Integration

3.1 The indefinite integral and the definite integral

3.1.1 The indefinite integral

Suppose that

\[ \frac{dy}{dx} = 1 \]

True or False The function \( y \) is given by

\[ y = x? \]
Differentiate the following functions

\[
y = x \quad \frac{dy}{dx} = \\
y = x + 1 \quad \frac{dy}{dx} = \\
y = x + 2 \quad \frac{dy}{dx} = \\
y = x + 3 \quad \frac{dy}{dx} = \\
\]

Suppose that

\[
\frac{dy}{dx} = 1
\]

Then

\[
y = \underline{_______}
\]
We know that

\[
\frac{d}{dx} (x^2) = \underline{______}
\]

**True or False**

Therefore, the integral of \(2x\) with respect to \(x\) is \(x^2\)?

This is written

\[
\int 2x \, dx = x^2
\]

where the symbols \(\int \ldots \, dx\) denote ‘the integral of \(\ldots\) with respect to \(x\)’.
3.1.2 The antiderivative

Definition

Suppose that \( \frac{d}{dx} \mathcal{F}(x) = f(x) \). The \( \mathcal{F} \) is called an antiderivative or indefinite integral or primitive of a function \( f \).

The antiderivative \( \mathcal{F} \) is usually denoted by

\[
\mathcal{F}(x) = \int f(x) \, dx
\]

More generally,

\[
\int f(x) \, dx = \mathcal{F}(x) + c
\]

where \( c \) is known as the constant of integration.
Evaluate the integrals

1. \[ \int 1 \cdot dx \]

2. \[ \int 2x \, dx \]

3. \[ \int 3x^2 \, dx \]

4. \[ \int \cos (x) \, dx \]

5. \[ \int \sec^2 (x) \, dx \]

6. \[ \int \frac{1}{x} \, dx \]

7. \[ \int e^x \, dx \]

8. \[ \int \sin (x) \, dx \]
\[ c = \text{constant} \]
\[ x + c \]
\[ x^2 + c \]
\[ x^3 + c \]
\[ \sin x + c \]
\[ \tan x + c \]
\[ \ln x + c \]
\[ e^x + c \]
\[ -\cos x + c \]
Suppose that
\[ \frac{d}{dx} \mathcal{F}(x) = f(x) \]

Hence
\[ \int f(x) \, dx = \mathcal{F} + c \]

Thus
\[ \frac{d}{dx} \mathcal{F}(x) = \frac{d}{dx} \left[ \int f(x) \, dx \right] \]
\[ = f(x) \]
i.e. \[ \frac{d}{dx} \left[ \int f(x) \, dx \right] = f(x) \]
3.2 The Definite Integral

3.2.1 Fixed end points

If $\mathcal{F}$ is the antiderivative of a function $f$, then the definite integral of $f$ is given by

$$\int_{a}^{b} f(x) \, dx = \left[ \mathcal{F}(x) \right]_{a}^{b} = \mathcal{F}(b) - \mathcal{F}(a)$$

**Example** Calculate

$$\int_{0}^{2} 1 \cdot dx = \left[ \_ \right]_{0}^{2} = \_$$

$a$ and $b$ are called the limits of integration, and $x$ the dummy variable of integration. The function $f(x)$ is called the integrand.
$x \quad 2 - 0 = 0$
Question. What does $\int_{a}^{b} f(x) \, dx$ ‘mean’?
Evaluate the definite integrals

1. \[ \int_{0}^{5} 1 \cdot dx \]
2. \[ \int_{1}^{5} x dx \]
3. \[ \int_{2}^{3} 6x^2 dx \]
4. \[ \int_{0}^{\pi/2} \cos (2x) dx \]
5. \[ \int_{0}^{\pi/4} \sec^2 (x) dx \]
6. \[ \int_{e}^{e^2} \frac{4}{x} dx \]
7. \[ \int_{0}^{2} 5e^x dx \]
$5 \left( e^2 - 1 \right)$
The value of the integral depends on the function to be integrated, not on the particular variable used, i.e.

\[
\int_{a}^{b} f(x)dx = \int_{a}^{b} f(t)dt = \int_{a}^{b} f(u)du
\]

\[
= \ldots = \mathcal{F}(b) - \mathcal{F}(a)
\]

e.g.

\[
\int_{1}^{2} 2xdx = \int_{1}^{2} 2tdt = \int_{1}^{2} 2udu
\]

\[
= \left[\underline{\quad}\right]_{1}^{2} = \underline{\quad}
\]
\[ x^2 \quad 4 - 1 = 3 \]
3.2.2 Variable Endpoints

A definite integral can take the following form \( \int_a^x f(x)\,dx \) where the upper limit is allowed to vary. For such integrals it is best to use a letter different from \( x \) for the variable of integration; thus we write

\[
\int_a^x f(t)\,dt \quad \text{rather than} \quad \int_a^x f(x)\,dx.
\]

Both endpoints may be functions, for example

\[
\int_{g(x)}^{h(x)} f(t)\,dt = [F(t)]_{g(x)}^{h(x)}
\]

\[
= F[h(x)] - F[g(x)]
\]

Motivation?
Evaluate the integrals

1. \[ \int_{t}^{t^2} 5 \cdot dx \]

