C^*-ALGEBRAS GENERATED BY 2-GRAPHS AND VARIATIONS UNDER FACTORISATION PROPERTIES

DAVID ROBERTSON

Abstract. These notes were written during an eight-week summer scholarship at the University of Newcastle in the summer of 2005/2006. They were written at the time as a project summary and had not been intended for distribution. However, some of the examples and conclusions of sections 3 and 4 arose in [4, Example 6.11]. We have therefore posted these notes electronically as support material for [4].

1. Introduction

We regard $\mathbb{N}$ as the set of all natural numbers including 0, and write $\mathbb{N}^k$ for the additive semigroup of k-tuples $n = (n_1, n_2, \dots n_k)$, with additive identity $0 = (0, 0, \dots, 0)$. We denote the canonical generators of $\mathbb{N}^k$ by $\{e_1, e_2, \dots e_k\}$. Given $m, n \in \mathbb{N}^k$, we say $m \le n$ if $m_i \le n_i$ for all $i = 1, 2, \dots, k$. We write $m \vee n$ for their coordinate-wise maximum, and $m \wedge n$ for their coordinate-wise minimum.

Definition 1.1. A *category* $\mathcal{C}$ is a sextuplet $(\mathrm{Obj}(\mathcal{C}), \mathrm{Mor}(\mathcal{C}), \mathrm{dom}, \mathrm{cod}, \mathrm{id}, \circ)$. $\mathrm{Obj}(\mathcal{C})$ are the *objects*, $\mathrm{Mor}(\mathcal{C})$ are the *morphisms* between objects, dom, cod are the *domain* and *codomain* functions from $\mathrm{Mor}(\mathcal{C})$ to $\mathrm{Obj}(\mathcal{C})$, id is the *identity* function from $\mathrm{Obj}(\mathcal{C})$ to $\mathrm{Mor}(\mathcal{C})$. $\circ$ is the *composition* function from $\mathrm{Mor}(\mathcal{C}) \times_{\mathrm{Obj}(\mathcal{C})} \mathrm{Mor}(\mathcal{C})$ to $\mathrm{Mor}(\mathcal{C})$ where $\mathrm{Mor}(\mathcal{C}) \times_{\mathrm{Obj}(\mathcal{C})} \mathrm{Mor}(\mathcal{C}) = \{(g, f) \in \mathrm{Mor}(\mathcal{C})^2 : \mathrm{dom}(g) = \mathrm{cod}(f)\}$ is the set of *composable pairs* in $\mathrm{Mor}(\mathcal{C})$. We say $\mathcal{C}$ is countable if $\mathrm{Mor}(\mathcal{C})$ is countable.

Definition 1.2. Let $k \in \mathbb{N}\backslash\{0\}$ A *k-graph* is a pair (Λ, d), where Λ is a countable category and $d : \Lambda \to \mathbb{N}^k$ is a functor which satisfies the *factorisation property*: if $\lambda \in \mathrm{Mor}(\Lambda)$ and $d(\lambda) = m + n$, then there are unique morphisms $\mu \in d^{-1}(m)$ and $\nu \in d^{-1}(n)$ such that $\lambda = \mu\nu$.

We refer to the elements of $\mathrm{Mor}(\Lambda)$ as *paths* and to elements of $\mathrm{Obj}(\Lambda)$ as *vertices* and we write r and s for the domain and codomain maps respectively. The factorisation property allows us to identify $\mathrm{Obj}(\Lambda)$ with $\{\lambda \in \mathrm{Mor}(\Lambda) : d(\lambda) = 0\}$. So we write $\lambda \in \Lambda$, and when $d(\lambda) = 0$, we regard

---

Date: October 15, 2007.

λ as a vertex of Λ.

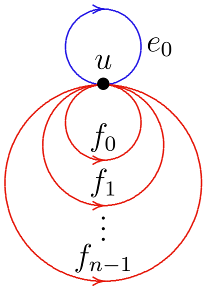
If $\lambda \in \Lambda$ with $d(\lambda) = l$, and $0 \le m \le n \le l$, then we can apply the factorisation property twice to obtain unique elements λ', λ'' and $\lambda''' \in \mathrm{Mor}(\Lambda)$ such that $d(\lambda') = m, d(\lambda'') = n - m$ and $d(\lambda''') = l - n$, and such that $\lambda = \lambda'\lambda''\lambda'''$. We write $\lambda(0, m), \lambda(m, n)$ and $\lambda(n, l)$ for λ', λ'' and λ''' respectively.

We write $\Lambda^n := \{\lambda \in \Lambda : d(\lambda) = n\}$ for the paths of length $n \in \mathbb{N}^k$. Hence $\Lambda^0 = \{\lambda \in \Lambda : d(\lambda) = 0\} = \mathrm{Obj}(\Lambda)$. Given $\lambda \in \Lambda$ and $E \subseteq \Lambda$, we write λE for the set $\{\lambda\mu : \mu \in E, s(\mu) = r(\lambda)\}$. In particular, if $v \in \Lambda^0$ then $vE = \{\lambda \in E : r(\lambda) = v\}$ and $Ev = \{\lambda \in E : s(\lambda) = v\}$.

Definition 1.3. A k-graph (Λ, d) is *row finite* if $v\Lambda^n$ is finite for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$, and Λ has *no sources* if $v\Lambda^n \neq \emptyset$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$. We say that Λ is *strongly connected* if $v\Lambda w \neq \emptyset$ for all $v, w \in \Lambda^0$, and that Λ is finite if Λ^0 is finite and each Λ^{e_i} is finite. Given $k \in \mathbb{N}\backslash\{0\}$ and k-graphs $\Lambda_1, d_1)$ and (Λ_2, d_2), we call a function $x : \Lambda_1 \to \Lambda_2$ a *graph morphism* if it satisfies $d_2 \circ x = d_1$.

Definition 1.4. Given $k \in \mathbb{N}\backslash\{0\}$, we write Ω_k for the k-graph given by $\mathrm{Obj}(\Omega_k) = \mathbb{N}^k$, $\mathrm{Mor}(\Omega_k) = \{(m, n) \in \mathbb{N}^k \times \mathbb{N}^k : m \le n\}$, $r(m, n) = m, s(m, n) = n, (m, n) \circ (n, p) = (m, p)$, and $d(m, n) = n - m$. Given a k-graph Λ, an *infinite path* in Λ is a graph morphism $x : \Omega_k \to \Lambda$. We write Λ^∞ for the collection of all infinite paths in Λ. For $p \in \mathbb{N}^k$, we write $\sigma^p : \Lambda^\infty \to \Lambda^\infty$ for the shift map determined by $\sigma^p(x)(n) = x(n + p)$, and we say that $x \in \Lambda^\infty$ is *aperiodic* if there do not exist $p, q \in \mathbb{N}^k, p \neq q$ and $\sigma^p(x) = \sigma^q(x)$. For the sake of notation, we write $x = \lambda_1\lambda_2\lambda_3\ldots$ where each $\lambda_j \in \Lambda^{e_i}$ for some $i \in \{1, 2, \ldots, k\}$.

