12 Linear Stability Analysis of Steady-State Solutions

12.1 Physical motivation of ‘stability’

Consider the differential equation
\[
\frac{dx}{dt} = f(x). \tag{12.1}
\]

The steady-state solutions, \( x^* \), of equation (12.1) are the solutions of
\[
f(x^*) = 0.
\]

At a steady-state the system does not change with time.

By carefully examining what happens to a system when it is near a steady-state we can gain an insight into the dynamic behaviour of equation (5.1).

To do this we distinguish between two types of steady-states by re-introducing the the concept of stability.

Since this is best described by analogy, see figure 12.1, which exemplifies three situations, two of which are steady states________; one steady state is stable, the other unstable.
A steady state is termed *stable* if neighbouring states are attracted to it and *unstable* if the converse is true.

As shown in figure 12.1, whilst an object balanced precariously on a hill may be in steady state, it will not return to this position if disturbed slightly. Rather, it may proceed on some lengthy excursion leading possibly to a second, more stable situation. (After Edelstein-Keshet, 1988)
Such distinctions have *practical implications* in applied mathematics.

When steady states are *unstable*, great changes may be about to happen: a population may *crash*, possibly becoming *extinct*, or a disease may *spread* through a population.

This *qualitative* information about whether change is imminent is potential of *great importance*.

With this motivation behind us, we turn to the analysis that permits us to make such predictions.

### 12.2 Linear stability analysis

Suppose that we have already found a steady-state solution $x^*$ of equation (12.1).

We now proceed to explore its stability by asking the following key question: given an initial condition $x(0)$ *close* to $x^*$, will the solution of the differential equation, i.e. the values $x(t)$, tend *toward* or *away* from the steady state $x^*$?

To answer this question we write

$$
\xi(t) = x(t) - x^*, \quad |\xi(t)| \ll 1,
$$
where $\xi(t)$ is the distance between the population size at time $t$ and the steady-state. If $\xi(t) > 0$ then $x(t) > x^*$ and conversely if $\xi(t) < 0$ then $x(t) < x^*$. Using equation (12.1) we can obtain a differential equation for $\xi(t)$.

$$\frac{d\xi}{dt} = \frac{dx}{dt} [x(t) - x^*]$$

(why?)

$$= f(x^* + \xi) + f'(x^*) \xi + O(\xi^2); \text{ (why?)}$$

: 1.

*: (why?)

(12.2)
Thus we have obtained a linear differential equation for the distance between the solution of our differential equation, $x(t)$, and the steady-state $x^*$. In this equation the parameter $\lambda$ is known as the eigenvalue of the steady-state $x^*$.

The solution of the differential equation

$$\frac{d\xi}{dt} = \lambda \xi$$

is

$$\xi(t) = \text{__________,}$$

$$\Rightarrow \lim_{t \to \infty} \xi(t) = \begin{cases} \text{________,} & \lambda < 0 \\ \text{________,} & \lambda > 0 \end{cases}$$

Thus the eigenvalue determines the stability of the steady-state.

**Definition 12.1**

*(Stable and unstable fixed point)* A steady-state $x^*$ is

stable if $f'(x^*) < 0$ \hspace{1cm} (12.3)

unstable if $f'(x^*) > 0$ \hspace{1cm} (12.4)

The stability of the steady-state solution $x^*$ is determined by the value of $f'(x^*)$. 

Thus the eigenvalue determines the stability of the steady-state.
Question 12.1 Why does \( \lambda < 0 \) mean that the steady-state solution \( x^* \) is stable?

Question 12.2 Is the distinction between a stable steady-state solution and an unstable steady-state solution of practical importance?

Comment 12.1 If you start near a stable steady-state how quickly do you approach it?

Suppose that \( x^* \) is a stable steady-state solution of the differential equation

\[
\frac{dx}{dt} = f(x).
\]

Then the stability of the steady-state can change in only one way: the eigenvalue \( \lambda = f'(x^*) \) increases/decreases through 0.

Question 12.3 How many ways can the fixed-point \( x^* \) of the difference equation

\[
x_{n+1} = f(x_n)
\]

lose stability?
12.3 Stability of the steady-states in the logistic equation

For the logistic differential equation we have

\[ \frac{dx}{dt} = rx \left(1 - \frac{x}{K}\right). \]  \hspace{1cm} (12.5)

Thus

\[ f(x) = rx \left(1 - \frac{x}{K}\right) \]

and

\[ \frac{df(x)}{dx} = \text{________}. \]

\[ \frac{df(x)}{dx} = r \left(1 - \frac{2x}{K}\right) \]

The steady-states of the logistic equation are given by

\[ f(x) = 0, \]
\[ rx \left(1 - \frac{x}{K}\right) = 0, \]
\[ x_1^* = 0 \quad \text{or} \quad x_2^* = \text{__}. \]

The steady-states and their corresponding eigenvalues \( \lambda \) are

\[ x_1^* = 0, \quad \lambda = f'(x_1^*) = \text{_______}. \]  \hspace{1cm} (12.6)

\[ x_2^* = K, \quad \lambda = f'(x_2^*) = \text{_______}. \]  \hspace{1cm} (12.7)
\[ x_1^* = 0, \quad \lambda = f'(0) = r, \]

\[ x_2^* = K, \quad \lambda = f'(K) = -r. \]

For \( r > 0 \) both of these steady-states are biologically meaningful, that is non-negative. With \( r > 0 \) the trivial fixed point is _______ whereas the non-trivial steady-state is _______.