2. \[ \int_{\cos t}^{\sin t} \frac{x}{2} \, dx \]

3. \[ \int_{0}^{t} \cos \left( \frac{x}{4} \right) \, dx \]

4. \[ \int_{e^t}^{e^{2t}} \frac{1}{2x} \, dx \]

5. \[ \int_{t}^{5t} \sin (x) \, dx \]
\[ 5t(t - 1) \]
\[ \frac{1}{4} \left( \sin^2 t - \cos^2 t \right) \]
\[ 4 \sin \left( \frac{t}{4} \right) \]
\[ \frac{t}{2} \]
\[ \cos (t) - \cos (5t) \]
We know that
\[ \int_a^x f(t) \, dt = \mathcal{F}(x) - \mathcal{F}(a) \]

Claim
\[ \frac{d}{dx} \int_a^x f(t) \, dt = f(x) \]

Proof
\[ \frac{d}{dx} \int_a^x f(t) \, dt = \frac{d}{dx} \left[ \mathcal{F}(x) - \mathcal{F}(a) \right] \]
\[ = \frac{d}{dx} \left[ \mathcal{F}(x) \right] - \frac{d}{dx} \left[ \mathcal{F}(a) \right] \]
\[ = \frac{d}{dx} \left[ \mathcal{F}(x) \right] \]
\[ = f(x). \]

Motivation?
A more general result can be found using the chain rule.

Claim

\[
\frac{d}{dx} \int_{a}^{g(x)} f(t) \, dt = f[g(x)] \cdot g'(x)
\]

Motivation?

**Example** Evaluate \( \frac{d}{dx} \left[ \int_{\frac{\pi}{4}}^{x^2} \cos t \, dt \right] \) by

1. Using the above formula
2. By first integrating and then differentiating
1. Use the formula

\[
\begin{align*}
  a &= \\
  f &= \\
  g &= \\
  g' &= \\

  \therefore \frac{d}{dx} \left[ \int_{\frac{\pi}{4}}^{x^2} \cos t \, dt \right] &= 
\end{align*}
\]
\[
\frac{\pi}{4} \cos t \\
x^2 \\
2x
\]
\[ \cos \left( x^2 \right) \cdot 2x \]
2. (i) First do the integration

\[ \int_{\pi/4}^{x^2} \cos t \, dt = \left[ \_ \right]_{\pi/4}^{x^2} \]

\[ = \_ \]

(ii) Now do the differentiation

\[ \frac{d}{dx} \left[ \int_{\pi/4}^{x^2} \cos t \, dt \right] = \_ \]

\[ = \_ \]
\[
\sin t \\
\sin (x^2) - \sin \left(\frac{\pi}{4}\right)
\]
\[
\frac{\mathrm{d}}{\mathrm{d}x} \sin \left( x^2 \right) - \frac{\mathrm{d}}{\mathrm{d}x} \sin \left( \frac{\pi}{4} \right)
\]

\[
2x \cos \left( x^2 \right)
\]
Example Evaluate
\[ \frac{d}{dx} \left[ \int_{0}^{x} (t^2 - 2t + 4) \, dt \right] . \]

\[ a = \_ \]

\[ f = \_ \]

\[ g = \_ \]

\[ g' = \_ \]

\[ \therefore \frac{d}{dx} \left[ \int_{0}^{x} (t^2 - 2t + 4) \, dt \right] = \]

\[ \_ \]
$x$

$t^2 - 2t + 4$

$x$

1
\((x^2 - 2x + 4) \cdot 1\)
Example Evaluate

\[
\frac{d}{dx} \left[ \int_a^{x^2} (t^2 - 2t + 4)^{3/4} \, dt \right]
\]

\[a = \_\]

\[f = \_\]

\[g = \_\]

\[g' = \_\]

\[
\therefore \frac{d}{dx} \left[ \int_a^{x^2} (t^2 - 2t + 4)^{3/4} \, dt \right]
\]

\[= \_\]

\[= \_
\]
\[ a \]
\[ \left( t^2 - 2t + 4 \right)^{3/4} \]
\[ x^2 \]
\[ 2x \]
\[
\left[(x^2)^2 - 2(x^2) + 4\right]^{3/4} \cdot 2x
\]

\[
(x^4 - 2x^2 + 4)^{3/4} \cdot 2x
\]
3.2.3 Properties of Integrals

(i) \( \int_a^b (f+g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx \)

(ii) \( \int_a^b (\alpha f)(x)dx = \int_a^b \alpha f(x)dx = \alpha \int_a^b f(x)dx \)

(iii) \( \int_a^a f(x)dx = 0 \)

(iv) \( \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \)

(v) \( \int_a^b f(x)dx = -\int_b^a f(x)dx \)

(vi) If \( f(x) \) is an odd function (i.e., 
\( f(x) = -f(-x) \) for all \( x \))
\( \int_{-a}^a f(x)dx = 0. \)

(vii) If \( f(x) \) is an even function (i.e., 
\( f(x) = f(-x) \) for all \( x \))
\( \int_{-a}^a f(x)dx = 2 \int_0^a f(x)dx. \)

(viii) If \( f(x) \leq g(x) \) for \( a \leq x \leq b \), then
\( \int_a^b f(x)dx \leq \int_a^b g(x)dx \)
3.2.4 Exercises