Example 1. Consider the following 1-skeleton of a 2-graph, (Λ, d):



Set $d(e_0) = (1, 0)$ and $d(f_i) = (0, 1)$ for all $i \in \mathcal{I} = \{0, 1, \ldots, n - 1\}$. We have the following possible factorisation properties:

$$\begin{array}{rcl} e_0f_0 & = & \left.\begin{array}{c} f_0e_0 \\ f_1e_0 \\ \vdots \\ f_{n-1}e_0 \end{array}\right\} (n \text{ possibilities}) \\ \\ e_0f_1 & = & \dots \ (n-1 \text{ possibilites}) \\ & \vdots & \\ e_0f_{n-1} & = & \dots \ (1 \text{ possibility}) \end{array}$$

So there are $n \times (n-1) \times \cdots \times 2 \times 1 = n!$ possible factorisation properties. Suppose we have

$$e_0f_i = f_{\phi(i)}e_0$$

where $i \in \mathcal{I}$ and $\phi : \mathcal{I} \to \mathcal{I}$ is a bijection. The period of some $i \in \mathcal{I}$ must be $p_i = 1, 2, \dots, n$ by the Pigeonhole principle. Hence, for each $i \in \mathcal{I}$, we have $p_i|n!$, so $\phi^{(n!)}(i) = i$ for all $i \in \mathcal{I}$, where $\phi^{(n!)}$ denotes the $n!$th iterate of ϕ. Consider the infinite path $x = e_0f_{i(1)}e_0f_{i(2)}e_0f_{i(3)}\dots$. Then

$$\begin{array}{rcl} \sigma^{(1,0}(x) & = & \sigma^{(1,0)}(e_0f_{i(1)}e_0f_{i(2)}e_0f_{i(3)}\cdots) \\ & = & f_{i(1)}e_0f_{i(2)}e_0f_{i(3)}\cdots \\ & = & e_0f_{\phi(i(1))}e_0f_{\phi(i(2))}e_0f_{\phi(i(3))}\cdots \text{ by the factorisation property} \end{array}$$

By an inductive argument, we may see that

$$\begin{array}{rcl} \sigma^{(n!,0)}(x) & = & \sigma^{(n!,0)}(e_0f_{i(1)}e_0f_{i(2)}e_0f_{i(3)}\cdots) \\ & = & \sigma^{(n!-1,0)}(e_0f_{\phi(i(1))}e_0f_{\phi(i(2))}e_0f_{\phi(i(3))}\cdots) \\ & \vdots & \\ & = & \sigma^{(1,0)}(f_{\phi^{(n!-1)}(i(1))}e_0f_{\phi^{(n!-1)}(i(2))}e_0f_{\phi^{(n!-1)}(i(3))}\cdots) \\ & = & e_0f_{\phi^{n!}(i(1))}e_0f_{\phi^{n!}(i(2))}e_0f_{\phi^{n!}(i(3))}\cdots \\ & = & e_0f_{i(1)}e_0f_{i(2)}e_0f_{i(3)}\cdots \text{ as } \phi^{n!}(i) = i \text{ for all } i \in \mathcal{I} \\ & = & x \end{array}$$

So we see that x has a period of $(n!, 0)$ Note that this is not the least possible period of x. Since the factorisation property and x were both arbitrary, we see that Λ does not admit any aperiodic path.

Definition 1.5. Let (Λ, d) be a row-finite k-graph with no sources. A Cuntz-Krieger Λ-family is a collection $\{t_\lambda : \lambda \in \Lambda\}$ of partial isometries satisfying

- $\{t_v : v \in \Lambda^0\}$ is a collection of mutually orthogonal projections;
- $t_\lambda t_\mu = t_{\lambda\mu}$ whenever $s(\lambda) = r(\mu)$;

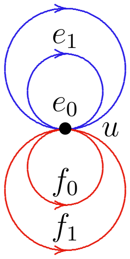
- $t_\lambda^* t_\lambda = t_{s(\lambda)}$ for all $\lambda \in \Lambda$; and
- $t_v = \sum_{\lambda \in \Lambda^n} t_\lambda t_\lambda^*$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$

The *Cuntz-Krieger Algebra* $C^*(\Lambda)$ is the C^*-algebra generated by a Cuntz-Krieger Λ-family $\{s_\lambda : \lambda \in \Lambda\}$ which is universal in the sense that for every Cuntz-Krieger Λ-family $\{t_\lambda : \lambda \in \Lambda\}$, there is a unique homomorphism π of $C^*(\Lambda)$ satisfying $\pi(s_\lambda) = t_\lambda$ for all $\lambda \in \Lambda$.

We say that a k-graph, Λ satisfies the *aperiodicity condition* if for each $v \in \Lambda^0$, there is an infinite path x with $r(x) = v$. It has been shown that if a k-graph satisfies the aperiodicty condition then the C^*-algebra arising from the Cuntz-Krieger Λ-family is simple, purely infinite and nuclear. However, satisfying the aperiodicity condition is not just dependent on the objects and morphisms of a particular graph, but also on the factorisation property associated with it.

2. 2-Graphs and Factorisation Properties

We now consider the 2-graph (Λ, d) given by



Set $d(e_0) = d(e_1) = (1,0)$ and $d(f_0) = d(f_1) = (0,1)$. There are $4! = 24$ different factorisation properties that we can associate to Λ. These are shown in the following table.

1.		**2.**		**3.**		**4.**		**5.**		**6.**	
$e_if_j = f_ke_l$		$e_if_j = f_ke_l$		$e_if_j = f_ke_l$		$e_if_j = f_ke_l$		$e_if_j = f_ke_l$		$e_if_j = f_ke_l$	
e_0f_0	f_0e_0	e_0f_0	f_0e_0	e_0f_0	f_0e_0	e_0f_0	f_0e_0	e_0f_0	f_0e_0	e_0f_0	f_0e_0
e_0f_1	f_0e_1	e_0f_1	f_0e_1	e_0f_1	f_1e_0	e_0f_0	f_1e_0	e_0f_1	f_1e_1	e_0f_1	f_1e_1
e_1f_0	f_1e_0	e_1f_0	f_1e_1	e_1f_0	f_0e_1	e_0f_0	f_1e_1	e_1f_0	f_0e_1	e_1f_0	f_1e_0
e_1f_1	f_1e_1	e_1f_1	f_1e_0	e_1f_1	f_1e_1	e_0f_0	f_0e_1	e_1f_1	f_1e_0	e_1f_1	f_0e_1

