Fundamentals Lecture Eight

Basic Trig Identities :

- Pythagorean Formulae (1.13, page 1-31)
- Sum and Difference of Angles Formulae (1.13, page 1-32)
- Double Angle Formulae (1.13, page 1-32)
- Half-angle Formulae (1.13, pages 1-32 & 1-33)
- Products as Sums and Differences (1.13, page 1-33 & 1-34)
- Half-Angle tangent formulae (1.14, page 1-34)
Basic Trigonometric Identities

(i) Pythagorean Formulae

\[
\begin{align*}
\sin^2 \theta + \cos^2 \theta &= 1 \\
1 + \cot^2 \theta &= \csc^2 \theta \\
\tan^2 \theta + 1 &= \sec^2 \theta
\end{align*}
\]

**Question**: How are the second and third of these formulae derived from the first?
Example

Show that

\[ (\sin \theta + \cos \theta)^2 + (\sin \theta - \cos \theta)^2 = 2 \]

Solution

\[
LHS = (\sin \theta + \cos \theta)^2 + (\sin \theta - \cos \theta)^2
\]

= 

= 

= 

= 

= RHS \quad \square
\[ + \sin^2 \theta + \cos^2 \theta + 2 \cos \theta \sin \theta \]
\[ + \sin^2 \theta + \cos^2 \theta - 2 \cos \theta \sin \theta \]
\[ 2 \sin^2 \theta + 2 \cos \theta \]
\[ 2 (\sin^2 \theta + \cos^2 \theta) \]
\[ 2 \]
Example

Verify that

\[
\frac{1 - \sin \theta}{\cos \theta} = \frac{\cos \theta}{1 + \sin \theta} \quad \text{Ex1N Q29(c)}
\]

Solution Multiply both sides of the equation by \( \cos \theta (1 + \sin \theta) \) to obtain
(1 − \sin \theta) (1 + \sin \theta) = \cos \theta \cos \theta

1 − \sin^2 \theta = \cos^2 \theta

1 = \cos^2 \theta + \sin^2 \theta

1 = 1

because \sin^2 \theta + \cos^2 \theta = 1 \text{ by the Pythagorean formula}
(ii) Sum and Difference of Angle Formulae

\[
\begin{align*}
\sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\
\cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \\
\tan(A \pm B) &= \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}
\end{align*}
\]

Example Find an exact value for \( \cos \frac{7\pi}{12} \)

Solution Now, \( \frac{7\pi}{12} = \frac{\pi}{3} + \frac{\pi}{4} \)

\[\therefore \cos \frac{7\pi}{12} = \cos \left( \frac{\pi}{3} + \frac{\pi}{4} \right) = \frac{1 - \sqrt{3}}{2\sqrt{2}}\]
\[
\cos \left( \frac{\pi}{3} \right) \cos \left( \frac{\pi}{4} \right) - \sin \left( \frac{\pi}{3} \right) \sin \left( \frac{\pi}{4} \right) \\
\frac{1}{2} \cdot \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}}
\]
**Example** Find an exact value for 
\[ \sin \frac{\pi}{12} \]

**Solution** Now, 
\[ \frac{7\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4} \]

\[ \therefore \sin \frac{\pi}{12} = \sin \left( \frac{\pi}{3} - \frac{\pi}{4} \right) \]

\[ = \]

\[ = \]

\[ = \]

\[ = \frac{\sqrt{2}}{4} \left( \sqrt{3} - 1 \right) \]
\[
\sin \left(\frac{\pi}{3}\right) \cos \left(\frac{\pi}{4}\right) - \cos \left(\frac{\pi}{3}\right) \sin \left(\frac{\pi}{4}\right) \\
\frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} - \frac{1}{2} \cdot \frac{1}{\sqrt{2}} \\
\frac{1}{2\sqrt{2}} \left(\sqrt{3} - 1\right)
\]
(iii) Double Angle Formulae

<table>
<thead>
<tr>
<th>Equation</th>
<th>Simplified Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin 2\theta )</td>
<td>( 2 \sin \theta \cos \theta )</td>
</tr>
<tr>
<td>( \cos 2\theta )</td>
<td>( \cos^2 \theta - \sin^2 \theta )</td>
</tr>
<tr>
<td></td>
<td>( 2 \cos^2 \theta - 1 )</td>
</tr>
<tr>
<td></td>
<td>( 1 - 2 \sin^2 \theta )</td>
</tr>
<tr>
<td>( \tan 2\theta )</td>
<td>( \frac{2 \tan \theta}{1 - \tan^2 \theta} )</td>
</tr>
</tbody>
</table>

These formulae can be obtained by letting \( A = B = \theta \) in (ii)
Example Prove that \( \frac{1 - \cos 2\theta}{\sin 2\theta} = \tan \theta \).

