2.6 INVERSE TRIGONOMETRIC FUNCTIONS

Before the next lecture you should read sections 2.7.1 2.7.3 — I will assume that you have done so.
2.6.1 The Inverse Sine Function

The graph of \( y = \sin x \) shows that \( \sin x \) is a many-to-one function i.e. there are many \( x \)-values that will give the same \( y \)-value. To find the inverse of \( \sin x \) we need to restrict its domain so that we have a 1-1 function.

**Figure 9: \( y = \sin x \)**

We will restrict the domain of \( y = \sin x \) to \( -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \).

The inverse sine function is denoted by \( y = \sin^{-1} x \) or \( y = \arcsin x \).
Let $y = \sin^{-1} x$ and take sine of both sides

$\therefore \sin y = \sin(\sin^{-1} x) = x$

**Definition**

If $y = \sin^{-1} x$, then $x = \sin y$, for

$-1 \leq x \leq 1$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.

In words we can describe $\sin^{-1} x$ as the angle whose sin is $x$.

**Note** $\sin^{-1} x$ is the inverse of $\sin x$ not the reciprocal.

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**The Graph of** $y = \sin^{-1} x$

From the graph note:

1. Dom $\sin^{-1} x = [-1, 1]$
2. Range $\sin^{-1} x = [-\frac{\pi}{2}, \frac{\pi}{2}]$

**Figure 10:** $y = \sin^{-1} x$

Note: Dom $\sin x = [-\frac{\pi}{2}, \frac{\pi}{2}]$ and Range $\sin x = [-1, 1]$. 
Example
Find exactly:

(a) \( \sin^{-1} \left( \frac{1}{2} \right) \)
(b) \( \sin^{-1} \left( -\frac{1}{2} \right) \)
(c) \( \sin^{-1} \left( \frac{\sqrt{3}}{2} \right) \)
(d) \( \sin^{-1} 1 \)

\[
\frac{1}{\sin x} = [\sin (x)]^{-1}
\]
Find exactly: (a) $\sin^{-1} \left( \frac{1}{2} \right)$

Let $y = \sin^{-1} \left( \frac{1}{2} \right)$, \quad $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

$\therefore \quad \sin y = \frac{1}{2}$

$\therefore \quad y = \frac{\pi}{6}$
(b) Find exactly \( \sin^{-1}\left(-\frac{1}{2}\right) \)

Let \( y = \sin^{-1}\left(-\frac{1}{2}\right), \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \).

\[ \therefore \quad \sin y = -\frac{1}{2} \]

Therefore \( y \) is in the fourth quadrant.

\[ \therefore \quad y = -\frac{\pi}{6} \]
Find exactly (c) $\sin^{-1} \left( \frac{\sqrt{3}}{2} \right)$

Let $y = \sin^{-1} \left( \frac{\sqrt{3}}{2} \right)$, $\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$

$\therefore \sin y = \frac{\sqrt{3}}{2}$

$\therefore y = \frac{\pi}{3}$
Find exactly (d) $\sin^{-1} 1$

Let $y = \sin^{-1} (1)$, $\quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

$\therefore \sin y = 1$

$\therefore y = \frac{\pi}{2}$
2.6.2 The Inverse Cosine Function

Like the sine function, the cosine function is not 1-1. We can make a 1-1 function by restricting the domain to $0 \leq x \leq \pi$.

The inverse cosine function is denoted by $y = \cos^{-1} x$ or $y = \arccos x$.

The Graph of $y = \cos^{-1} x$

From the graph note:
1. $\text{Dom } \cos^{-1} x = [-1, 1]$
2. $\text{Range } \cos^{-1} x = [0, \pi]$

Figure 12: $y = \cos^{-1} x$

Note: Dom $\cos x = [0, \pi]$ and Range $\cos x = [-1, 1]$. 
Let \( y = \cos^{-1} x \) and take cosine of both sides
\[
\therefore \cos y = \cos (\cos^{-1} x) = x
\]

Definition
If \( y = \cos^{-1} x \), then \( x = \cos y \), for
\[-1 \leq x \leq 1 \text{ and } 0 \leq y \leq \pi.
\]
In words we can describe \( \cos^{-1} x \) as
the angle whose cosine is \( x \).

**Example**
Find exactly:
(a) \( \cos^{-1} \left( \frac{1}{2} \right) \)  
(b) \( \cos^{-1} \left( \frac{-1}{\sqrt{2}} \right) \)
(c) \( \cos^{-1} \left( \frac{\sqrt{3}}{2} \right) \)  
(d) \( \cos^{-1} 0 \)
Find exactly: (a) $\cos^{-1} \left( \frac{1}{2} \right)$

Let $y = \cos^{-1} \left( \frac{1}{2} \right)$

$$\therefore \cos y = \frac{1}{2}, \quad 0 \leq y \leq \pi$$

$$\therefore y = \frac{\pi}{3}$$
Find exactly: (b) \( \cos^{-1} \left( \frac{-1}{\sqrt{2}} \right) \)

Let \( y = \cos^{-1} \left( \frac{-1}{\sqrt{2}} \right) \)

\[ \therefore \cos y = -\frac{1}{\sqrt{2}} \quad 0 \leq y \leq \pi \]

cosine is negative so \( y \) is in the second quadrant.

\[ \therefore y = \frac{3\pi}{4} \]
Find exactly: (c) $\cos^{-1} \left( \frac{\sqrt{3}}{2} \right)$