7.		**8.**		**9.**		**10.**		**11.**		**12.**	
$e_if_j = f_ke_l$		$e_if_j = f_ke_l$		$e_if_j = f_ke_l$		$e_if_j = f_ke_l$		$e_if_j = f_ke_l$		$e_if_j = f_ke_l$	
e_0f_0	f_0e_1	e_0f_0	f_0e_1	e_0f_0	f_0e_1	e_0f_0	f_0e_1	e_0f_0	f_0e_1	e_0f_0	f_0e_1
e_0f_1	f_0e_0	e_0f_1	f_0e_0	e_0f_1	f_1e_0	e_0f_0	f_1e_0	e_0f_1	f_1e_1	e_0f_1	f_1e_1
e_1f_0	f_1e_0	e_1f_0	f_1e_1	e_1f_0	f_0e_0	e_0f_0	f_1e_1	e_1f_0	f_0e_0	e_1f_0	f_1e_0
e_1f_1	f_1e_1	e_1f_1	f_1e_0	e_1f_1	f_1e_1	e_0f_0	f_0e_0	e_1f_1	f_1e_0	e_1f_1	f_0e_0

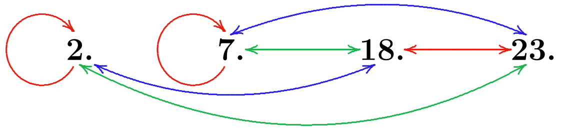
13.		**14.**		**15.**		**16.**		**17.**		**18.**	
$e_if_j = f_ke_l$		$e_if_j = f_ke_l$		$e_if_j = f_ke_l$		$e_if_j = f_ke_l$		$e_if_j = f_ke_l$		$e_if_j = f_ke_l$	
e_0f_0	f_1e_0	e_0f_0	f_1e_0	e_0f_0	f_1e_0	e_0f_0	f_1e_0	e_0f_0	f_1e_0	e_0f_0	f_1e_0
e_0f_1	f_0e_0	e_0f_1	f_0e_0	e_0f_1	f_0e_1	e_0f_0	f_0e_1	e_0f_1	f_1e_1	e_0f_1	f_1e_1
e_1f_0	f_0e_1	e_1f_0	f_1e_1	e_1f_0	f_0e_0	e_0f_0	f_1e_1	e_1f_0	f_0e_0	e_1f_0	f_0e_1
e_1f_1	f_1e_1	e_1f_1	f_0e_1	e_1f_1	f_1e_1	e_0f_0	f_0e_0	e_1f_1	f_0e_1	e_1f_1	f_0e_0

19.		**20.**		**21.**		**22.**		**23.**		**24.**	
$e_if_j = f_ke_l$		$e_if_j = f_ke_l$		$e_if_j = f_ke_l$		$e_if_j = f_ke_l$		$e_if_j = f_ke_l$		$e_if_j = f_ke_l$	
e_0f_0	f_1e_1	e_0f_0	f_1e_1	e_0f_0	f_1e_1	e_0f_0	f_1e_1	e_0f_0	f_1e_1	e_0f_0	f_1e_1
e_0f_1	f_0e_0	e_0f_1	f_0e_0	e_0f_1	f_0e_1	e_0f_1	f_0e_1	e_0f_1	f_1e_0	e_0f_1	f_1e_0
e_1f_0	f_0e_1	e_1f_0	f_1e_0	e_1f_0	f_0e_0	e_1f_0	f_1e_0	e_1f_0	f_0e_0	e_1f_0	f_0e_1
e_1f_1	f_1e_0	e_1f_1	f_0e_1	e_1f_1	f_1e_0	e_1f_1	f_0e_0	e_1f_1	f_0e_1	e_1f_1	f_0e_0

There are some changes that we can make to Λ that while not effectively changing the graph, they will change the factorisation property

- Swap the labels of e_0 and e_1
- Swap the labels of f_0 and f_1
- Swap e_0 with f_0 and e_1 with f_1.

Example 2. The following diagram shows the relationship between factorisation properties **2.**, **7.**, **18.** and **23.**.



The Blue arrows indicate swapping e_0 and e_1, the Green arrows indicate swapping f_0 and f_1, and the Red arrows indicate swapping e_0 with f_0 and

e_1 with f_1. In contrast to this, factorisation property **3.** is invariant under all three changes.

Let $F = \{\mathbf{1.}, \mathbf{2.}, \dots, \mathbf{24.}\}$ be the set of factorisation properties, and define an equivalence relation $\sim$ on F, where

$$X \sim Y \iff \begin{cases} X = Y \text{ or} \\ X \mathrel{\color{blue}{\longleftrightarrow}} Y, \text{ or} \\ X \mathrel{\color{green}{\longleftrightarrow}} Y, \text{ or} \\ X \mathrel{\color{red}{\longleftrightarrow}} Y, \text{ or} \\ \exists\, Z \in F \text{ such that } X \mathrel{\color{green}{\longleftrightarrow}} Z \mathrel{\color{blue}{\longleftrightarrow}} Y \text{ or } X \mathrel{\color{blue}{\longleftrightarrow}} Z \mathrel{\color{green}{\longleftrightarrow}} Y \end{cases}$$

where $X, Y \in F$. Then the set of equivalence classes, $F/\sim$, has nine elements:

$$F/\sim = \left\{ \begin{array}{l} \{\mathbf{1.}, \mathbf{24.}\}, \{\mathbf{2.}, \mathbf{7.}, \mathbf{18.}, \mathbf{23.}\}, \{\mathbf{3.}\}, \{\mathbf{4.}, \mathbf{5.}, \mathbf{9.}, \mathbf{13.}\}, \{\mathbf{6.}, \mathbf{10.}, \mathbf{15.}, \mathbf{19.}\}, \\ \{\mathbf{8.}, \mathbf{17.}\}, \{\mathbf{11.}, \mathbf{14.}\}, \{\mathbf{12.}, \mathbf{16.}, \mathbf{20.}, \mathbf{21.}\}, \{\mathbf{22.}\} \end{array} \right\}$$

We denote an equivalence class by $[X]$ where $X \in F$.

3. DUALS OF HIGHER-RANK GRAPHS

Definition 3.1. Let (Λ, d) be a k-graph. Let $p\Lambda := \{\lambda \in \Lambda : d(\lambda) \geq p\}$. Define range and source maps on $p\Lambda$ by $r_p(\lambda) := \lambda(0, p)$, and $s_p(\lambda) := \lambda(d(\lambda) - p, d(\lambda))$ for all $\lambda \in \Lambda$, and define composition by $\lambda \circ_p \mu := \lambda\mu(p, d(\mu)) = \lambda(0, d(\lambda) - p)\mu$ whenever $s_p(\lambda) = r_p(\mu)$. Also, define a degree map d_p on $p\Lambda$ by $d_p(\lambda) = d(\lambda) - p$ whenever $\lambda \in p\Lambda$.

