HSC Mathematics - Extension 1

Workshop 2

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Mixed Topics

Trigonometric functions

Problem:

$$\int \cos^2 x \; dx \;\; = ? \tag{1}$$

Note:

- Always check a solution - how can I prove the result is correct?

- Formulae are useful but can lead you astray. Be able to derive from first principles.

1. **Exercise:** Find $\int \sin^2 5x \; dx$

   The previous results used the double angle result for *cos*.
   It is useful to remember the double angle result for *sin* in reverse:

   $$\sin 2x = 2 \sin x \cos x \tag{2}$$

   So

   $$\sin x \cos x = \frac{1}{2} \sin 2x \tag{3}$$

2. **Exercise:** How many ways can we calculate $\int \sin x \cos x \; dx$?

   A special limit

   We know

   $$\lim_{x \to 0} \frac{\sin x}{x} = 1 \tag{4}$$

   Do we know why???

   What is

   $$\lim_{x \to 0} \frac{\sin ax}{bx} \; ? \tag{5}$$

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3. **Exercise:** Find $\lim_{x\to 0} \frac{\sin 5x}{3x}$ showing working.

   t results

   The t results are derived from the tan double angle formula.

   Trigonometric equations

4. **Exercise:**

   Solve for $0 \le \theta \le 2\pi$,

$$\sin\theta + \sqrt{3}\cos\theta = 1$$

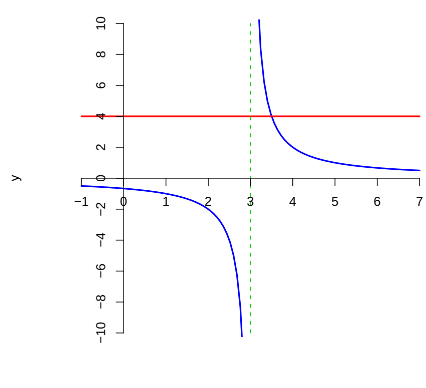
   (a) by expressing the LHS in the form $R\sin(\theta+\alpha)$
   (b) using the t results

Hyperbolic inequalities

Solve $\frac{2}{x-3} < 4$ and $\frac{x+3}{x-3} < 4$

- The usual method of solution is to multiply both sides by the square of the denominator.
- There are other methods which are simpler.

Solving $\frac{2}{x-3} < 4$ graphically



Now look at the slightly more complicated inequality:

$$\text{Solve } \frac{x+3}{x-3} < 4$$

5. **Exercise:** Solve (a) $\frac{-3}{x+2} \leq -2$ (b) $\frac{2x+1}{2-x} \geq 4$

Angle between two lines

The standard method uses the equation $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$
There is a simpler way!

Example (a) Find the acute angle between the lines $y = 5x - 1$ and $y = 2x + 1$.
(b) The angle between two lines is 1.05 radians. If the equation of one line is $y = -2x + 6$, find the gradient of the other.

6. **Exercise:** Find the two possible gradients of the line that intersects with $x - \sqrt{3}y + 2 = 0$ at $45°$.

7. **Exercise:** Find the acute angle between the tangents to the curves $y = x^3$ and $y = (x-2)^2$ at their point of intersection.

Division of an interval

We can either use the standard formulae or diagrams of similar triangles.

Suppose we have two points A(2,5) and B(8,14) and we wish to divide AB in the ratio 1:2 (a) internally and (b) externally.

8. **Exercises:** Find the point that divides the interval AB, where A is (-2,-3) and B is (1,2), in the ratio 3:1 (a) internally (b) externally.

Integration by substitution

Suppose we have to evaluate $\int_a^b f(x)dx$ by substitution, using the given substitution $x = f(u)$.

The steps are:

(a) replace every x by $f(u)$

(b) calculate $\frac{dx}{du}$ and then substitute for dx

(c) calculate the new limits ie u values when $x = (a, b)$

Example: Evaluate $\int_1^2 \frac{1}{1+\sqrt{x}} dx$ using the substitution $x = u^2$ for $u > 0$.

9. **Exercises:** Find (a) $\int_0^1 \frac{dx}{1+x^2}$ using $x = \tan u$
(b) $\int_0^7 \frac{dx}{1+\sqrt[3]{1+x}}$ using $x = u^3 - 1$

Circle properties

A circle property that commonly appears in exam questions is:

The angle between a tangent and a chord is equal to the angle in the alternate segment.

The three properties of intersecting chords are just three variations of the one property. The chords can intersect internally or externally or one chord can be a tangent.

10. **Exercise:** Prove the relationship for chords that intersect externally.

Induction

There are three types of statement that can be proved inductively:

(a) sum of series

(b) divisibility

(c) inequality

The 4 steps required for a proof:

(a) Prove true for the specified lowest value of n (usually 1)

(b) Assume true for $n = k$

(c) Prove true for $n = k + 1$, using the assumed statement in (2)

(d) Conclusion - the statement is true for $n = k+1$ if it is true for $n = k$. The statement was true for the lowest value of n and hence must be true for all values of n.

11. **Exercises:** Prove by induction: (a) $2 + 6 + 10 + ...(4n - 2) = 2n^2$ for $n \geq 1$
(b) $7^n - 2^n$ is divisible by 5 for $n \geq 1$
(c) $n^3 \geq 5n - 4$ for $n \geq 1$

Parametric equations and Locus problems

No need to remember any formulae here as everything has to be derived from first principles.

12. **Exercise:** Let $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ be two points on the parabola $x^2 = 4ay$.
(i) Find the equation of PQ and find the condition for PQ to be a focal chord.
(ii) Find the locus of the points of intersection of the normals at P and Q given that PQ is a focal chord.

Polynomials

The Factor Theorem
If the polynomial $P(x)$ has $(x - a)$ as a factor then $P(a) = 0$

The Remainder Theorem
If the polynomial $P(x)$ is divided by $(x - a)$ the remainder is $P(a)$

13. **Exercise:** $(x - 2)$ is a factor of the polynomial $P(x)$ and the remainder after dividing by $(x + 2)$ is -12.
When $P(x)$ is divided by $(x - 2)(x + 2)$ the remainder is $R(x)$.
(i) Explain why the general form of $R(x)$ is $R(x) = ax + b$
(ii) Find values of a and b

Relationship between roots of a cubic

Know the relationships between the coefficients and the sum of the roots taken 1,2,3,... at a time.

14. **Exercise:** It is known that one of the roots of $x^3 - 9x^2 + 20x + k = 0$ is twice the sum of the other two roots. Find the value of k.