MATH 111 — Applied Mathematical Modelling I

Spring Session 2004

Mid-Session Test

Student Name: ___________________________ Student Number: __________

Instructions

Time Allowed: 90 minutes
Number of questions: 9.

1. Each question is to be attempted.
2. The questions are not of equal value. The value of each question is indicated in square brackets.
3. The examination paper is printed on both sides.
4. WORKING (including all necessary reasoning) is to be shown for all solutions.
5. Working is to be done in the exam paper.

Examination Materials/Aids Allowed
Non-alphanumeric calculators are permitted.

A one-page, A4-sized, double-sided summary sheet is permitted.

Examination Materials/Aids to be supplied
None.

This examination paper must NOT be removed from the examination room.
1. Consider the difference equation

\[ y_{n+1} = f(y_n), \quad y_0 = 1, \quad n = 0, 1, \ldots \]

where \( f(y_n) \) is some unspecified function of \( y_n \). With reference to this equation explain what is meant by the word ‘dynamics’. \[1\]

2. (a) Give an example of an autonomous difference equation and a non-autonomous difference equation, explaining why your equation is autonomous/non-autonomous. \[2\]

(b) Identify if the following difference equations are linear or non-linear. You must justify your answer. \[2\]

(i) \( n_{y+2} = n_{y+1} y \)

(ii) \( y_{n+1} = 2y_{n+1} + \sin(n) \)
3. Consider the problem of modelling the number of chickens in Mr & Mrs Tweedy’s farm. Each week the following activities occur:

- The number of chickens increases through natural growth by 10%.
- A fraction, \( \alpha \), of the chickens are killed by foxes.
- A constant number of chickens are converted into chicken pies.

(a) Write down a \textbf{word} equation that defines this problem. \hspace{1cm} [2]

(b) Write down, formally, the difference equation that describes the above scenario. Define \textbf{all} variables and explain your terms. \hspace{1cm} [2]

4. How long will it take \$1500 to accumulate to at least \$2000 at 5.0\% simple interest? \hspace{1cm} [1]
5. Lien borrows $20,000 to have a MATH111 chip implanted in her head so that everything makes sense. Interest is compounded monthly at 9% p.a.

(a) In the first year Lien makes no repayments. How much does she owe at the end of the year? \[\text{[1]}\]
(b) Starting in the second year Lien makes a repayment at the end of each month. If the loan is to be repaid after a further nine years what is the monthly repayment? \[\text{[2]}\]
6. You have been given $1000 to invest for one year. You have a choice of three bank accounts.

- ‘You Beaut’ bank offers you 10\% p.a. compounded annually. At the end of the year you will pay $20 in fees.
- ‘Fair Go’ bank offers you 10\% p.a. compounded quarterly. At the end of the year you will pay $30 in fees.
- ‘Ocker’ bank offers you 11\% p.a. compounded every four months. At the end of the year you will pay $25 in fees.

Which bank should you put your money into (justify your answer)?
7. Patient flow in a department of geriatric medicine is modelled by the difference equation,

\[ x_n = N + (1 - \alpha - \beta - \gamma) x_{n-1}, \quad n = 1, 2, 3 \ldots \]

where \( x_n \) is the number of patients in the department in the \( n \)th month, \( N \) is the number of new patients admitted each month, \( \alpha \) is the fraction of current patients who are discharged each month, \( \beta \) is fraction of current patients who, unfortunately, die each month and \( \gamma \) is the fraction of the current patients who are transferred to another section each month. For convenience we write

\[ a = 1 - \alpha - \beta - \gamma \]

and assume that \( 0 < a < 1 \).

(a) Find the general solution of the patient flow model, simplifying as far as possible. \[2\]

(b) What is the number of patients in the department in the limit \( n \to \infty \)? \[2\]
(c) A new geriatric ward is added to a hospital. When the ward is opened there are no patients \( (x_0 = 0) \).

(i) Suppose that the new ward has 100 beds and it is anticipated that 50 patients are admitted a month.

- Explain why if \( a \leq 0.5 \) the ward never overfills. [2]

(ii) The ward operates with \( a = 0.51 \). During which month does the ward have to start turning patients away? [2]

(d) We have assumed that the parameters \( N \) & \( a \) are constant. Briefly discuss if this is reasonable. [2]
8. Consider the logistic equation with fixed harvesting

\[ x_{n+1} = rx_n (1 - x_n) - h, \quad n = 0, 1, 2 \ldots \]

where \(1 < r < 4\) and \(0 \leq h \leq 1\).

(a) Show that harvesting is only sustainable if

\[ h \leq \frac{(r - 1)^2}{4r}. \]
(b) The stable fixed point of the harvesting model (should it exist) is given by

\[ x^* = \frac{- (1 - r) + \sqrt{(1 - r)^2 - 4rh}}{2r} \]

Find the fixed point (to four decimal places), and the associated eigenvalue, when

(i) \( r = 1.5 \) and \( h = 0.015 \).
(ii) \( r = 1.6 \) and \( h = 0.05625 \).
(iii) \( r = 2 \) and \( h = 0.045 \).
(c) Gollum Fresh Fish (motto ‘fish fresh from the sea, three times a day’) has the choice to send its fishing fleet to one of three fisheries. The value for \( h \) is regulated by Mordor Moguls. The long-term yearly profit (\( P \)) for fishing in a fishery is

\[
P = ax^* - b
\]

where \( a \) and \( b \) are parameters that depend upon the fishery and \( x^* \) is the steady fixed point of the harvesting model for the specified values of \( h \) and \( r \). The numbers associated with each fishery are

**Fishery one:** \( r = 1.5, h = 0.015, a = 3, b = 0.6 \).

**Fishery two:** \( r = 1.6, h = 0.05625, a = 4, b = 0.45 \).

**Fishery three:** \( r = 2, h = 0.045, a = 1, b = 0.2 \).

Which fishery should Gollum Fresh Fish use and why?

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9. Find the fixed points of the Ricker difference equation.

\[
x_{n+1} = x_n \exp \left[ r \left( 1 - x_n \right) \right], \quad n = 0, 1, \ldots
\]