Theorem 3.2 (Theorem 3.5, from [2])**.** *Let (Λ, d) be a row finite k-graph with no sources, and let $p \in \mathbb{N}^k$. Let $\{s_\lambda : \lambda \in \Lambda\}$ denote the universal generating Cuntz-Krieger Λ-family in $C^*(\Lambda)$, and let $\{t_\lambda : \lambda \in \Lambda\}$ be the universal generating Cuntz-Krieger $p\Lambda$-family in $C^*(p\Lambda)$. For all $\lambda \in p\Lambda$, define $r_\lambda := s_\lambda s^*_{s_p(\lambda)}$. Then there is an isomorphism $\phi : C^*(p\Lambda) \to C^*(\lambda)$ such that $\phi(t_\lambda) = r_\lambda$ for all $\lambda \in p\Lambda$*

Using this theorem, we can determine whether a k-graph Λ satisfies the aperiodicity condition by seeing whether a particular dual graph $p\Lambda$ satisfies the aperiodicity condition for some $p \in \mathbb{N}^k$.

Lemma 3.3. *(Lemma **3.7** from* [1]*) Let (Λ, d) be a k-graph, and let $p \in \mathbb{N}^k$. For each $n \in \mathbb{N}^k$ with $n \leq p$ and $v, w \in p\Lambda^0 = \Lambda^p$, there is at most one $\lambda \in v(p\Lambda^n)w$.*

Consider the previous 2-graph. For each of the 9 classes of equivalent factorisation properties, we can construct a unique dual graph, $\mathbf{1\Lambda}$. To determine whether each of these satisfy the aperiodicity condition, we use methods outlined in Robertson and Steger's paper ([5]).

Notation 3.4. Let (Λ, d) be a k-graph. We write $M_i^\Lambda, 1 \le i \le k$ for the matrices in $M_{\Lambda^0}(\mathbb{N})$ determined by $(M_i^\Lambda)_{v,w} := |w\Lambda^{e_i}v|$ for $w, v \in \Lambda^0$, and we refer to these matrices as the *coordinate matrices* of Λ.

Definition 3.5. Let (Λ, d) be a k-graph, and suppose $\mu, \nu \in \Lambda$. We say that λ is a *common extension* of μ and ν if $\lambda(0, d(\mu)) = \mu$ and $\lambda(0, d(\nu)) = \nu$ (it necessarily follows that $d(\lambda) \ge d(\mu) \vee d(\nu)$). We call λ a *minimal common extension* of μ and ν if it is a common extension of μ and ν and also satisfies $d(\lambda) = d(\mu) \vee d(\nu)$. We denote the collection of all minimal common extensions of μ and ν by $\mathrm{MCE}(\mu, \nu)$, and we use the notation $\Lambda^{\min}(\mu, \nu)$ for the collection

$$\Lambda^{\min}(\mu, \nu) := \{(\alpha, \beta) \in \Lambda \times \Lambda : \mu\alpha = \nu\beta \in \mathrm{MCE}(\mu, \nu)\}$$

Definition 3.6. By applying theory outlined in [5], we associate the following properties to the coordinate matrices of a k-graph, Λ.

- **(H0):** Each M_i is a non-zero, $\{0, 1\}$ matrix;
- **(H1):** Let $\mu \in \Lambda^m$ and $\nu \in \Lambda^n$. If $r(\mu) = s(\nu)$, then there exists a unique $\lambda \in \Lambda^{m+n}$ such that
$$\lambda(0, m) = \mu \text{ and } \lambda(m, m+n) = \nu$$
- **(H2):** The 1-skeleton of Λ is irreducible
- **(H3):** Let $p \in \mathbb{Z}^r, p \ne 0$. Then there exists some $\lambda \in \Lambda$ which is not p-periodic.

We can use condition **(H3)** to construct an aperiodic path, so it essentially says that Λ satisfies the aperiodicity condition. Since it is impossible to check directly, with p ranging over all of $\mathbb{Z}^r$ we introduce another condition:

- **(H3*):** Let $j, k \in \{1, 2\}$ with $j \ne k$, and let $n \in \mathbb{N}$. Let $\lambda \in \Lambda^{ne_k}$. Then there exist $\alpha, \alpha' \in \Lambda^{e_j}$ such that $r(\alpha) = r(\lambda) = r(\alpha')$ and $\mathrm{MCE}(\alpha\lambda)$ and $\mathrm{MCE}(\alpha', \lambda)$ are both non-empty and not equal to each other.

Robertson and Steger showed in [5] that

Lemma 3.7. *(Lemma 2.1 from* [5]*) Conditions* **(H0)** - **(H2)** *and* **(H3*)** *imply condition* **(H3)**.

However, it is still quite hard to check condition **(H3*)** since we can have $n \in \mathbb{N}$ be anything. We want to restrict it to $n = 1$.

Lemma 3.8. *In our* 2*-graph with some factorisation property,* $|(\mathbf{1}\Lambda^{e_i})v| = 2$ *for all* $v \in \mathbf{1}\Lambda^0, i = 1, 2$.

Proof. Let $i = 1$ and fix $v \in \mathbf{1}\Lambda^0$. Then

$$\begin{aligned} |(\mathbf{1}\Lambda^{e_i})v| &= \{e_k v : k \in \{0, 1\}\} \\ &= 2 \end{aligned}$$

The case for $i = 2$ can be shown similarly. $\square$

Proposition 3.9. *Consider the* 2*-graph with some factorisation property from Section 2. Then condition* **(H3*)** *is equivalent to, for all* $e \in \mathbf{1}\Lambda^{e_1}$ *and for all* $f \in \mathbf{1}\Lambda^{e_2}$ with $r_{\mathbf{1}}(e) = r_{\mathbf{1}}(f)$, $\mathrm{MCE}(e, f) \neq \emptyset$.

Proof. ($\Longrightarrow$) Suppose condition **(H3*)** holds. Let $e \in \mathbf{1}\Lambda^{e_1}$. Then there exists $f, f' \in r_{\mathbf{1}}(e)\mathbf{1}\Lambda^{e_2}$ such that $\mathrm{MCE}(e, f)$ and $\mathrm{MCE}(e, f')$ are non-empty. But by Lemma 3.8, f and f' are unique, so we have that $\mathrm{MCE}(e, f) \neq \emptyset$ for all $f \in r(f)\mathbf{1}\Lambda^{e_2}$. The case for $e \in \mathbf{1}\Lambda^{e_2}$ follows by symmetry.
($\Longleftarrow$) Suppose that given $e \in \mathbf{1}\Lambda^{e_1}, \mathrm{MCE}(e, f) \neq \emptyset$ for all $f \in r_{\mathbf{1}}(e)\mathbf{1}\Lambda^{e_2}$. We wish to prove that this holds for $e \in \mathbf{1}\Lambda^{ne_1}$ where $n \in \mathbb{N}$ is any natural number. Suppose it is true for $n \leq k$. Fix $\lambda \in \mathbf{1}\Lambda^{(k+1)e_1}$, and write $\lambda = \lambda_k\lambda_1$ where $d(\lambda_k) = ke_1$ and $d(\lambda_1) = e_1$. By the inductive hypothesis, $\mathrm{MCE}(\lambda, f)$ and $\mathrm{MCE}(\lambda, f')$ are non-empty where f, f' are the 2 distinct elements of $r_{\mathbf{1}}(\lambda)\mathbf{1}\Lambda^{e_2}$. Say $\lambda_k g \in \mathrm{MCE}(\lambda, f)$ and $\lambda_k g' \in \mathrm{MCE}(\lambda, f')$. Since $f\lambda \neq f'\lambda$, we have $\lambda g \neq \lambda g'$ which implies that $g \neq g'$. Hence g and g' are the 2 distinct elements of $r_{\mathbf{1}}(\lambda)\mathbf{1}\Lambda^{e_2} = s_{\mathbf{1}}(\lambda_k)\mathbf{1}\Lambda^{e_2}$. By hypothesis, we have $\mathrm{MCE}(\lambda_1, g)$ and $\mathrm{MCE}(\lambda_1, g')$ both being non-empty, say $\lambda_1 h \in \mathrm{MCE}(\lambda_1, g)$ and $\lambda_1 h' \in \mathrm{MCE}(\lambda_1, g')$. Now, consider λh.

