n \) such that \( \lambda = \lambda(0, m) \lambda(m, n) \); that is, \( (\lim(\Lambda_i, p_i), \tilde{d}) \) is a second-countable small category with a degree functor satisfying the factorisation property.

The identity \( \tilde{d}(\lambda_i)_{i=1}^\infty = d(\lambda_i) \) for all \( i \geq 1 \) implies that \( Z(\lambda_1, \ldots, \lambda_j) \) is empty unless \( d(\lambda_1) = \cdots = d(\lambda_j) \), and it follows that \( \tilde{d} \) is continuous.

We claim that \( \tilde{r} \) and \( \tilde{s} \) are local homeomorphisms. To see this, fix a cylinder set \( Z(v_1, \ldots, v_j) \subset \lim(\Lambda_i, p_i)^0 \), and for \( \lambda \in v_1 \Lambda_1 \) and \( 2 \leq l \leq j \), let \( v_lp_1^{-1}_l(\lambda) \) be the unique element of \( v_1 \Lambda_l \) such that \( p_1 \circ p_2 \circ \cdots \circ p_{l-1}(v_lp_1^{-1}_l(\lambda)) = \lambda \). Then

\[ \tilde{r}^{-1}(Z(v_1, \ldots, v_j)) \cap \lim(\Lambda_i, p_i)^n := \sqcup_{\lambda \in v_1 \Lambda_1} Z(\lambda, v_2p_1^{-1}_1(\lambda), \ldots, v_jp_1^{-1}_1(\lambda)) \]

which is clearly open, showing that \( \tilde{r} \) is continuous. Moreover, this same formula shows that for \( \lambda = (\lambda_i)_{i=1}^\infty \in \lim(\Lambda_i, p_i) \), the restriction of \( \tilde{r} \) to \( Z(\lambda_1) \) is a homeomorphism, and \( \tilde{r} \) is a local homeomorphism as claimed. A similar argument shows that \( \tilde{s} \) is also a local homeomorphism.
It is easy to see that the inverse image under composition of the cylinder set
\[ Z(\lambda_1, \ldots, \lambda_j) \in \bigcup_{p+q=n} Z(\lambda_1(0,p), \ldots, \lambda_j(0,p)) \times Z(\lambda_1(p,q), \ldots, \lambda_j(p,q)) \]
of cartesian products of cylinder sets and hence is open, so that composition is
continuous, and it follows that \((\lim_{\infty}(\Lambda_i, p_i), \overline{d})\) is a topological \(k\)-graph in the sense of Yeend [21, 20].

Let \(\lim_{\infty}(\Lambda_n; p_n)\) be as described in Section 2.3 and let \(P_0\) denote the full projection
\[ \sum_{v \in \Lambda^0} s_v \in M(C^*(\lim_{\infty}(\Lambda_n; p_n))). \]
For the following proposition, we need to describe
\[ P_0 C^*(\lim_{\infty}(\Lambda_n; p_n)) P_0 \] in detail. For \(n \geq m \geq 1\), we write \(p_{m,n} : \Lambda_n \to \Lambda_m\) for the
covering map \(p_{m,n} := p_m \circ \cdots \circ p_{m-1}\), with the convention that \(p_{n,n}\) is the identity
map on \(\Lambda_n\). For \(v \in \Lambda^0_n\), and \(l \leq m\), we denote by \(\alpha_{m,n}(v)\) the unique path in
\[ \lim_{\infty}(\Lambda_n; p_n)^{(m-1)k+1} \] whose source is \(v\) and whose range is \(p_{l,m}(v)\). In particular,
\(\alpha_{1,m}(v)\) the unique path in \(\lim_{\infty}(\Lambda_n; p_n)^{(m-1)k+1}\) whose source is \(v\) with range in \(\Lambda_1\).
For \(\lambda \in \Lambda_m\),
\[ s_{\alpha_{1,m}(r(m))} s_{\alpha_{1,m}(r(m))} s_{p_{1,1}(1)} = s_{\alpha_{1,m}(r(m))} s_{\alpha_{1,m}(s(m))} s_{\alpha_{1,m}(s(m))}. \]
Furthermore, \(P_0 C^*(\lim_{\infty}(\Lambda_n, p_n)) P_0\) is equal to the closed span
\[ P_0 C^*(\lim_{\infty}(\Lambda_n, p_n)) P_0 = \text{span}\{s_{\alpha_{1,m}(r(m))} s_{\alpha_{1,m}(s(m))} : m \geq 1, \lambda \in \Lambda_m\}. \]