Solution

\[
\text{LHS} = \frac{1 - \cos 2\theta}{\sin 2\theta}
\]

Now \( \cos 2\theta = 1 - 2\sin^2 \theta \) (Double Angle Formulae)

\[
\therefore \text{LHS} =
\]

\[
= 
\]

\[
= 
\]

\[
= \text{RHS} \quad \Box
\]

because \( \sin 2\theta = 2 \sin \theta \cos \theta \) (Double Angle Formulae)
\[
\frac{1 - (1 - 2\sin^2 \theta)}{\sin 2\theta} \\
\frac{2\sin^2 \theta}{\sin 2\theta} \\
\frac{2\sin^2 \theta}{2\sin \theta \cos \theta} \\
\frac{\sin \theta}{\cos \theta}
\]
Half-angle Formulae

\[
\begin{align*}
\sin^2 \frac{\theta}{2} &= \frac{1 - \cos \theta}{2} \\
\cos^2 \frac{\theta}{2} &= \frac{1 + \cos \theta}{2} \\
\tan \frac{\theta}{2} &= \frac{1 - \cos \theta}{\sin \theta} = \frac{\sin \theta}{1 + \cos \theta}
\end{align*}
\]

Note: From the half-angle formulae

\[
\begin{align*}
\sin \frac{\theta}{2} &= \pm \sqrt{\frac{1 - \cos \theta}{2}} \\
\cos \frac{\theta}{2} &= \pm \sqrt{\frac{1 + \cos \theta}{2}}
\end{align*}
\]

The ‘+’ or ‘-’ sign is chosen to be compatible with the known quadrant of \( \frac{\theta}{2} \).
Example

1. Explain why $\cos \frac{5\pi}{3} = \frac{1}{2}$

2. Hence find an exact value for $\cos \frac{5\pi}{6}$ using the half-angle formulae.

Solution We need to use the fact that

$\frac{5\pi}{6} = \frac{1}{2} \cdot \left( \frac{5\pi}{3} \right)$ and that

$\cos^2 \left( \frac{\theta}{2} \right) = \frac{1 + \cos \theta}{2}$

$\Rightarrow \cos \frac{\theta}{2} = \pm \sqrt{\frac{1 + \cos \theta}{2}}$
\[
\cos \frac{\theta}{2} = \pm \sqrt{\frac{1 + \cos \theta}{2}}
\]

Note that \(\frac{5\pi}{6}\) is in the 2nd quadrant so that \(\cos \frac{5\pi}{6}\) is negative. Let \(\theta = \frac{5\pi}{3}\)

\[
\therefore \quad \cos \frac{5\pi}{6} = \quad = \quad = \quad =
\]
\[- \left[ \frac{1 + \cos \left( \frac{5\pi}{3} \right)}{2} \right]^{1/2} \]
\[- \left[ \frac{1}{2} + \frac{1}{4} \right]^{1/2} \]
\[- \left[ \frac{3}{4} \right]^{1/2} \]
\[- \frac{\sqrt{3}}{2} \]
Products as Sums or Differences

\[
\begin{align*}
2 \cos A \cos B &= \cos(A+B) + \cos(A-B) \\
2 \sin A \sin B &= \cos(A-B) - \cos(A+B) \\
2 \sin A \cos B &= \sin(A+B) + \sin(A-B) \\
2 \cos A \sin B &= \sin(A+B) - \sin(A-B)
\end{align*}
\]

These formulae are obtained by the addition or subtraction of the formulae in (ii)

**Examples**

Write the following products as a sum or difference.

a. \(2 \cos 7x \cos 5x\)

b. \(\cos 4x \sin x\)
Solutions

a Let $A = 7x$ and $B = 5x$. Then as

$$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$$

we have

$$2 \cos 7x \cos 5x = \cos(7x+5x) + \cos(7x-5x) = \cos(12x) + \cos(2x)$$

b Let $A = 4x$ and $B = x$. Then as

$$\cos A \sin B = \frac{1}{2} [\sin(A+B) - \sin(A-B)]$$

we have

$$\cos 4x \sin x = \frac{1}{2} [\sin(4x+x) - \sin(4x-x)] = \frac{1}{2} [\sin(5x) - \sin(3x)]$$
Half-angle tangent formula

Let \( t = \tan \frac{x}{2} \),

then \( \sin x = \frac{2t}{1 + t^2} \),

\( \cos x = \frac{1 - t^2}{1 + t^2} \),

\( \tan x = \frac{2t}{1 - t^2} \).

This substitution is useful in integrating rational functions of sine and cosine (MATH 142).
Example Find an expression for \( \frac{1}{1 + \cos x} \) by letting \( t = \tan \frac{x}{2} \).

Solution Now, \( \cos x = \frac{1 - t^2}{1 + t^2} \). Hence

\[
\frac{1}{1 + \cos x} = \quad = \\
= \\
= 
\]
\[
\frac{1}{1 + \frac{1-t^2}{1+t^2}}
\]

\[
\frac{1}{\frac{1+t^2}{1+t^2} + \frac{1-t^2}{1+t^2}}
\]

\[
\frac{1+t^2}{2}
\]
Exercises on Trig Identities

Exercise 1.13.7
pages 44 & 45.

For each of the questions do as many of the subquestions as you require in order to gain mastery of the basic technique.

You need to do these exercises!