$$\begin{aligned}
\lambda h &= \lambda_k(\lambda_1 h) \\
&= \lambda_k(g\alpha) \ \text{ for some } \alpha \in s_{\mathbf{1}}(g)\Lambda^{e_1} \\
&= (f\beta)\alpha \ \text{ for some } \beta \in s_{\mathbf{1}}(f)\Lambda^{e_1}
\end{aligned}$$

So $\lambda h \in \mathrm{MCE}(\lambda, f)$. Similarly, $\lambda h' \in \mathrm{MCE}(\lambda, f')$. In particular, $\mathrm{MCE}(\lambda, f)$ and $\mathrm{MCE}(\lambda, f')$ are non-empty. By induction we can extend this to any $\lambda \in \mathbf{1}\Lambda^{ne_1}$ for any $n \in \mathbb{N}$. By symmetry, we have it for any $\lambda \in \mathbf{1}\Lambda^{ne_2}$ and we have **(H3*)**. $\square$

Proposition 3.10. *Consider the* 2*-graph together with a factorisation property* $X \in F$ *from Section 2. Then* **(H3*)** *is equivalent to, there exists a pair* $(e, f) \in \mathbf{1}\Lambda^{e_1} \times \mathbf{1}\Lambda^{e_2}$ *with* $r_{\mathbf{1}}(e) = r_{\mathbf{1}}(f)$ *such that* $\mathrm{MCE}(e, f)$ *is a singleton.*

Proof. ($\Longrightarrow$) Suppose condition **(H3*)** holds. Fix $v \in \mathbf{1}\Lambda^0$. Let $M_v := v\mathbf{1}\Lambda^{e_1} \times v\mathbf{1}\Lambda^{e_2}$. Then for $(e, f) \in M_v$, elements of $\mathrm{MCE}(e, f)$ are of the form $e_i f_j e_k f_l$ as an element of $\Lambda^{(2,2)}$ where $i, j, k, l \in \{0, 1\}$ and $e_i f_j = v$. So we have that

$$\bigcup_{(e,f)\in M_v} \mathrm{MCE}(e, f) = \{e_i f_j e_k f_l : k, l \in \{0, 1\}\}$$

has exactly 4 elements since i and j are fixed. Since M_v has 4 elements if $\mathrm{MCE}(e, f)$ has more than one element for some $(e, f) \in M_v$, there must exist some $(e', f') \in M_v$ such that $\mathrm{MCE}(e', f') = \emptyset$. But condition **(H3*)** holds, so $\mathrm{MCE}(e, f)$ must be a singleton for all $(e, f) \in M_v$.

($\Longleftarrow$). Now, suppose there exists some $v \in \mathbf{1}\Lambda^0$ and $(e,f) \in M_v$ such that $\mathrm{MCE}(e,f)$ is a singleton. Then by the argument above $\mathrm{MCE}(e,f)$ is a singleton for all pairs $(e,f) \in M_v$. Let $v = e_i f_j$ where $e_i, f_j \in \Lambda$. Now, consider $e_k \in \Lambda^{e_1}$ and $f_l \in \Lambda^{e_2}$. We have

$$\mathrm{MCE}(e_i f_j e_k, e_i f_j f_l) = \{e_i f_j \lambda : \lambda \in \mathrm{MCE}(e_k, f_l)\}$$

Since $r_{\mathbf{1}}(e_i f_j e_k) = r_{\mathbf{1}}(e_i f_j f_l) = v$, we have that $\mathrm{MCE}(e_i f_j e_k, e_i f_j f_l)$ is a singleton which implies that $\mathrm{MCE}(e_k, f_l)$ is a singleton. Since e_k and f_l were arbitrary, we have that $\mathrm{MCE}(e,f)$ is a singleton for all $e \in \Lambda^{e_1}$ and $f \in \Lambda^{e_2}$. Now let $\mu \in \mathbf{1}\Lambda^{e_1}$ and $\nu \in \mathbf{1}\Lambda^{e_2}$ with $r_{\mathbf{1}}(\mu) = r_{\mathbf{1}}(\nu)$. Write $\mu = \mu_{\mathbf{1}}\mu_{e_1}$ where $\mu_{\mathbf{1}} \in \Lambda^{\mathbf{1}} = r_{\mathbf{1}}(\mu)$ and $\mu_{e_1} \in \Lambda^{e_1}$, and write ν similarly. Then

$$\mathrm{MCE}(\mu,\nu) = \{\mu_{\mathbf{1}}\lambda : \lambda \in \mathrm{MCE}(\mu_{e_1}, \nu_{e_2})\}$$

which is a singleton since $\mathrm{MCE}(\mu_{e_1}, \nu_{e_2})$ is. Since μ and ν were arbitrary, we have that $\mathrm{MCE}(e,f) \neq \emptyset$ for all $e \in \mathbf{1}\Lambda^{e_1}$ and $f \in \mathbf{1}\Lambda^{e_2}$ with $r_{\mathbf{1}}(e) = r_{\mathbf{1}}(f)$, and hence we have condition **(H3*)** satisfied by Proposition 3.9. So we have equivalence. $\square$

4. C^*-Algebra Generated by Specific Examples

Consider the 2-graph, (Λ, d), from Section 2, together with some factorisation property $X \in F$ such that Λ_X does not satisfy condition **(H3*)**. Let $x = e_{i_1} f_{j_1} e_{i_2} f_{j_2} \dots e_{i_k} f_{j_k} \dots$ be an infinite path in Λ. Consider the following functions:

$$\mathcal{E} : \mathbb{Z}_2^\infty \times \mathbb{Z}_2^\infty \to \mathbb{Z}_2^\infty \times \mathbb{Z}_2^\infty \text{ defined by}$$
$$\mathcal{E}(\{i_k\},\{j_l\}) = (\{j'_m\},\{i'_n\}) \text{ whenever } f_{i_k} e_{j_{k+1}} = e_{j'_k} f_{i'_k}$$

and

$$\varphi : \mathbb{Z}_2^\infty \times \mathbb{Z}_2^\infty \to \mathbb{Z}_2^\infty \times \mathbb{Z}_2^\infty \text{ defined by}$$
$$\varphi(\{i_k\},\{j_l\}) = (\{j'_m\},\{i'_n\}) \text{ whenever } e_{j_{k+1}} f_{i_{k+1}} = f_{i'_k} e_{j'_{k+1}}$$

Where $\mathbb{Z}_2^\infty := \{i = \{i_k\}_{k=1}^\infty : i_k \in \{0,1\} \text{ for each } k = 1,2,\dots\}$ is an additive group with identity $0 = (0,0,0,\dots)$, $i^{-1} = i$ for all i. Note that in $\mathbb{Z}_2$, $1+1=0$. The $\mathcal{E}$ and φ functions for each $X \in F$ not satisfying condition **(H3*)** are given in the table below.