**Proposition 6.1.** Let \((\Lambda_n, \Lambda_{n+1}, p_n)_{n=1}^\infty\) be a sequence of row-finite coverings of \(k\)-graphs with no sources, and let \(\lim_{\infty}(\Lambda_n, p_n)\) be the associated \((k+1)\)-graph as in [12]. Let \(P_0 := \sum_{v \in \Lambda^0_n} s_v \in M(C^*(\lim_{\infty}(\Lambda_n; p_n)))\). Let \((\lim_{\infty}(\Lambda_n, p_n), \overline{d})\) be the topological \(k\)-graph defined above. Then there is a unique isomorphism
\[ \pi : P_0 C^*(\lim_{\infty}(\Lambda_n, p_n)) P_0 \to C^*(\lim_{\infty}(\Lambda_n, p_n)) \]
such that for \(\lambda \in \Lambda_m\),
\[ \pi(s_{\alpha_{1,m}(r(m))} s_{\alpha_{1,m}(s(m))}) = X Z(p_{1,1}(1), p_{2,1}(1), \ldots, p_{m-1,1}(1), 1). \]
In particular, with the notation and assumptions [11], the \(C^*-\)algebra \(C^*(\lim_{\infty}(\Lambda_n, p_n))\)
of the topological \(k\)-graph \(\lim_{\infty}(\Lambda_n, p_n)\) is isomorphic to the coaction crossed-product
\(C^*(\Lambda) \rtimes G\).

**Proof.** The final statement will follow from Theorem 4.3 once we establish the first statement.

To prove the first statement we will use Allen’s gauge-invariant uniqueness theorem for corners in \(k\)-graph algebras [11]. For this, we adopt Allen’s notation: for \(\mu, \nu \in \Lambda_n^0\), we let \(t_{\mu, \nu} := s_{p_{1,1}(1)} s_{\alpha_{1,m}(r(m))} s_{\alpha_{1,m}(s(m))}\) for some \(m \geq 1\) and \(\mu', \nu' \in \Lambda_m\) with \(s(\mu') = s(\nu')\). By [11] Corollary 3.7, there is an isomorphism \(\theta\) of \(P_0 C^*(\lim_{\infty}(\Lambda_n; p_n)) P_0\) onto Allen’s universal algebra \(C^*(\lim_{\infty}(\Lambda_n; p_n), \Lambda^0_1)\) (see Definition 3.1 and the following paragraphs in [11]) which satisfies \(\theta(t_{\mu, \nu}) = T_{\mu, \nu}\) for all \(\mu, \nu\). It therefore suffices here to show that there is an isomorphism \(\psi : C^*(\lim_{\infty}(\Lambda_n; p_n), \Lambda^0_1) \to C^*(\lim_{\infty}(\Lambda_n, p_n))\) such that
\[ \psi(T_{\alpha_1,m}(r(\mu)) \mu,\alpha_1,m(r(\nu)) \nu) = \chi z(p_{1,m}(\mu),...,m)z(p_{1,m}(\nu),...,\nu) \] for all \( m \geq 1 \) and \( \mu, \nu \in \Lambda_m \) with \( s(\mu) = s(\nu) \); the composition \( \pi := \psi \circ \theta \) clearly satisfies (3), and it is uniquely specified by (3) because the elements \( \{ t_{\alpha_1,m}(r(\lambda))\lambda,\alpha_1,m(s(\lambda)) : m \geq 1, \lambda \in \Lambda_m \} \) generate \( P_0c^*(\lim(\Lambda_n;p_n))P_0 \) as a \( C^* \)-algebra.

Let \( \hat{\Gamma} \) denote the topological \( k \)-graph \( \lim(\Lambda_i,p_i) \). Since \( \Gamma \) is row-finite and has no sources, \( \partial \Gamma = \Gamma^\infty \). As in [21], for open subsets \( U, V \subset \Gamma \), let \( Z_{\hat{\Gamma}}(U \times_s V, m) \) denote the set \( \{ (\mu x, m, \nu x) : \mu \in U, \nu \in V, x \in \Gamma^\infty, s(\mu) = s(\nu) = r(x) \} \). Then \( \mathcal{G}_\Gamma \) is the locally compact Hausdorff topological groupoid

\[ \mathcal{G}_\Gamma = \{ (x, m - n, y) : x, y \in \Gamma^\infty, m, n \in \mathbb{N}, \sigma^m(x) = \sigma^n(y) \} \]

where the \( Z_{\hat{\Gamma}}(U \times_s V, m) \) form a basis of compact open sets for the topology.

