2.6 INVERSE TRIGONOMETRIC FUNCTIONS
Before the next lecture you should read sections 2.7.1–2.7.3 — I will assume that you have done so.
2.6.1 The Inverse Sine Function

The graph of $y = \sin x$ shows that $\sin x$ is a many-to-one function i.e. there are many $x$-values that will give the same $y$-value. To find the inverse of $\sin x$ we need to restrict its domain so that we have a 1-1 function.

Figure 9: $y = \sin x$
We will restrict the domain of \( y = \sin x \) to \(-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\).

The **inverse sine function** is denoted by \( y = \sin^{-1} x \) or \( y = \arcsin x \).
The Graph of $y = \sin^{-1} x$

From the graph note:
1. Dom $\sin^{-1} x = [-1, 1]$
2. Range $\sin^{-1} x = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Figure 10: $y = \sin^{-1} x$

Note: Dom $\sin x = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and Range $\sin x = [-1, 1]$. 
Let \( y = \sin^{-1} x \) and take sine of both sides
\[
\therefore \sin y = \sin (\sin^{-1} x) = x
\]

**Definition**
If \( y = \sin^{-1} x \), then \( x = \sin y \), for
\[-1 \leq x \leq 1 \text{ and } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}.
\]

In words we can describe \( \sin^{-1} x \) as
the angle whose \( \sin \) is \( x \).

**Note** \( \sin^{-1} x \) is the **inverse** of \( \sin x \) **not**
the reciprocal.
\[
\frac{1}{\sin x} = [\sin (x)]^{-1}
\]
Examples

Find exactly:

(a) $\sin^{-1} \left( \frac{1}{2} \right)$  
(b) $\sin^{-1} \left( -\frac{1}{2} \right)$

(c) $\sin^{-1} \left( \frac{\sqrt{3}}{2} \right)$  
(d) $\sin^{-1} 1$
Find exactly: (a) $\sin^{-1} \left( \frac{1}{2} \right)$
Let \( y = \sin^{-1} \left( \frac{1}{2} \right) \), \(-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}\)

\[ \therefore \sin y = \frac{1}{2} \]

\[ \therefore y = \frac{\pi}{6} \]
(b) Find exactly $\sin^{-1} \left( -\frac{1}{2} \right)$
Let \( y = \sin^{-1}\left(-\frac{1}{2}\right) \), \(-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}\)

\[ \therefore \quad \sin y = -\frac{1}{2} \]

Therefore \( y \) is in the fourth quadrant

\[ \therefore \quad y = -\frac{\pi}{6} \]
Find exactly $(c) \sin^{-1} \left( \frac{\sqrt{3}}{2} \right)$
Let \( y = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) \), \( -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \)

\[ \therefore \sin y = \frac{\sqrt{3}}{2} \]

\[ \therefore y = \frac{\pi}{3} \]
Find exactly (d) $\sin^{-1} 1$
Let \( y = \sin^{-1} (1) \), \[ -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \]

\[ \therefore \sin y = 1 \]

\[ \therefore y = \frac{\pi}{2} \]
2.6.2 The Inverse Cosine Function

Like the sine function, the cosine function is not 1-1. We can make a 1-1 function by restricting the domain to $0 \leq x \leq \pi$. The inverse cosine function is denoted by $y = \cos^{-1} x$ or $y = \arccos x$.

Figure 11: $y = \cos x$
The Graph of $y = \cos^{-1} x$

From the graph note
1. $\text{Dom } \cos^{-1} x = [-1, 1]$
2. $\text{Range } \cos^{-1} x = [0, \pi]$

**Figure 12:** $y = \cos^{-1} x$

Note: $\text{Dom } \cos x = [0, \pi]$
and $\text{Range } \cos x = [-1, 1]$. 
Let \( y = \cos^{-1} x \) and take cosine of both sides
\[
\therefore \cos y = \cos (\cos^{-1} x) = x
\]

**Definition**

If \( y = \cos^{-1} x \), then \( x = \cos y \), for \( -1 \leq x \leq 1 \) and \( 0 \leq y \leq \pi \).

In words we can describe \( \cos^{-1} x \) as the angle whose cosine is \( x \).
Examples
Find exactly:
(a) $\cos^{-1} \left( \frac{1}{2} \right)$  (b) $\cos^{-1} \left( \frac{-1}{\sqrt{2}} \right)$
(c) $\cos^{-1} \left( \frac{\sqrt{3}}{2} \right)$  (d) $\cos^{-1} 0$
Find exactly: (a) $\cos^{-1} \left( \frac{1}{2} \right)$
Let \( y = \cos^{-1} \left( \frac{1}{2} \right) \)

\[
\therefore \quad \cos y = \frac{1}{2} \quad 0 \leq y \leq \pi
\]

\[
\therefore \quad y = \frac{\pi}{3}
\]
Find exactly: (b) \( \cos^{-1} \left( \frac{-1}{\sqrt{2}} \right) \)
Let \( y = \cos^{-1} \left( \frac{-1}{\sqrt{2}} \right) \)

\[\therefore \cos y = -\frac{1}{\sqrt{2}}, \quad 0 \leq y \leq \pi\]

cosine is negative so \( y \) is in the second quadrant.

\[\therefore y = \frac{3\pi}{4}\]
Find exactly: \( (c) \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) \)
Let $y = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$\therefore \cos y = \frac{\sqrt{3}}{2}, \quad 0 \leq y \leq \pi$

$\therefore y = \frac{\pi}{6}$
Find exactly: (d) $\cos^{-1} 0$
Let \( y = \cos^{-1} (0) \)

\[
\therefore \quad \cos y = 0 \quad 0 \leq y \leq \pi
\]

\[
\therefore \quad y = \frac{\pi}{2}
\]
We restrict the tangent function to the domain \(-\frac{\pi}{2} < x < \frac{\pi}{2}\) so that we have a 1-1 function.