X	$\mathcal{E}$	φ
[1.]	$\mathcal{E}(\{i_k\},\{j_k\}) = (\{j_k\},\{i_{k+1}\})$	$\varphi(\{i_k\},\{j_k\}) = (\{j_k\},\{i_{k+1}\})$
[2.]	$\mathcal{E}(\{i_k\},\{j_k\}) = (\{j_k\},\{i_{k+1}+j_k\})$	$\varphi(\{i_k\},\{j_k\}) = (\{i_k+j_k\},\{i_{k+1}\})$
[8.]	$\mathcal{E}(\{i_k\},\{j_k\}) = (\{j_k\},\{i_{k+1}+1\})$	$\varphi(\{i_k\},\{j_k\}) = (\{j_k+1\},\{i_{k+1}\})$
[22.]	$\mathcal{E}(\{i_k\},\{j_k\}) = (\{i_{k+1}+1\},\{j_k+1\})$	$\varphi(\{i_k\},\{j_k\}) = (\{i_k+1\},\{j_{k+1}+1\})$

Lemma 4.1. *Consider Λ_X for some $X \in F$. Then if $\sigma^{(m,n)}(x) = x$ for all $x \in \Lambda_X^\infty$, the sign of m is not equal to the sign of n; that is if one is positive, the other is negative.*

Proof. Noticing that if $\sigma^{(m,n)}(x)$ if and only if $\sigma^{(-m,-n)}(x) = x$ and every $x \in \Lambda^\infty$ has a trivial period of $(0,0)$, we need only consider the case for $(m,n) \in \mathbb{N}^2 \backslash \{(0,0)\}$. So let $(m,n) \in \mathbb{N}^2 \backslash \{(0,0)\}$. Fix some $x \in \Lambda^\infty$ such that $x(0,e_1) = e_1$ and $x((m,n),(m+1,n) = e_2$. Such a path exists as there is only one $v \in \Lambda^0$. Then $\sigma^{(m,n)}(x)(0,e_1) = e_2$ but $x(0,e_1) = e_1$, and hence $\sigma^{(m,n)}(x) \neq x$. Since this is true for arbitrary $(m,n) \in \mathbb{N}^2 \backslash \{(0,0)\}$, we have that there is some $x \in \Lambda^\infty$ such that $\sigma^{(m,n)}(x) \neq x$ for all $(m,n) \in \mathbb{N}^2 \backslash \{(0,0)\}$. $\square$

Proposition 4.2. *Consider Λ_X for some $X \in F$. Let $x = e_{i_1} f_{j_1} e_{i_2} f_{j_2} e_{i_3} f_{j_3} \in \Lambda^\infty$. Then $\sigma^{(m,n)}(x) = x$ if and only if $\mathcal{E}^m(\{i_k\},\{j_k\}) = \varphi^n(\{i_k\},\{j_k\})$.*

Proof. We have that

$$\begin{array}{rcl}
x & = & e_{i_1} f_{j_1} e_{i_2} f_{j_2} \dots e_{i_k} f_{j_k} \dots \\
& = & e_{i_1} (f_{j_1} e_{i_2} \dots f_{j_k} e_{i_{k+1}} \dots) \\
& = & e_{i_1} \left(e_{i_1^{(1)}} f_{j_1^{(1)}} \dots e_{i_k^{(1)}} f_{j_k^{(1)}} \dots \right) \quad \text{where } (\{i_k^{(1)}\},\{f_k^{(1)}\}) = \mathcal{E}(\{i_k\},\{j_k\}) \\
& = & e_{i_1} e_{i_1^{(1)}} \left(f_{j_1^{(1)}} e_{i_2^{(1)}} \dots f_{j_k^{(1)}} e_{i_{k+1}^{(1)}} \dots \right) \\
& = & e_{i_1} e_{i_1^{(1)}} \left(e_{i_1^{(2)}} f_{j_1^{(2)}} \dots e_{i_k^{(2)}} f_{j_k^{(1)}} \dots \right) \quad \text{where } (\{i_k^{(2)}\},\{f_k^{(2)}\}) = \mathcal{E}^2(\{i_k\},\{j_k\}) \\
& = & \dots \\
& = & e_{i_1} e_{i_1^{(1)}} \dots e_{i_1^{(m-1)}} \left(e_{i_1^{(m)}} f_{j_1^{(m)}} \dots e_{i_k^{(m)}} f_{j_k^{(m)}} \dots \right) \\
& & \text{where } (\{i_k^{(m)}\},\{f_k^{(m)}\}) = \mathcal{E}^m(\{i_k\},\{j_k\})
\end{array}$$

Similarly, we have

$$\begin{array}{rcl}
x & = & e_{i_1} f_{j_1} e_{i_2} f_{j_2} \dots e_{i_k} f_{j_k} \dots \\
& = & f_{j_0'^{(1)}} e_{i_1'^{(1)}} \dots f_{j_k'^{(1)}} e_{i_{k+1}'^{(1)}} \dots \\
& = & f_{j_0'^{(1)}} \left(e_{i_1'^{(1)}} f_{j_1'^{(1)}} \dots e_{i_k'^{(1)}} f_{j_k'^{(1)}} \dots \right) \quad \text{where } (\{i_k'^{(1)}\},\{j_k'^{(1)}\}) = \varphi(\{i_k\},\{j_k\}) \\
& = & f_{j_0'^{(1)}} f_{j_0'^{(2)}} \left(e_{i_1'^{(2)}} f_{j_1'^{(2)}} \dots e_{i_k'^{(2)}} f_{j_k'^{(2)}} \dots \right) \quad \text{where } (\{i_k'^{(2)}\},\{j_k'^{(2)}\}) = \varphi^2(\{i_k\},\{j_k\}) \\
& = & \dots \\
& = & f_{j_0'^{(1)}} f_{j_0'^{(2)}} \dots f_{j_0'^{(n)}} \left(e_{i_1'^{(n)}} f_{j_1'^{(n)}} \dots e_{i_k'^{(n)}} f_{j_k'^{(n)}} \dots \right) \\
& & \text{where } (\{i_k'^{(n)}\},\{j_k'^{(n)}\}) = \varphi^n(\{i_k\},\{j_k\})
\end{array}$$

