HSC Mathematics

Workshop 3

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Optimization problems

In *Optimization problems* (or *Max-min problems*) we have a quantity that we want to optimize, subject to a constraint that needs to be satisfied by two dependent variables.

For example we may want to maximise the area of a rectangular field subject to the constraint that the perimeter of the field (effectively the sum of the length and width) is a given fixed amount.

Method of solution:

1. Since there are 2 dependent variables we need 2 equations
2. Form one equation from the constraint
3. Form another equation from the quantity to be optimized
4. Make one of the variables the subject of the constaint
5. Substitute this expression into the second equation, differentiate, set equal to 0 and solve, thus finding the stationary point
6. Determine the nature of the stationary point
7. Determine the optimum value, if required.

A simple example

Find the maximimum area enclosed by a rectangle with perimeter 60m.

1. **Exercise:**

   Rectangular airconditioning ducting is to be placed inside a circular pipe of diameter 1 metre. Find the dimensions of the ducting to maximise the volume of air carried.

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2. **Exercise:**

   A triangular prism is to have a volume of $100cm^3$. Find the dimensions of the prism to minimise the surface area, given that the cross-section is an isosceles triangle.

3. **Exercise:**

   The cost *per hour* of running a ship is comprised of two components: fixed costs (wages, insurance etc) of \$5000 and variable costs (fuel) of $20v^2$ where v is the speed in km/h. At what speed should the ship travel to complete a journey of 500km at minimum cost? Does the optimum speed change if the ship cannot travel faster than $20km/h$ and must complete the journey in no more than 30 hours?

4. **Exercise:**

   A structural beam is in the shape of a rectangular prism. The strength of the beam is proportional to the product of the cross-sectional area and the square of the length. If the sum of the dimensions of the beam is 10 metres and the cross-section is a square, find the dimensions that produce the strongest beam.

Integration

We have the following basic results:

(a) $\int ax^n dx = \frac{ax^{n+1}}{n+1} + C$

(b) $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$

(c) $\int_a^b f(x)dx = F(b) - F(a)$ where $F(x)$ is the primitive of $f(x)$.

5. **Exercise:**

   Evaluate:
   (i) $\int \frac{5}{9\sqrt{x^3}}dx$ ii) $\int (x^2+4)^2 dx$ (iii) $\int \frac{4x^3+3x^2+10x}{2x}dx$

6. **Exercise:**

   Using the graph in Figure 1 of the function $y = f(x)$, determine
   (a) $\int_0^5 f(x)dx$ (b) the area bounded by the function and the x-axis

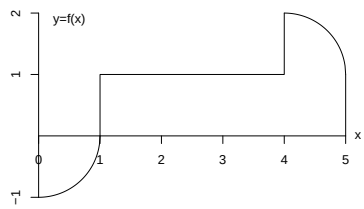

Figure 1: Function for example 6

7. **Exercise:**

   Find the area bounded by the functions $f(x) = (x+2)^2$ and $g(x) = |2x+1|$ and the x-axis.

8. **Exercise:**

   Find the area bounded by the functions $f(x) = -x^2 + 2x + 3$ and $g(x) = x^2 - 8x + 11$.

Points to note:

In **Example 6** the definite integral calculates the *net area* (taking account of signs) whereas the area ignores the sign.
In **Example 7** we need two separate integrals with different limits and we *add* the integrals.
In **Example 8** we *subtract* the functions within one integral.
In **Examples 6 and 7** we can use ordinary area formulae when the functions are straight lines.

Volumes of solids of revolution

(a) For rotation around the x-axis: $V = \pi \int_a^b y^2 dx$ where a, b are values on the x-axis

(b) For rotation around the y-axis: $V = \pi \int_a^b x^2 dy$ where a, b are values on the y-axis

9. **Exercise:**

   Find the volume of the solid of revolution formed when $y = x^3$ is rotated around the x-axis from $x = 2$ to $x = 4$.

10. **Exercise:**

    Find the volume of the solid of revolution formed when $y = x^3$ is rotated around the y-axis from $y = 2$ to $y = 4$.

Areas by approximation

There are functions for which it is not possible to find a primitive and therefore we cannot calculate the area bounded by the function and x-axis eg you do not yet know the primitive for $\frac{1}{x}$.

In these situations we use approximate area formulae.

Trapezoidal rule
We approximate the area with trapezia. The area of a trapezium with parallel sides of length a and b that are distance h apart is

$$A = \frac{h}{2}[a + b] \tag{1}$$

If two points on the curve are (x_0, y_0) and (x_1, y_1) then the area of the trapezium is

$$A = \frac{x_1 - x_0}{2}[y_0 + y_1] \tag{2}$$

If we have a 3 points on the curve, equally spaced on the x-axis then the total area of the trapezia is:

$$\begin{aligned} A &= \frac{x_1 - x_0}{2}[y_0 + y_1] + \frac{x_2 - x_1}{2}[y_1 + y_2] \\ &= \frac{h}{2}[y_0 + 2y_1 + y_2] \end{aligned} \tag{3}$$

where h is the constant difference between x-values.

We can use a table for the calculations, best illustrated by an example.

Simpson's rule

Simpson's rule fits a parabola through the endpoints $(x_0, f(x_0)), (x_1, f(x_1))$ and the point on the curve whose abscissa is the midpoint of x_0 and x_1. Then the approximate area under the curve is given by:

$$A \approx \frac{x_1 - x_0}{6}[f(x_0) + 4f(\frac{x_0 + x_1}{2}) + f(x_1)] \tag{4}$$

Note that there are 3 function values with weightings $(\frac{1}{6}, \frac{4}{6}, \frac{1}{6})$ and $x_1 - x_0 = 2h$.

We set out the calculations in a table similar to the Trapezoidal Rule.

Then

$$A \approx \frac{h}{3} \sum wy \tag{5}$$

11. **Exercise:**

    Evaluate $\int_2^4 \frac{1}{x} dx$ using 5 function values with (a) Trapezoidal Rule (b) Simpson's Rule.

12. **Exercise:**

    The diagram (given on the slide) shows a cross-section of a river. Find the approximate area of the cross-section using Simpson's Rule.

13. **Exercise:**

    Calculate the volume of the solid of revolution when $y = \frac{1}{\sqrt{1+x^2}}$ is rotated around the x-axis between $x = 0$ and $x = 2$ using Simpson's Rule with 5 function values to evaluate the integral.