The **inverse tangent function** is denoted by \( y = \tan^{-1} x \) or \( y = \arctan x \).
The Graph of $y = \tan^{-1} x$

From the graph note
1. $\text{Dom } \tan^{-1} x = \mathbb{R}$
2. $\text{Range } \tan^{-1} x = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Figure 14: $y = \tan^{-1} x$

Note: $\text{Dom } \tan x = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
and $\text{Range } \tan x = \mathbb{R}$.
Let $y = \tan^{-1} x$ and take tangent of both sides
\[ \therefore \tan y = \tan (\tan^{-1} x) = x. \]

**Definition**

If $y = \tan^{-1} x$, then $x = \tan y$, for $-\infty < x < \infty$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

In words we can describe $\tan^{-1} x$ as the angle whose tangent is $x$. 
Examples
Find exactly:
(a) \( \tan^{-1} 1 \)  
(b) \( \tan^{-1} (-\sqrt{3}) \)  
(c) \( \tan^{-1} (-1) \)  
(d) \( \tan^{-1} \left( \frac{1}{\sqrt{3}} \right) \)
Find exactly: (a) \( \tan^{-1} 1 \)
Let \( y = \tan^{-1} 1 \)

\[
\therefore \quad \tan y = 1 \quad -\frac{\pi}{2} < y < \frac{\pi}{2}
\]

\[
\therefore \quad y = \frac{\pi}{4}
\]
Find exactly: (b) $\tan^{-1}(-\sqrt{3})$
Let \( y = \tan^{-1} -\sqrt{3} \)

\[ \therefore \tan y = -\sqrt{3} \quad -\frac{\pi}{2} < y < \frac{\pi}{2} \]

tangent is negative in the fourth quadrant

\[ \therefore y = -\frac{\pi}{3} \]
Find exactly: (c) $\tan^{-1}(-1)$
Let $y = \tan^{-1} - 1$

$\therefore \tan y = -1 \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$

tangent is negative in the fourth quadrant

$\therefore \quad y = -\frac{\pi}{4}$
Find exactly: (d) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$
Let \( y = \tan^{-1} \frac{1}{\sqrt{3}} \)

\[
\therefore \tan y = \frac{1}{\sqrt{3}} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}
\]

\[
\therefore \quad y = \frac{\pi}{6}
\]
Further properties of the inverse trigonometric functions

\[
\begin{align*}
\sin^{-1}(\sin x) &= x \quad \text{if } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\
\sin(\sin^{-1} x) &= x \quad \text{if } -1 \leq x \leq 1 \\
\cos^{-1}(\cos x) &= x \quad \text{if } 0 \leq x \leq \pi \\
\cos(\cos^{-1} x) &= x \quad \text{if } -1 \leq x \leq 1 \\
\tan^{-1}(\tan x) &= x \quad \text{if } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\
\tan(\tan^{-1} x) &= x \quad \text{if } -\infty < x < \infty
\end{align*}
\]
Example Simplify the function $\cos (\sin^{-1} x)$ for $|x| \leq 1$.

Solution
Method 1

Recall \( \sin^2 \theta + \cos^2 \theta = 1 \)

\[ \Rightarrow \cos \theta = \pm \sqrt{1 - \sin^2 \theta} \]

Let \( \theta = \sin^{-1} x \)

\[ :. \quad \cos \left( \sin^{-1} x \right) = \pm \sqrt{1 - \sin^2 \left( \sin^{-1} x \right)} \]

Now \( -\frac{\pi}{2} \leq \sin^{-1} x \leq \frac{\pi}{2} \)

and \( \cos > 0 \) in this interval

also \( \sin \left( \sin^{-1} x \right) = x, \) if \( |x| \leq 1 \)

\[ :. \quad \cos \left( \sin^{-1} x \right) = \sqrt{1 - x^2} \]
Example Simplify the function
\( \cos (\sin^{-1} x) \) for \( |x| \leq 1 \).

Solution Method 2.
\[ y = \sin^{-1} x \implies \sin(y) = x. \]

From the triangle we have
\[ d = \sqrt{1 - x^2} \]
\[ \cos y = \frac{\sqrt{1 - x^2}}{1} \]
\[ \cos (\sin^{-1} x) = \sqrt{1 - x^2}. \]

Take the positive root for the same reasons as in method 1.
2.6.4 Revision Questions

The following questions are about the key ideas in this section.

1. Sketch the following graphs: (a) $y = \sin^{-1} x$ (b) $y = \cos^{-1} x$ (c) $y = \tan^{-1} x$.

2. What are the domain and range of: (a) $y = \sin^{-1} x$ (b) $y = \cos^{-1} x$ (c) $y = \tan^{-1} x$.

3. Why do we need to restrict the domain of the inverse trigonometric functions?
2.6.5 Exercises

1. Simplify the following functions.

2. Evaluate (if possible) the following expressions (don’t use a calculator!)