Now, $d\left(e_{i_1}e_{i_1^{(1)}}\dots e_{i_1^{(m-1)}}\right)=(m,0)$, so we must have

$$\sigma^{(m,0)}(x)=e_{i_1^{(m)}}f_{j_1^{(m)}}\dots e_{i_k^{(m)}}f_{j_k^{(m)}}\dots$$

and similarly $d\left(f_{j_0^{\prime(1)}}f_{j_0^{\prime(2)}}\dots f_{j_0^{\prime(n)}}\right)=(0,n)$ so we have

$$\sigma^{(0,n)}(x)=e_{i_1^{\prime(n)}}f_{j_1^{\prime(n)}}\dots e_{i_k^{\prime(n)}}f_{j_k^{\prime(n)}}\dots$$

Hence, if $\sigma^{(m,0)}(x)=\sigma^{(0,n)}(x)\iff\sigma^{(m,-n)}(x)=x$, then we must have $(\{i_k^{(m)}\},\{j_k^{(m)}\})=(\{i_k^{\prime(n)}\},\{j_k^{\prime(n)}\})$; that is $\mathcal{E}^m(\{i_k\},\{j_k\})=\varphi^n(\{i_k\},\{j_k\}$. □

Proposition 4.3. *Consider* $(\Lambda_{[\mathbf{2.}]},d)$. *There exists an* $x=e_{i_1}f_{j_1}e_{i_2}f_{j_2}e_{i_3}f_{j_3}\dots\in\Lambda^\infty_{[\mathbf{2.}]}$ *such that* $\mathcal{E}^m(\{i_k\},\{j_k\})\neq\varphi^n(\{i_k\},\{j_k\})$ *for any* $m,n\in\mathbb{N}$.

Proof. We know from the table above that $\mathcal{E}(\{i_k\},\{j_k\})=(\{j_k\},\{i_{k+1}+j_k\})$ and $\varphi(\{i_k\},\{j_k\})=(\{i_k+j_k\},\{i_{k+1}\})$. Let $\{\alpha_{m,r}\},\dots\{\eta'_{n,s}\}\in c_0(\mathbb{Z}_2)$ and $a_{m,0},\dots b'_{m,o}\in\mathbb{Z}_2^\infty$ so we can write

$$\mathcal{E}^m(\{i_k\},\{j_k\})=(\{a_{m,0}+\textstyle\sum_{r=k}^\infty\alpha_{m,r}i_r+\beta_{m,r}j_r\},\{a'_{m,0}+\sum_{s=k}^\infty\alpha'_{m,s}i_s+\beta'_{m,s}j_s\})$$

and

$$\varphi^n(\{i_k\},\{j_k\}=(\{b_{n,0}+\textstyle\sum_{r=k}^\infty\zeta_{n,r}i_r+\eta_{n,r}j_r\},\{b'_{n,0}+\sum_{s=k}^\infty\zeta'_{n,s}i_s+\eta'_{n,s}j_s\})$$

Suppose $a_{m,0}=a'_{m,0}=0$ for some $m\geq1$. Then $a_{m+1,0}=a'_{m,0}=0$ and $a;_{m+1,0}=a_{m+1,0}+a'_{m,0}=0+0=0$. Since $a_{1,0}=a'_{1,0}=0$ by the definition of $\mathcal{E}$, we have $a_{m,0}=a'_{m,0}=0$ for all $m\geq1$ by induction on m. Similarly, suppose $\alpha_{m,k}=\alpha'_{m,k}=0$ for some $m\geq1$ Then $\alpha_{m+1,k}=\alpha'_{m,k}=0$, and $\alpha'_{m+1,k}=\alpha'_{m,k}=0$. Since $\alpha_{1,k}=\alpha'_{1,k}=0$ by the definition of $\mathcal{E}$, we have $\alpha_{m,k}=\alpha'_{m,k}=0$ for all $m\geq1$ by induction on m. Now suppose $b_{m,0}=b'_{m,0}=0$ for some $m\geq0$. Then $b_{m+1,0}=b_{m,0}+b'_{m,0}=0+0=0$ and $b'_{m,0}=b_{m+1,0}=0$. As $b_{1,0}=b'_{1,0}=0$, we have $b_{m,0}=b'_{m,0}=0$ for all $m\geq1$ by induction on m. Similarly, suppose $\zeta_{m,k}=1$ for some $m\geq1$. Then $\zeta_{m+1,k}=\zeta_{m,k}+\zeta'_{m,k}=1$ provided $\zeta'_{m,k}=0$. But since $\zeta'_{m,s}$ is only defined for $s\geq k+1$, we have $\zeta_{m_0+1,k}=\zeta_{m_0,k}=1$. Since $\zeta_{1,k}=1$ by the definition of φ, we have that $\zeta_{m,s}=1$ for all $m\geq1$ by induction on m. Then

$$\begin{aligned}\mathcal{E}^m(\{i_k\},\{j_k\}) &= (\{a_{m,0}+\textstyle\sum_{r=k}^\infty\alpha_{m,r}i_r+\beta_{m,r}j_r\},\{a'_{m,0}+\sum_{s=k}^\infty\alpha'_{m,s}i_s+\beta'_{m,s}j_s\})\\ &= (\{a_{m,0}+\alpha_{m,1}\},\{a'_{m,0}+\alpha'_{m,1}\})\ \text{ since } i_1=1, i_k=0 \text{ for all } k\geq2,\\ &\qquad\qquad\text{and } j_k=0 \text{ for all } k\geq1\end{aligned}$$

and

$$\begin{aligned}\varphi^n(\{i_k\},\{j_k\}) &= (\{b_{n,0}+\textstyle\sum_{r=k}^{\infty}\zeta_{n,r}i_r+\eta_{n,r}j_r\}, b'_{n,0}+\textstyle\sum_{s=k}^{\infty}\zeta'_{n,s}i_s+\eta'_{n,s}j_s\})\\ &= (\{b_{m,0}+\zeta_{m,1}\},\{b'_{m,0}+\zeta'_{m,1}\}) \text{ since } i_1=1, i_k=0 \text{ for all } k\geq 2,\\ &\qquad\qquad\qquad\qquad\text{and } j_k=0 \text{ for all } k\geq 1\end{aligned}$$

