Consider the following graphs:

- A function is one-to-one if its graph is cut only once by any horizontal line.

2.5.1 **One-to-one Functions**

2.5 **Inverse One-to-one and**
The function $f$ is a two-to-one function as each $y$-value has two $x$-values. e.g. if $y = a^2$, then $x = a$ or $x = -a$. An example of a many-to-one function is $f(x) = \sin x$ e.g. if $y = \frac{1}{2}$, then we have $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \ldots$.

2.5.2 Inverse Functions

The functions $f(x) = 2x$ and $g(x) = \frac{1}{2}x$ have the property that each is the inverse of the other.

\[
\begin{align*}
  f(g(x)) &= f\left(\frac{1}{2}x\right) = 2 \left(\frac{1}{2}x\right) = x \\
  g(f(x)) &= g(2x) = \frac{1}{2} \cdot 2x = x
\end{align*}
\]

Similarly, the functions $f(x) = x^{1/3}$ and $g(x) = x^3$ are the reverse of each other.

\[
\begin{align*}
  (g(x)) &= f(x^3) = \left(x^3\right)^{1/3} = x \\
  g(f(x)) &= g(x^{1/3}) = \left(x^{1/3}\right)^3 = x
\end{align*}
\]
We say that \( f \) is an \underline{inverse} of \( g \) and \( g \) is an \underline{inverse} of \( f \).

We can also say that \( f \) and \( g \) are \( f \) and \( g \) are inverse functions.

Thus \( f(x) = 2x \) and \( g(x) = \frac{1}{2}x \) are inverse functions as are \( f(x) = x^{1/3} \) and \( g(x) = x^3 \).

**Notation**

The \underline{inverse} of \( f \) is commonly denoted as \( f^{-1} \) (read ‘\( f \) inverse’) thus

\[
f(f^{-1}(x)) = f^{-1}(f(x)) = x
\]

So, we can write our inverse functions

\[
f(x) = 2x \quad \text{and} \quad g(x) = \frac{1}{2}x
\]

\[
f(x) = 2x \quad \text{and} \quad f^{-1}(x) = \frac{1}{2}x
\]

\[
g(x) = \frac{1}{2}x \quad \text{and} \quad g^{-1}(x) = 2x.
\]

**BEWARE:** \( f^{-1}(x) \) denotes the \underline{inverse} of \( f(x) \) \underline{not} the \underline{reciprocal}.

**Question.** Do all functions have inverses?

**Definition** A function \( f \) has an inverse if and only if it is \underline{one-to-one}.
2.5.2.1 Exercises  Do the following functions have inverses?
(a) \( f(x) = x^2, \ x \leq 0 \)
(b) \( g(x) = \sqrt{3x - 2} \)

Hint: Sketch the functions.

2.5.3 Domain and Range of an Inverse

Consider the function \( f(x) = 2x \), let \( \text{Dom} \ f = \{x : x = 2, 4, 6, 8\} \) then \( \text{Range} \ f = \{y : y = 4, 8, 12, 16\} \).

Consider also the function \( f^{-1}(x) = \frac{1}{2}x \) and let \( \text{Dom} \ f^{-1} = \{x : x = 4, 8, 12, 16\} \) then \( \text{Range} \ f^{-1} = \{y : y = 2, 4, 6, 8\} \).

If we compare we see that

\[
\begin{align*}
\text{Dom} \ f^{-1} &= \text{Range} \ f \\
\text{Range} \ f^{-1} &= \text{Dom} \ f
\end{align*}
\]

This is true for all \( f \) and \( f^{-1} \).
\[
\begin{array}{c}
\text{Dom } f^{-1} = \text{ Range } f \\
\text{Range } f^{-1} = \text{ Dom } f \\
\end{array}
\]

2.5.3.1 Exercises  Find Dom \( f^{-1} \) and Range \( f^{-1} \) given \( f \)

(a) \( f(x) = 2x - 4 \)

(b) \( f(x) = x^2 - 1, \ x \geq 0 \)

(c) \( f(x) = \sqrt{x - 3}, \ x > 3 \)
Dom $f(x) = x \in (-\infty, +\infty)$

Range $f(x) = y \in (-\infty, +\infty).$  Thus

Dom $f^{-1}(x) = x \in (-\infty, +\infty)$

Range $f^{-1}(x) = y \in (-\infty, +\infty)$

Find Dom $f^{-1}$ and Range $f^{-1}$

(b) $f(x) = x^2 - 1$, $x \geq 0.$ Draw the function.
Dom $f(x) = x \in [0, +\infty)$

Range $f(x) = y \in [-1, +\infty)$. Thus

Dom $f^{-1}(x) = x \in [-1, +\infty)$

Range $f^{-1}(x) = y \in [0, +\infty)$

Find Dom $f^{-1}$ and Range $f^{-1}$ (c)

$f(x) = \sqrt{x - 3}, \ x > 3$. Draw the function.
2.5.4 Finding $f^{-1}(x)$

If we let $y = f(x)$, we can find the inverse, $f^{-1}$, in terms of $y$, by solving for $x$.

Let $y = f(x)$

taking $f^{-1}$ of both sides

$$f^{-1}(y) = f^{-1}(f(x)) = x$$

i.e. $x = f^{-1}(y)$ if and only if $y = f(x)$. 