For \( m \geq 1 \) and \( \lambda \in \Lambda_m \), let \( U_{m,\lambda} := Z(p_{1,m}(\lambda),\ldots,\lambda) \subset \Gamma \). So the \( U_{m,\lambda} \) are a basis for the topology on \( \Gamma = \lim(\Lambda_i,p_i) \). Now for \( m \geq 1 \) and \( \mu, \nu \in \Lambda_m \) with \( s(\mu) = s(\nu) \), let

\[ u_{\alpha_1,m}(r(\mu)) \mu,\alpha_1,m(r(\nu)) \nu := \chi z(U_{m,\mu} \star U_{m,\nu},d(\mu) - d(\nu)) \in C_*(\mathcal{G}_\Gamma) \]

Tedious but routine calculations using the definition of the convolution product and the involution on \( C_*(\mathcal{G}_\Gamma) \subset C^*(\mathcal{G}_\Gamma) \) show that \( \{ u_{\alpha_1,m}(r(\mu)) \mu,\alpha_1,m(r(\nu)) \nu : m \geq 1, \mu, \nu \in \Lambda_m, s(\mu) = s(\nu) \} \) is a Cuntz-Krieger (\( \lim(\Lambda_n;p_n), \Lambda^0 \)) family in \( C^*(\mathcal{G}_\Gamma) \). By the universal property of \( C^*(\lim(\Lambda_n;p_n), \Lambda^0) \) (see [1, Section 3]), there is a homomorphism \( \psi : C^*(\lim(\Lambda_n;p_n), \Lambda^0) \to C^*(\mathcal{G}_\Gamma) \) such that

\[ \psi(T_{\alpha_1,m}(r(\mu)) \mu,\alpha_1,m(r(\nu)) \nu) = u_{\alpha_1,m}(r(\mu)) \mu,\alpha_1,m(r(\nu)) \nu \]

for each \( \mu, \nu \). The canonical gauge action \( \beta : \mathbb{T}^k \to \text{Aut}(C^*(\mathcal{G}_\Gamma)) \) determined by \( \beta_z(f)(x, m, y) := z^m f(x, m, y) \) satisfies \( \psi \circ \gamma_z = \beta_z \circ \psi \) for all \( z \in \mathbb{T}^k \), where \( \gamma \) is the gauge action on \( C^*(\lim(\Lambda_n;p_n), \Lambda^0) \). Proposition 4.3 of [21] shows that each \( u_{\alpha_1,m}(r(\mu)) \mu,\alpha_1,m(r(\nu)) \mu \) is nonzero, and it follows from the gauge-invariant uniqueness theorem [1, Theorem 3.5] that \( \psi \) is injective. The topologies on \( \Gamma^{(0)} \) and on \( \mathcal{G}_\Gamma \) are generated by the collections \( \{ U_{m,\lambda} : m \geq 1, \lambda \in \Lambda_m \} \) and \( \{ U_{m,\mu} \star U_{m,\nu} : m \geq 1, \mu, \nu \in \Lambda_m, s(\mu) = s(\nu) \} \) respectively of compact open sets. Since \( C^*(\{ u_{\alpha_1,m}(r(\mu)) \mu,\alpha_1,m(r(\nu)) \nu : m \geq 1, \mu, \nu \in \Lambda_m, s(\mu) = s(\nu) \}) \subset C^*(\mathcal{G}_\Gamma) \) contains the characteristic functions of these sets, it follows that \( \psi \) is also onto, and this completes the proof.

**Remark 6.2.** The final statement of Proposition 6.1 suggests that \( \lim(\Lambda_i,p_i) \) should be thought of as a skew-product of \( \Lambda \) by \( G \).

To make this precise, note that for \( \lambda \in \Lambda \), \( c(\lambda) := (c_n(\lambda))_{n=1}^\infty \) belongs to \( G \), and \( c : \Lambda \to G \) is then a cocycle. There is a natural bijection between the cartesian product \( \Lambda \times G \) and the topological \( k \)-graph \( \lim(\Lambda_i,p_i) \), so we may view \( \Lambda \times G \) as a topological \( k \)-graph by pulling back the structure maps from \( \lim(\Lambda_i,p_i) \). What we obtain coincides with the natural definition of the skew-product \( \Lambda \times_c G \).

With this point of view, we can regard Proposition 6.1 as a generalisation of [15, Theorem 7.1(ii)] to profinite groups and topological \( k \)-graphs: \( C^*(\Lambda \times_c G) \cong C^*(\Lambda) \times_s G \).

**Example 6.3 (Example 3.3 continued).** Resume the notation of Examples 3.3 and 4.2. The resulting projective limit \( \lim(\Lambda_n,p_n) \) is the topological 1-graph \( E \) associated to
the odometer action of $\mathbb{Z}$ on the Cantor set as in [21, Example 2.5(3)]. That is, $E$ can be realised as the skew-product of $B_1^1$ by the $2$-adic integers $\mathbb{Z}_2$ with respect to the functor $c : B_1^1 \to \mathbb{Z}_2$ determined by $c(f) = (1, 1, 1, \ldots)$, where $f$ is the loop edge generating $B_1^1$.

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