2.10 IMPLICIT DIFFERENTIATION

2.10.1 The Chain Rule Revisited

Recall definition.
If $g$ is differentiable at $x$ and $f$ is differentiable at $g(x)$, then
$$[f(g(x))]' = f'(g(x)) \cdot g'(x).$$

If we introduce another variable, say $u$, we can write the chain rule in another form.

Given $y = f(g(x)$, let $u = g(x)$ and so $y = f(u)$.

Now, if $y = f(u)$, then $\frac{dy}{du} = f'(u)$ and if $u = g(x)$, then $\frac{du}{dx} = g'(x)$.

If we use these substitutions then the derivative of $y = f(g(x)$, where $u = g(x)$, is

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$$
$$= f'(u) \cdot \frac{du}{dx} \quad \text{let } u = g(x)$$
$$= \frac{dy}{du} \cdot \frac{du}{dx}.$$
2.10.2 Alternative Definition of the Chain Rule.

If $g$ is differentiable at $x$ and $f$ is differentiable at $g(x)$ and also if $y = f(g(x))$ and $u = g(x)$ then

$y = f(u)$ and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

This formula is easy to remember if we note that the LHS is exactly what we get if we ‘cancel’ the $du$’s on the right.

2.10.2.1 Exercises on the chain rule

1. Differentiate:
   a. $y = (3x + 5)^4$
   b. $w = \sqrt{4 + 3\sqrt{t}}$
   c. $y = \cos(\cos\theta)$
   d. $y = \sqrt{x\tan^3(\sqrt{x})}$

2. In each of the following find $\frac{dy}{dx}$ by first making an appropriate choice for $u$.
   a. $y = (x^3 + 2x)^{37}$
   b. $y = \left(x^3 - \frac{7}{x}\right)^{-2}$
   c. $y = \sin\left(\frac{1}{x^2}\right)$
   d. $y = \cos^3(\sin 2x)$
1a. Differentiate \( y = (3x + 5)^4 \).

Let, \( u = 3x + 5 \). \( \frac{du}{dx} = 3 \)

Then, \( y = u^4 \),

\[
\frac{dy}{du} = 4u^3 \\
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \\
= 4u^3 \cdot 3 \\
= 12 (3x + 5)^3
\]
1b. Differentiate \( w = \sqrt{4 + 3\sqrt{t}} \).

Let, \( u = 4 + 3\sqrt{t} \).

\[
\frac{du}{dt} = \frac{3}{2} t^{-1/2}
\]

Then, \( w = u^{1/2} \),

\[
\frac{dw}{du} = \frac{1}{2} u^{-1/2}
\]

\[
\frac{dw}{dt} = \frac{dw}{du} \cdot \frac{du}{dt} = \frac{1}{2} u^{-1/2} \cdot \frac{3}{2} t^{-1/2} = \frac{3}{4} t^{-1/2} \cdot \left(4 + 3\sqrt{t}\right)^{-1/2} = \frac{3}{4\sqrt{t} \cdot \sqrt{4 + 3\sqrt{t}}}
\]
1c. Differentiate \( w = \cos(\cos \theta) \)

Let, \( u = \cos \theta \).

\[
\frac{du}{d\theta} = -\sin \theta
\]

Then, \( w = \cos u \),

\[
\frac{dw}{du} = -\sin u
\]

\[
\frac{dw}{d\theta} = \frac{dw}{du} \cdot \frac{du}{d\theta}
\]

\[
= - (\sin u) \cdot (-\sin \theta)
\]

\[
= \sin (\cos \theta) \cdot \sin \theta
\]
1d. Differentiate \( y = \sqrt{x} \tan^3(\sqrt{x}) \).

Let, \( u = (x)^{1/2} \).

\[
\frac{du}{dx} = \frac{1}{2} (x)^{-1/2}
\]

\( y = u \tan^3 u \),

\[
\frac{dy}{du} = \tan^3 u + u \frac{d}{du} \tan^3 u
\]

\[
= \tan^3 u + 3u \tan^2 u \sec^2 u (*)
\]

\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}
\]

\[
= (\tan^3 u + 3u \tan^2 u \sec^2 u) \cdot \frac{1}{2} (x)^{-1/2}
\]

(*) I missed out some calculations here. You should be able to do this!

