MATH111 – Spring 2007
Tutorial Sheet – Week 4

This tutorial sheet covers chapter 4 of the notes.

Chapter 4

Revision of Key Ideas

1. The average birth rate of a species per individual per year is \( r \). The fractional death rate of the species is \( d \) per year. The initial number of animals in the population is given by \( N_0 \).
   
   (a) Write down a word-equation for this model.
   (b) Write down the associated discrete equation model for the size of the species population in year \( n \).
   (c) Write down the closed-form solution to your model. How does the long-term population size depend upon the parameters \( r \) and \( d \)?
   (d) What is the flaw in the linear population model and how is it rectified?
   (e) Name two additional processes that might be included in a more general model.

2. The size of a population is given by the solution of the equation

   \[
y_{n+1} = f(y_n).
   \]

   (a) Sketch the curve \( y = f(y_n) \) for a biological realistic function \( f(y_n) \),
   (b) Identify the four important features that make \( f(y_n) \) a realistic function to describe population dynamics.

3. Consider the difference equation

   \[
x_{n+1} = f(x_n), \quad x_0 = X
   \]

   (a) How are the fixed points of this equation found?
   (b) Why are fixed points important?

4. The population of a species is governed by the difference equation

   \[
x_{n+1} = f(x_n), \quad x_0 = X
   \]

   Explain how cobwebbing is used to determine the dynamics of this model for the specified initial condition.

5. Consider the logistic difference equation

   \[
x_n = r x_n (1 - x_n), \quad 0 < r < 4, \quad 0 < x_0 < 1.
   \]

   (a) By drawing a cobwebbing diagram show that if \( 0 < r < 1 \) then \( \lim_{n \to \infty} x_n = 0 \).
   (b) What is the biological meaning of this result?
Exercises

1. Consider the following map

\[ x_{n+1} = \frac{27rx_n^2 (1 - x_n)}{16} \]

(a) Show that if \( 0 \leq r \leq 4 \) and \( 0 \leq x_n \leq 1 \) then \( 0 \leq x_{n+1} \leq 1 \).

(b) Show that there is only one fixed point \( (x^* = 0) \) for \( 0 \leq r < \frac{64}{27} \), two fixed points when \( r = \frac{64}{27} \) and three fixed points for \( \frac{64}{27} \leq r \leq 4 \). Give a formulae for the new pair \( x^* \).

2. Consider the map

\[ x_{n+1} = rx_n (1 - x_n^2) \]

(a) Show that if \( 0 \leq x_n \leq 1 \) then \( 0 \leq x_{n+1} \leq 1 \) provided that \( 0 \leq r \leq \frac{3\sqrt{3}}{2} \).

(b) Solve the fixed point equation, and show that there is only one fixed point \( (x^* = 0) \) for \( 0 \leq r \leq 1 \) and three fixed points when \( 1 < r \). Give a formulae for the new pair \( x^* \).

(c) Only one of the new pair of solutions is biologically meaningful: which one is it?

Worked solutions to the exercise questions are available as follows:

1. This question appears in the questions section of Chapter 4. The worked solution appears in the corresponding appendix.

2. Assignment Week 6 (Spring 2004).