2.5 ONE-TO-ONE AND INVERSE FUNCTIONS

2.5.1 One-to one functions

Definition
A function $f$ is one-to-one if its graph is cut only once by any horizontal line. Consider the following graphs:
(a) $f(x)$  \hspace{0.5cm} (b) $g(x)$  \hspace{0.5cm} (c) $h(x)$

$g$ and $h$ are examples of one-to-one (often denoted 1-1) functions, $f$ is not 1-1. The functions $g$ and $h$ have for each $y$ value, only one $x$ value.
The function $f$ is a two-to-one function as each $y$-value has two $x$-values. e.g. if $y = a^2$, then $x = a$ or $x = -a$. An example of a many-to-one function is $f(x) = \sin x$ e.g. if $y = \frac{1}{2}$, then we have $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \ldots$. 
2.5.2 Inverse Functions

The functions $f(x) = 2x$ and $g(x) = \frac{1}{2}x$ have the property that each is the inverse of the other.

\[
f(g(x)) = f\left(\frac{1}{2}x\right) = 2 \left(\frac{1}{2}x\right) = x
\]

\[
g(f(x)) = g(2x) = \frac{1}{2}2x = x
\]

Similarly, the functions $f(x) = x^{1/3}$ and $g(x) = x^3$ are the reverse of each other.

\[
(g(x)) = f(x^3) = (x^3)^{1/3} = x
\]

\[
g(f(x)) = g(x^{1/3}) = (x^{1/3})^3 = x
\]
We say that $f$ is an \underline{inverse} of $g$ and $g$ is an \underline{inverse} of $f$.

We can also say that $f$ and $g$ are \underline{$f$ and $g$ are inverse functions}. Thus $f(x) = 2x$ and $g(x) = \frac{1}{2}x$ are inverse function s as are $f(x) = x^{1/3}$ and $g(x) = x^3$.

\textit{Notation}

The \textit{inverse of $f$} is commonly denoted as $f^{-1}$ (read ‘$f$ inverse’) thus

$$f \left( f^{-1}(x) \right) = f^{-1}(f(x)) = x$$
So, we can write our inverse functions 
\( f(x) = 2x \) and \( g(x) = \frac{1}{2}x \) as 
\[ f(x) = 2x \text{ and } f^{-1}(x) = \frac{1}{2}x \text{ or } \]
\[ g(x) = \frac{1}{2}x \text{ and } g^{-1}(x) = 2x. \]

**BEWARE:** \( f^{-1}(x) \) denotes the inverse of \( f(x) \) *not* the reciprocal.

**Question.** Do all functions have inverses?

*Definition* A function \( f \) has an inverse if and only if it is one-to-one.
2.5.2.1 Exercises  Do the following functions have inverses?

(a) \( f(x) = x^2, \ x \leq 0 \)

(b) \( g(x) = \sqrt{3x - 2} \)

**Hint:** Sketch the functions.
2.5.3 Domain and Range of an Inverse

Consider the function $f(x) = 2x$, let
Dom $f = \{x : x = 2, 4, 6, 8\}$ then
Range $f = \{y : y = 4, 8, 12, 16\}$.

Consider also the function $f^{-1}(x) = \frac{1}{2}x$
and let Dom $f^{-1} = \{x : x = 4, 8, 12, 16\}$
then Range $f^{-1} = \{y : y = 2, 4, 6, 8\}$.

If we compare we see that

$$\text{Dom } f^{-1} = \text{Range } f$$
$$\text{Range } f^{-1} = \text{Dom } f$$

This is true for all $f$ and $f^{-1}$
\[
\begin{align*}
\text{Dom } f^{-1} &= \text{ Range } f \\
\text{Range } f^{-1} &= \text{ Dom } f
\end{align*}
\]

### 2.5.3.1 Exercises
Find \( \text{Dom } f^{-1} \) and \( \text{Range } f^{-1} \) given \( f \)

(a) \( f(x) = 2x - 4 \)

(b) \( f(x) = x^2 - 1, \ x \geq 0 \)

(c) \( f(x) = \sqrt{x - 3}, \ x > 3 \)
\begin{align*}
\text{Dom } f^{-1} &= \text{Range } f \\
\text{Range } f^{-1} &= \text{Dom } f
\end{align*}

Find \text{Dom } f^{-1} \text{ and Range } f^{-1}:

(a) \( f(x) = 2x - 4 \). Draw the function.
Dom $f(x) = x \in (-\infty, +\infty)$

Range $f(x) = y \in (-\infty, +\infty)$. Thus

Dom $f^{-1}(x) = x \in (-\infty, +\infty)$

Range $f^{-1}(x) = y \in (-\infty, +\infty)$
\[
\begin{align*}
\text{Dom } f^{-1} &= \text{Range } f \\
\text{Range } f^{-1} &= \text{Dom } f
\end{align*}
\]

Find \( \text{Dom } f^{-1} \) and \( \text{Range } f^{-1} \)

(b) \( f(x) = x^2 - 1, \ x \geq 0 \). Draw the function.
\begin{align*}
\text{Dom } f(x) &= x \in [0, +\infty) \\
\text{Range } f(x) &= y \in [-1, +\infty). \\
\text{Thus} \\
\text{Dom } f^{-1}(x) &= x \in [-1, +\infty) \\
\text{Range } f^{-1}(x) &= y \in [0, +\infty)
\end{align*}
\[ \text{Dom } f^{-1} = \text{Range } f \]
\[ \text{Range } f^{-1} = \text{Dom } f \]

Find \( \text{Dom } f^{-1} \) and \( \text{Range } f^{-1} \) (c)

\( f(x) = \sqrt{x - 3}, \ x > 3 \). Draw the function.
\begin{align*}
\text{Dom } f(x) &= x \in (3, +\infty) \\
\text{Range } f(x) &= y \in (0, +\infty). \\
\text{Dom } f^{-1}(x) &= x \in (0, +\infty) \\
\text{Range } f^{-1}(x) &= y \in (3, +\infty)
\end{align*}

Thus
2.5.4 Finding $f^{-1}(x)$

If we let $y = f(x)$, we can find the inverse, $f^{-1}$, in terms of $y$, by solving for $x$.