\[
\therefore \frac{dy}{dx} = \frac{\tan^3 (\sqrt{x}) + 3\sqrt{x} \tan^2 (\sqrt{x}) \cdot \sec^2 (\sqrt{x})}{2\sqrt{x}}
\]
2a. Find \( \frac{dy}{dx} \) by first making an appropriate choice for \( u \).

\[
y = (x^3 + 2x)^{37}
\]

Let, \( u = x^3 + 2x \)

\[
\frac{du}{dx} = 3x^2 + 2
\]

Then, \( y = u^{37} \)

\[
\frac{dy}{du} = 37u^{36}
\]

\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}
\]

\[
= 37u^{36} \cdot (3x^2 + 2)
\]

\[
= 37 \left( 3x^2 + 2 \right) (x^3 + 2x)^{36}
\]
2b. Find $\frac{dy}{dx}$ by first making an appropriate choice for $u$.

$$y = \left( x^3 - \frac{7}{x} \right)^{-2}$$

Let, $u = x^3 - \frac{7}{x}$

$$\frac{du}{dx} = 3x^2 + 7x^{-2}$$

Then, $y = (u)^2$

$$\frac{dy}{du} = -2(u)^{-3}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= -2(u)^{-3} \cdot (3x^2 + 7x^{-2})$$

$$= -2 \left( 3x^2 + 7x^{-2} \right) \left( 3x^3 - 7x^{-1} \right)^{-3}$$
2c. find $\frac{dy}{dx}$ by first making an appropriate choice for $u$.

\[ y = \sin \left( \frac{1}{x^2} \right) \]

Let, \[ u = x^{-2} \]

\[ \frac{du}{dx} = -2x^{-3} \]

Then, \[ y = \sin u \]

\[ \frac{dy}{du} = \cos u \]

\[ \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \]

\[ = (\cos u) \cdot -2x^{-3} \]

\[ = -2x^{-3} \cos (x^{-2}) \]
2d. Find $\frac{dy}{dx}$ by first making an appropriate choice for $u$.

$y = \cos^3(\sin 2x)$.

Let, $u = \sin 2x$ \hspace{1cm} $\frac{du}{dx} = 2\cos 2x$

Then, $y = \cos^3 u$

We need to find $\frac{dy}{du}$. We can’t do this directly so we make a second substitution

Let, $w = \cos u$ \hspace{1cm} $\frac{dw}{du} = -\sin u$

$y = w^3$ \hspace{1cm} $\frac{dy}{dw} = 3w^2$

$\frac{dy}{du} = \frac{dy}{dw} \cdot \frac{dw}{du}$

$= 3w^2 \cdot -\sin u$

$= -3(\sin u)(\cos^2 u)$
\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}
\]
\[
= 3 (\sin u) (\cos^2 u) \cdot 2 \cos 2x
\]
\[
= -6 (\sin [\sin 2x]) (\cos^2 [\sin 2x]) \cdot \cos 2x
\]
2.10.3 Implicit Differentiation

So far our equations have generally been expressed in explicit form. For example,
\[ y = 3x - 5, \]
\[ s = -16t^2 + 20t \] and \[ u = 2w - w^2 \] where
\[ y \] is explicitly in terms of \( x \),
\[ s \] is explicitly in terms of \( t \) and
\[ u \] is explicitly in terms of \( w \).

What happens if we are unable to write \( y \) in terms of \( x \), for example
\[ x^2 - 2y^3 + 4y = 2 \]
How do we find \( \frac{dy}{dx} \) in this example?

You are expected to work through section 2.11.2 before the lecture.
To find $\frac{dy}{dx}$ we use implicit differentiation, in which we assume that $y$ is a differentiable function of $x$.

To find $\frac{dy}{dx}$ we differentiate each term with respect to $x$. When we differentiate a term involving $x$ alone, we can differentiate as usual. But when we differentiate a term involving $y$ we must apply the Chain Rule.