Let $y = \cos^{-1} \left( \frac{\sqrt{3}}{2} \right)$

$\therefore \cos y = \frac{\sqrt{3}}{2} \quad 0 \leq y \leq \pi$

$\therefore y = \frac{\pi}{6}$
Find exactly: (d) $\cos^{-1} 0$

Let $y = \cos^{-1} (0)$

$\therefore \cos y = 0 \quad 0 \leq y \leq \pi$

$\therefore y = \frac{\pi}{2}$
2.6.3 The Inverse Tangent Function

We restrict the tangent function to the domain
\[-\frac{\pi}{2} < x < \frac{\pi}{2}\]
so that we have a 1-1 function.
The inverse tangent function is denoted by 
\[y = \tan^{-1} x\]
or 
\[y = \arctan x.\]

Figure 13: \(y = \tan x\)

The Graph of \(y = \tan^{-1} x\)

From the graph note
1. \(\text{Dom} \tan^{-1} x = \mathbb{R}\)
2. \(\text{Range} \tan^{-1} x = (-\frac{\pi}{2}, \frac{\pi}{2})\)

Figure 14: \(y = \tan^{-1} x\)

Note: \(\text{Dom} \tan x = (-\frac{\pi}{2}, \frac{\pi}{2})\)
and \(\text{Range} \tan x = \mathbb{R}\).
Let \( y = \tan^{-1} x \) and take tangent of both sides
\[
\therefore \tan y = \tan (\tan^{-1} x) = x.
\]

**Definition**
If \( y = \tan^{-1} x \), then \( x = \tan y \), for
\[-\infty < x < \infty \] and \( -\frac{\pi}{2} < y < \frac{\pi}{2} \).

In words we can describe \( \tan^{-1} x \) as the angle whose tangent is \( x \).

**Examples**
Find exactly:
(a) \( \tan^{-1} 1 \)  
(b) \( \tan^{-1} (-\sqrt{3}) \)
(c) \( \tan^{-1} (-1) \)  
(d) \( \tan^{-1} \left( \frac{1}{\sqrt{3}} \right) \)
Find exactly: (a) $\tan^{-1} 1$

Let $y = \tan^{-1} 1$

$\therefore \tan y = 1 \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$

$\therefore y = \frac{\pi}{4}$
Find exactly: (b) $\tan^{-1}(-\sqrt{3})$

Let $y = \tan^{-1}(-\sqrt{3})$

$\therefore \tan y = -\sqrt{3} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$

tangent is negative in the fourth quadrant

$\therefore y = -\frac{\pi}{3}$
Find exactly: (c) $\tan^{-1}(-1)$

Let $y = \tan^{-1}(-1)$

$\therefore \tan y = -1 \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$

tangent is negative in the fourth quadrant

$\therefore \quad y = -\frac{\pi}{4}$
Find exactly: (d) $\tan^{-1} \left( \frac{1}{\sqrt{3}} \right)$

Let $y = \tan^{-1} \frac{1}{\sqrt{3}}$

$\therefore \tan y = \frac{1}{\sqrt{3}} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$

$\therefore y = \frac{\pi}{6}$
Further properties of the inverse trigonometric functions

\[
\begin{align*}
\sin^{-1} (\sin x) &= x \quad \text{if } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\
\sin (\sin^{-1} x) &= x \quad \text{if } -1 \leq x \leq 1 \\
\cos^{-1} (\cos x) &= x \quad \text{if } 0 \leq x \leq \pi \\
\cos (\cos^{-1} x) &= x \quad \text{if } -1 \leq x \leq 1 \\
\tan^{-1} (\tan x) &= x \quad \text{if } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\
\tan (\tan^{-1} x) &= x \quad \text{if } -\infty < x < \infty
\end{align*}
\]

Example Simplify the function \(\cos (\sin^{-1} x)\) for \(|x| \leq 1\).

Solution
Method 1

Recall \( \sin^2 \theta + \cos^2 \theta = 1 \)

\( \implies \cos \theta = \pm \sqrt{1 - \sin^2 \theta} \)

Let \( \theta = \sin^{-1} x \)

\[ \therefore \cos (\sin^{-1} x) = \pm \sqrt{1 - \sin^2 (\sin^{-1} x)} \]

Now \( -\frac{\pi}{2} \leq \sin^{-1} x \leq \frac{\pi}{2} \)

and \( \cos > 0 \) in this interval

also \( \sin (\sin^{-1} x) = x, \) if \( |x| \leq 1 \)

\[ \therefore \cos (\sin^{-1} x) = \sqrt{1 - x^2} \]
\[ y = \sin^{-1} x \implies \sin (y) = x. \]

From the triangle we have
\[ d = \sqrt{1 - x^2} \]
\[ \cos y = \frac{\sqrt{1 - x^2}}{1} \]
\[ \cos (\sin^{-1} x) = \sqrt{1 - x^2}. \]

Take the positive root for the same reasons as in method 1.

### 2.6.4 Revision Questions

The following questions are about the key ideas in this section.

1. Sketch the following graphs: (a) \( y = \sin^{-1} x \) (b) \( y = \cos^{-1} x \) (c) \( y = \tan^{-1} x \).

2. What are the domain and range of: (a) \( y = \sin^{-1} x \) (b) \( y = \cos^{-1} x \) (c) \( y = \tan^{-1} x \).

3. Why do we need to restrict the domain of the inverse trigonometric functions?
2.6.5 Exercises

1. Simplify the following functions.

2. Evaluate (if possible) the following expressions (don’t use a calculator!)