Now, consider the infinite path $x=e_1f_0e_0f_0e_0f_0\cdots\in\Lambda^{\infty}_{[2.]}$. Then we have $\{i_k\}=(1,0,0,0,\dots)$ and $\{j_k\}=(0,0,0,0,\dots)$. Then, for any $m,n\in\mathbb{N}\backslash\{0\}$ we have

$$\begin{aligned}\mathcal{E}^m(\{i_k\},\{j_k\}) &= (\{a_{m,0}+\alpha_{m,1}i_k\},\{a'_{m,0}+\alpha'_{m,1}i_k\})\\ &= ((0,0,0\dots),(0,0,0\dots)) \text{ since }\dots\end{aligned}$$

but

$$\begin{aligned}\varphi^n(\{i_k\},\{j_k\}) &= (\{b_{m,0}+\zeta_{m,1}\},\{b'_{m,0}+\zeta'_{m,1}\})\\ &= ((1,0,0,\dots),(0,0,0,\dots)) \text{ since }\dots\\ &\neq \mathcal{E}^m(\{i_k\},\{j_k\}) \text{ for any } m\in\mathbb{N}\backslash\{0\}\end{aligned}$$

Hence, for any $m,n\in\mathbb{N}\backslash\{0\}$, there exists an $x=e_{i_1}f_{j_1}e_{i_2}f_{j_2}e_{i_3}f_{j_3}\cdots\in\Lambda^{\infty}_{[22.]}$ such that $\mathcal{E}^m(\{i_k\},\{j_k\})\neq\varphi^n(\{i_k\},\{j_k\})$. □

Proposition 4.4. *Consider* $(\Lambda_{[\mathbf{22.}]},d)$. *There exists an* $x=e_{i_1}f_{j_1}e_{i_2}f_{j_2}e_{i_3}f_{j_3}\cdots\in\Lambda^{\infty}_{[\mathbf{22.}]}$ *such that* $\mathcal{E}^m(\{i_k\},\{j_k\})\neq\varphi^n(\{i_k\},\{j_k\})$ *for any* $m,n\in\mathbb{N}$.

Proof. This time we have $\mathcal{E}(\{i_k\},\{j_k\})=(\{i_{k+1}+1\},\{j_k+1\})$ and $\varphi(\{i_k\},\{j_k\})=(\{i_k+1\},\{j_{k+1}+1\})$. Consider $\mathcal{E}^m(\{i_k\},\{j_k\})$ and $\varphi^m(\{i_k\},\{j_k\})$ written as in the proof of Proposition 4.3. For each m, $\alpha_{m,r}$ is only defined for $r\geq k+m$, and so $\alpha_{m,k}=0$ for all $m\geq 1$. By the definition of φ, $\zeta_{m+1,k}=\zeta_{m,k}$, and since $\zeta_{1,k}=1$ we have $\zeta_{m,k}=1$ for all $m\geq 1$ by induction on m. Also, by the definition of $\mathcal{E}$, $a_{m+1,0}=a_{m,0}+1$, and since $1+1=0$ in $\mathbb{Z}_2$, we have that $a_{m,0}=0$ if m is even and $a_{m,0}=1$ if m is odd. Similarly, by the definition of φ we have $b_{m,0}=0$ if m is even and $b_{m,0}=1$ if m is odd. So, we have $a_{m,0}=b_{n,0}$ if both m and n are even or odd, and $a_{m,0}\neq b_{n,0}$ if one is odd and the other is even. So, let $m,n\in\mathbb{N}\backslash\{0\}$ so that either they are both odd or they are both even. Consider $x=e_1f_0e_0f_0e_0f_0\cdots\in\Lambda^{\infty}_{[22.]}$. Then $\{i_k\}=(1,0,0,\dots)$ and $\{j_k\}=(0,0,0,\dots)$. Then

$$\begin{aligned}\mathcal{E}^m(\{i_k\},\{j_k\}) &= ((a_{m,0}+\alpha_{m,1},0,0,\dots),\{j'_k\}) \text{ because } i_k=0 \text{ for all } k\geq 2,\\ &\qquad\qquad\qquad\qquad\text{and where } \{j'_k\}\in\mathbb{Z}_2^{\infty}\\ &= ((a_{m,0},0,0,\dots),\{j'_k\}) \text{ as } \alpha_{m_k}=0 \text{ for all } m\geq 1\end{aligned}$$

and

$$\begin{aligned}\varphi^n(\{i_k\},\{j_k\}) &= ((b_{n,0}+\zeta_{n,1},0,0,\dots),\{j''_k\}) \text{ because } i_k = 0 \text{ for all } k \geq 2,\\ &\qquad\qquad\qquad\qquad\qquad \text{and where } \{j''_k\} \in \mathbb{Z}_2^\infty\\ &= ((b_{n,0}+1,0,0,\dots),\{j''_k\}) \text{ as } \zeta_{n,k} = 1 \text{ for all } n \geq 1\\ &= ((a_{m,0}+1,0,0,\dots),\{j''_k\} \text{ since } m \text{ and } n \text{ are both even or odd}\\ &\neq \mathcal{E}^m(\{i_k\},\{j_k\})\end{aligned}$$

Now, let $m, n \in \mathbb{N}$ such that one is even and one is odd. Consider $x = e_0 f_0 e_0 f_0 e_0 f_0 \cdots \in \Lambda^\infty_{[22.]}$, such that $\{i_k\} = (0,0,0,\dots)$ and $\{j_k\} = (0,0,0,\dots)$. Then

$$\mathcal{E}^m(\{i_k\},\{j_k\}) = ((a_{m,0},0,0,\dots),\{j'_k\}) \text{ because } i_k = 0 \text{ for all } k \geq 1,$$

and

$$\begin{aligned}\varphi^n(\{i_k\},\{j_k\}) &= ((b_{n,0},0,0,\dots),\{j''_k\}) \text{ because } i_k = 0 \text{ for all } k \geq 1,\\ &\neq \mathcal{E}^m(\{i_k\},\{j_k\} \text{ since } a_{m,0} \neq b_{n,0}\end{aligned}$$

Hence, for any $m, n \in \mathbb{N}\backslash\{0\}$, there exists an $x = e_{i_1} f_{j_1} e_{i_2} f_{j_2} e_{i_3} f_{j_3} \cdots \in \Lambda^\infty_{[22.]}$ such that $\mathcal{E}^m(\{i_k\},\{j_k\}) \neq \varphi^n(\{i_k\},\{j_k\})$. □

References

[1] S. Allen, *Dual Graph Constructions for Higher Rank Graphs,* University of Newcastle.

[2] S. Allen, D. Pask & A. Sims, *A Dual Graph Construction for The Higher-Rank Graphs, and K-Theory for finite 2-Graphs*, University Of Newcastle. (2004)

[3] A. Kumjian & D. Pask, *Higher Rank Graph C^*-Algebras,* New York J. Math. **6** (2000) 1-20

[4] A. Kumjian, D. Pask & A. Sims, *k-morphs*, in preparation.

[5] G. Robertson & T. Steger, *Affine Buildings, Tiling Systems and Higher-Rank Cuntz-Krieger Algebras,* University Of Newcastle. (1999)

[6] A. Sims, *C^*-Algebras Associated to Higher-Rank Graphs*, Phd. Thesis, University Of Newcastle. (2003)