In general,

$$\frac{d}{dx} [g(y)] = \frac{d}{dy} [g(y)] \cdot \frac{dy}{dx}$$

$$= g'(y) \frac{dy}{dx}$$

**Example** Differentiate the following with respect to $x$:

1. $3x^2$
2. $2y^3$
3. $x + 3y$
4. $xy^2$
1. Differentiate with respect to $x$. $3x^2$

2. Differentiate with respect to $x$. $2y^3$

\[
\frac{d}{dx} = 6x
\]
\[
\frac{d}{dx} (2y^3) = \frac{d}{dy} (2y^3) \frac{dy}{dx}
\]
\[
= 6y^2 \cdot \frac{dy}{dx}
\]

3. Differentiate with respect to \(x\).

\(x + 3y\)

4. Differentiate with respect to \(x\).

\(xy^2\)
\[
\frac{d}{dx} (x + 3y) = \frac{d}{dx} x + \frac{d}{dx} 3y \\
= 1 + \frac{d}{dy} (3y) \cdot \frac{dy}{dx} \\
= 1 + 3 \frac{dy}{dx}
\]

\[
\frac{d}{dx} (xy^2) = \frac{d}{dx} (y^2) \cdot x + \frac{d}{dx} (x) \cdot y^2 \\
= \frac{d}{dy} (y^2) \cdot \frac{dy}{dx} \cdot x + y^2 \\
= 2yx \cdot \frac{dy}{dx} + y^2
\]
Given an **equation** involving $x$ and $y$ we can find $\frac{dy}{dx}$ as follows.

1. Differentiate all terms of the equations with respect to $x$.

2. Collect all terms involving $\frac{dy}{dx}$ on the left hand side of the equation and move all other terms to the right side.

3. Take $\frac{dy}{dx}$ out as a factor on the left hand side of the equation.

4. Solve for $\frac{dy}{dx}$ by dividing both sides of the equation by the left hand factor that multiplies $\frac{dy}{dx}$.

**Examples**

1. find $\frac{dy}{dx}$ given $x^2 + y^2 - 3x - 6y = -5$. 
   \[
   \frac{dy}{dx} = \frac{3 - 2x}{2y - 6}
   \]

2. Find $\frac{dy}{dx}$ given $x^2 + xy + 3y^2 = 4$. 
   \[
   \frac{dy}{dx} = \frac{-x}{x + 6y}
   \]

3. Find $\frac{dy}{dx}$ given $5y^2 + \sin y^2 = x^2$. 
   \[
   \frac{dy}{dx} = \frac{x}{y(5 + 2\cos y^2)}
   \]

4. Use implicit differentiation to find $\frac{d^2y}{dx^2}$ if $4x^2 - 2y^2 = 9$. 
   \[
   \frac{d^2y}{dx^2} = \frac{-9}{y^3}
   \]

5. Show, using implicit differentiation, that $x(x^2 + 3y^2) = c$ is the solution to the differential equation $x^2 + y^2 + 2xy \frac{dy}{dx} = 0$. 

1. Find \( \frac{dy}{dx} \) given
\[ x^2 + y^2 - 3x - 6y = -5. \]

\[
\frac{d}{dx} x^2 + \frac{d}{dx} y^2 - \frac{d}{dx} 3x - \frac{d}{dx} 6y = \frac{d}{dx} (-5)
\]

\[
2x + \frac{d}{dy} (y^2) \cdot \frac{dy}{dx} - 3 - 6 \frac{d}{dy} (y) \cdot \frac{dy}{dx} = 0
\]

\[
2x - 3 + 2y \cdot \frac{dy}{dx} - 6 \frac{dy}{dx} = 0
\]

\[
(2y - 6) \frac{dy}{dx} = 3 - 2x
\]

\[
\frac{dy}{dx} = \frac{3 - 2x}{2y - 6}
\]
2. Find \( \frac{dy}{dx} \) given \( x^2 + xy + 3y^2 = 4 \).

\[ \frac{d}{dx} x^2 + \frac{d}{dx} (xy) + \frac{d}{dx} (3y^2) = \frac{d}{dx} 4 \]

\[ 2x + x \frac{d}{dx} (y) + y \frac{d}{dx} (x) + \frac{d}{dy} (3y^2) \cdot \frac{dy}{dx} = 0 \]

\[ 2x + x \frac{dy}{dx} + y + 6y \frac{dy}{dx} = 0 \]

\[ (x + 6y) \frac{dy}{dx} = -(y + 2x) \]

\[ \frac{dy}{dx} = \frac{-(y + 2x)}{x + 6y} \]
3. Find \( \frac{dy}{dx} \) given \( 5y^2 + \sin y^2 = x^2 \).

\[
\frac{d}{dx} (5y^2) + \frac{d}{dx} (\sin y^2) = \frac{d}{dx} (x^2)
\]

\[
\frac{d}{dy} (5y^2) \cdot \frac{dy}{dx} + \frac{d}{dy} (\sin y^2) \cdot \frac{dy}{dx} = 2x.
\]