Dom $f(x) = x \in (3, +\infty)$

Range $f(x) = y \in (0, +\infty)$. Thus

Dom $f^{-1}(x) = x \in (0, +\infty)$

Range $f^{-1}(x) = y \in (3, +\infty)$
To find the inverse of the function $f(x)$.

1. Check $f$ is 1-1. Why?
2. Let $y = f(x)$.
3. Solve for $x$ to obtain $x = f^{-1}(y)$.
4. Write $f^{-1}$ in terms of $x$.

\subsection*{2.5.4.1 Exercises}

1. Find the inverse, if it exists, of each of the following:
   
   (a) $f(x) = 2x + 4$
   (b) $f(x) = 7x - 6$
   (c) $f(x) = 3x^3 - 5$
   (d) $f(x) = \frac{3}{x^2}$
   (e) $f(x) = \frac{3}{x^2}, \ x < 0$
1. Check $f$ is 1-1.
2. Let $y = f(x)$.
3. Solve for $x$ to obtain $x = f^{-1}(y)$.
4. Write $f^{-1}$ in terms of $x$.

(1a) Find the inverse, if it exists of:

$f(x) = 2x + 4$

\[
y = 2x + 4 \\
y - 4 = 2x \\
x = \frac{1}{2} (y - 4) \\
f^{-1}(x) = \frac{x - 4}{2}.
\]
1. Check $f$ is 1-1.
2. Let $y = f(x)$.
3. Solve for $x$ to obtain $x = f^{-1}(y)$.
4. Write $f^{-1}$ in terms of $x$.

(1b) Find the inverse, if it exists of:

$f(x) = 7x - 6$

\[
\begin{align*}
y &= 7x - 6 \\
y + 6 &= 7x \\
x &= \frac{1}{7} (y + 6) \\
\therefore f^{-1}(x) &= \frac{x + 6}{7}.
\end{align*}
\]
1. Check \( f \) is 1-1.
2. Let \( y = f(x) \).
3. Solve for \( x \) to obtain \( x = f^{-1}(y) \).
4. Write \( f^{-1} \) in terms of \( x \).

(1c) Find the inverse, if it exists of:
\[ f(x) = 3x^3 - 5 \]

\[
\begin{align*}
y &= 3x^3 - 5 \\
y + 5 &= 3x^3 \\
x^3 &= \frac{y + 5}{3} \\
x &= \left( \frac{y + 5}{3} \right)^{1/3} \\
f^{-1}(x) &= \left( \frac{x + 5}{3} \right)^{1/3}
\end{align*}
\]
1. Check $f$ is 1-1.

2. Let $y = f(x)$.

3. Solve for $x$ to obtain $x = f^{-1}(y)$.

4. Write $f^{-1}$ in terms of $x$.

(1d) Find the inverse, if it exists of: $f(x) = \frac{3}{x^2}$

The inverse does not exist because the function is not one-to-one.
1. Check \( f \) is 1-1.

2. Let \( y = f(x) \).

3. Solve for \( x \) to obtain \( x = f^{-1}(y) \).

4. Write \( f^{-1} \) in terms of \( x \).

(1e) Find the inverse, if it exists of:
\( f(x) = \frac{3}{x^2}, \ x < 0 \)

\[
\begin{align*}
  y &= \frac{3}{x^2}, \quad x < 0 \\
  x^2 &= \frac{3}{y}, \quad x < 0 \\
  x &= -\sqrt{\frac{3}{y}} \\
  f^{-1}(x) &= -\sqrt{\frac{3}{x}}
\end{align*}
\]
2. The temperature at which water boils will, to a point, decrease linearly as the altitude increases. Let the boiling point be a function of altitude such that $T_B = f(h)$, where $T_B$ is the boiling point in °C and $h$ is the altitude in metres. What is the meaning in practical terms of $f^{-1}(90)$? If $f^{-1}(90) = 3000$ evaluate $f^{-1}(85)$.

Hint. What is the boiling point of water at sea (ground) level ($h = 0$)?

We know that

$$T_B = A - Bh.$$ 

The boiling point of water at sea level is $100°C$. Thus

$$100 = A.$$ 

The question tells us that

$$f^{-1}(90) = 3000.$$ 

Thus

$$90 = A - B(3000),$$ 

$$\Rightarrow 90 = 100 - B(3000),$$ 

$$\Rightarrow B = \frac{10}{3000} = \frac{1}{300}.$$
Hence

\[ T_B = 100 - \frac{1}{300}h \]

The solution \( f^{-1}(85) \) is given by

\[ 85 = 100 - \frac{1}{300}h, \]

\[ \Rightarrow h = 15 \cdot (300) = 4500. \]

### 2.5.5 Revision of key ideas

The following questions are about the key ideas in this section.

1. What is meant by the expression that ‘\( f \) is a 1-to-1 function’?

2. Suppose that \( f \) and \( g \) are functions. What is meant by the expression ‘\( f \) is the inverse of \( g \)’?

3. What is the requirement for a function \( f \) to have an inverse?

4. How do the domain and range of a function \( f \) relate to the domain and range of its inverse function \( f^{-1} \)?