Let $y = f(x)$

taking $f^{-1}$ of both sides

$$f^{-1}(y) = f^{-1}(f(x)) = x$$

i.e. $x = f^{-1}(y)$ if and only if $y = f(x)$. 
To find the inverse of the function $f(x)$.

1. Check $f$ is 1-1.  

2. Let $y = f(x)$.  

3. Solve for $x$ to obtain $x = f^{-1}(y)$.  

4. Write $f^{-1}$ in terms of $x$.  

Why?
2.5.4.1 Exercises

1. Find the inverse, if it exists, of each of the following:

(a) \( f(x) = 2x + 4 \)
(b) \( f(x) = 7x - 6 \)
(c) \( f(x) = 3x^3 - 5 \)
(d) \( f(x) = \frac{3}{x^2} \)
(e) \( f(x) = \frac{3}{x^2}, \quad x < 0 \)
1. Check $f$ is 1-1.

2. Let $y = f(x)$.

3. Solve for $x$ to obtain $x = f^{-1}(y)$.

4. Write $f^{-1}$ in terms of $x$.

(1a) Find the inverse, if it exists of:

$$f(x) = 2x + 4$$
\[
\begin{align*}
y &= 2x + 4 \\
y - 4 &= 2x \\
x &= \frac{1}{2} (y - 4) \\
f^{-1}(x) &= \frac{x - 4}{2}.
\end{align*}
\]
1. Check $f$ is 1-1.

2. Let $y = f(x)$.

3. Solve for $x$ to obtain $x = f^{-1}(y)$.

4. Write $f^{-1}$ in terms of $x$.

(1b) Find the inverse, if it exists of:
$f(x) = 7x - 6$
\[ y = 7x - 6 \]
\[ y + 6 = 7x \]
\[ x = \frac{1}{7} (y + 6) \]
\[ f^{-1}(x) = \frac{x + 6}{7}. \]
1. Check $f$ is 1-1.

2. Let $y = f(x)$.

3. Solve for $x$ to obtain $x = f^{-1}(y)$.

4. Write $f^{-1}$ in terms of $x$.

(1c) Find the inverse, if it exists of:

$$f(x) = 3x^3 - 5$$
\[ \begin{align*}
  y &= 3x^3 - 5 \\
  y + 5 &= 3x^3 \\
  x^3 &= \frac{y + 5}{3} \\
  x &= \left( \frac{y + 5}{3} \right)^{1/3} \\
  f^{-1}(x) &= \left( \frac{x + 5}{3} \right)^{1/3}
\end{align*} \]
1. Check $f$ is 1-1.

2. Let $y = f(x)$.

3. Solve for $x$ to obtain $x = f^{-1}(y)$.

4. Write $f^{-1}$ in terms of $x$.

(1d) Find the inverse, if it exists of: $f(x) = \frac{3}{x^2}$
The inverse does not exist because the function is not one-to-one.
1. Check $f$ is 1-1.

2. Let $y = f(x)$.

3. Solve for $x$ to obtain $x = f^{-1}(y)$.

4. Write $f^{-1}$ in terms of $x$.

(1e) Find the inverse, if it exists of:

$$f(x) = \frac{3}{x^2}, \quad x < 0$$
\[ y = \frac{3}{x^2}, \quad x < 0 \]
\[ x^2 = \frac{3}{y}, \quad x < 0 \]
\[ x = -\sqrt{\frac{3}{y}} \]
\[ f^{-1}(x) = -\sqrt{\frac{3}{x}} \]
2. The temperature at which water boils will, to a point, decrease linearly as the altitude increases. Let the boiling point be a function of altitude such that \( T_B = f(h) \), where \( T_B \) is the boiling point in °C and \( h \) is the altitude in metres. What is the meaning in practical terms of \( f^{-1}(90) \)? If \( f^{-1}(90) = 3000 \) evaluate \( f^{-1}(85) \).

**Hint.** What is the boiling point of water at sea (ground) level \((h = 0)\)?
We know that

\[ T_B = A - Bh. \]

The boiling point of water at sea level is \(100^\circ C\). Thus

\[ 100 = A. \]

The question tells us that

\[ f^{-1}(90) = 3000. \]

Thus

\[ 90 = A - B(3000), \]
\[ \Rightarrow 90 = 100 - B(3000), \]
\[ \Rightarrow B = \frac{10}{3000} = \frac{1}{300}. \]
Hence

\[ T_B = 100 - \frac{1}{300} h \]

The solution \( f^{-1}(85) \) is given by

\[ 85 = 100 - \frac{1}{300} h, \]

\( \Rightarrow h = 15 \times 300 = 4500. \)
2.5.5 Revision of key ideas

The following questions are about the key ideas in this section.

1. What is meant by the expression that ‘$f$ is a 1-to-1 function’?

2. Suppose that $f$ and $g$ are functions. What is meant by the expression ‘$f$ is the inverse of $g$’?

3. What is the requirement for a function $f$ to have an inverse?

4. How do the domain and range of a function $f$ relate to the domain and range of its inverse function $f^{-1}$?