We need to calculate \( \frac{d}{dy} (\sin y^2) \). Consider

\[
z = \sin y^2
\]

Let, \( w = y^2 \), \( \frac{dw}{dy} = 2y \)

\[
z = \sin w \quad \frac{dz}{dw} = \cos w
\]

\[
\frac{dz}{dy} = \frac{dz}{dw} \cdot \frac{dw}{dy}
\]

\[
= (\cos w) \cdot (2y)
\]

\[
= 2y \cos (y^2)
\]
\[ 10y \frac{dy}{dx} + 2y \cos \left( y^2 \right) \frac{dy}{dx} = 2x \]

\[ [10y + 2y \cos \left( y^2 \right)] \frac{dy}{dx} = 2x \]

\[ \frac{dy}{dx} = \frac{2x}{10y + 2y \cos \left( y^2 \right)} \]

\[ = \frac{x}{y \left[ 5 + \cos \left( y^2 \right) \right]} \]
4. Use implicit differentiation to find \( \frac{d^2y}{dx^2} \) if \( 4x^2 - 2y^2 = 9 \).

First of all we must find \( \frac{dy}{dx} \).

\[
\frac{d}{dx} (4x^2) - \frac{d}{dx} (2y^2) = \frac{d}{dx} (9)
\]

\[
8x - \frac{d}{dy} (2y^2) \cdot \frac{dy}{dx} = 0
\]

\[
8x - 4y \cdot \frac{dy}{dx} = 0
\]

\[
\frac{dy}{dx} = \frac{8x}{4y} = \frac{2x}{y}
\]
Now we calculate \( \frac{d^2 y}{dx^2} \) using the quotient rule

\[
\frac{d^2 y}{dx^2} = \frac{y \frac{d}{dx} (2x) - 2x \frac{d}{dx} (y)}{y^2}
\]

\[
= \frac{2y - 2x \frac{dy}{dx} (y) \cdot \frac{dy}{dx}}{y^2}
\]

\[
= \frac{2y - 2x \frac{dy}{dx}}{y^2}
\]

\[
= \frac{2y}{y^2}
\]

\[
= \frac{2 \cdot \frac{2x}{y}}{y^2}
\]

\[
= \frac{2y \frac{y}{y} - 2x \frac{2x}{y}}{y^2}
\]

\[
= \frac{2y^2 - 2x^2 \frac{1}{y}}{y^2}
\]

\[
= \frac{2y^2}{y^3}
\]

\[
= \frac{4x^2 - 2y^2}{y^3}
\]

\[
= \frac{(4x^2 - 2y^2)}{y^3}
\]

\[
= \frac{-9}{y^3}
\]
Show, using implicit differentiation, that $x(x^2 + 3y^2) = c$ is the solution to the differential equation

$$x^2 + y^2 + 2xy \frac{dy}{dx} = 0.$$ 

We need to differentiate the equation $x(x^2 + 3y^2) = c$

$$x^3 + 3xy^2 = c$$

$$\frac{d}{dx} (x^3) + 3 \frac{d}{dx} (xy^2) = \frac{d}{dx} (c)$$

$$3x^2 + 3y^2 \frac{d}{dx} (x) + 3x \frac{d}{dx} (y^2) = 0$$

$$3x^2 + 3y^2 + 3x \frac{dy}{dy} (y^2) \cdot \frac{dy}{dx} = 0$$

$$3x^2 + 3y^2 + 3x \cdot (2y) \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{(x^2 + y^2)}{2xy}$$
We now substitute our expression for $\frac{dy}{dx}$ into the differential equation
\[
x^2 + y^2 + 2xy \frac{dy}{dx} = 0
\]
\[
x^2 + y^2 + 2xy - \frac{(x^2 + y^2)}{2xy} = 0
\]
\[
x^2 + y^2 - x^2 - y^2 = 0.
\]
2.10.4 Revision Questions

The following questions are about the key ideas in this section.

1. Suppose that $y = f(u)$ where $u = g(x)$. What is $\frac{dy}{dx}$?

2. Given the *implicit* equation $f(y, x) = 0$ write down the procedure to find $\frac{dy}{dx}$.

2.10.5 Exercises

**Hint.** There’s always an exam question on implicit differentiation.