### 12.4 Stability of the steady-state in the Gompertz model

For the Gompertz model we have

\[
\frac{dx}{dt} = rx \ln \frac{K}{x}
\]

Thus

\[ f(x) = rx \ln \frac{K}{x} \]

and

\[
\frac{df(x)}{dx} = 
\]

\[
\]
12.5 Steady-state diagrams and bifurcations

Many problems of practical interest contain a control, or bifurcation, parameter \( \lambda \). Instead of writing

\[
\frac{dx}{dt} = f(x),
\]

such problems are written in the form

\[
\frac{dx}{dt} = f(x, \lambda). \tag{12.8}
\]

The steady-state solutions are found by solving the equation

\[
f(x, \lambda) = 0.
\]
In such problems the value of any steady-state solutions and their stability is usually a function of the control parameter. This information is useful expressed in a steady-state, or response, diagram.

**Definition 12.2 Steady-state diagram**
The graph of \( x \) versus \( \lambda \) is called a steady-state diagram or a response curve. This shows how the steady-state solutions of equation (12.8), \( x \), depend upon the control (bifurcation) parameter \( \lambda \).

In such a diagram steady-states are indicated by using a solid line whilst unstable steady-states are indicated by a dotted line. See figures 12.2–12.4 for examples.

Of particular interest on steady-state diagrams are **bifurcation points**. Loosely speaking, a bifurcation point is a point \( (x, \lambda) \) on a steady-state diagram where two solution branches with distinct tangents intersect. At such a point the number of steady-state solutions to the differential equation changes.

**Definition 12.3 (Bifurcation point)**
The point \( (x, \lambda) \) is called a bifurcation point if the number of steady-state solutions of equation (12.8) in the neighbourhood of \( x \) is not constant for any arbitrary small change of \( \lambda \).

In sections 12.5.1 12.5.3 we analyse three differential equations. The steady-state diagram for each equation contains a different type of bifurcation point.
12.5.1 The limit-point bifurcation

Consider the differential equation
\[
\frac{dx}{dt} = \mu - x^2. \tag{12.9}
\]

The set of steady-state solutions is given by
\[ x = \pm \sqrt{\mu}, \quad \mu \geq 0. \]

There are no steady-state solutions when \( \mu < 0 \), one steady-state solution when \( \mu = 0 \) and two steady-state solutions when \( \mu > 0 \).

The point \( (\mu = 0, x = 0) \) is therefore a bifurcation point and the value \( \mu = 0 \) is a bifurcation value.

Question 12.4 Show that the steady-state \( x = +\sqrt{\mu} \) is stable \((\mu > 0)\) whilst the steady-state \( x = -\sqrt{\mu} \) is unstable \((\mu > 0)\).

The steady-state diagram for equation (12.9) is shown in figure 12.2.
Figure 12: Steady-state diagram for the differential equation \( \frac{dx}{dt} = \mu - x^2 \): a solid line indicates a stable solution whilst a dotted line indicates an unstable solution.

The particular type of bifurcation occurring in figure 12.2 (i.e., where on one side of a parameter value there are \( \text{no} \) fixed points and on the other side there are \( \text{two} \) fixed points) is known as a saddle-node bifurcation or a limit-point bifurcation.
2.5.2 The transcritical bifurcation

Consider the differential equation

\[
\frac{dx}{dt} = \mu x - x^2.
\]

The set of steady-state solutions is given by

\[
x_1 = 0, \\
x_2 = \mu.
\]

There are two steady-state solutions for \( \mu \neq 0 \) and one steady-state solution when \( \mu = 0 \). Thus the number of steady-state solutions changes at the point \((\mu, x) = (0, 0)\). Thus this point is a bifurcation point.

Question 12.5

1. Show that the steady-state branch \( x_1 = 0 \) is stable for \( \mu < 0 \) and unstable for \( \mu > 0 \).

2. Show that the steady-state branch \( x_2 = \mu \) is unstable for \( \mu < 0 \) and stable for \( \mu > 0 \).
The steady-state diagram for equation (2.110) is shown in figure 2.13. Note that at the point $(\mu, x) = (0, 0)$, which we have already identified as a bifurcation point, two solution branches with distinct tangents intersect.

Figure 2.13: Steady-state diagram for the differential equation $\frac{dx}{dt} = \mu x - x^2$: a solid line indicates a stable solution whilst a dotted line indicates an unstable solution.
2.5.3 The pitchfork bifurcation

Consider the differential equation
\[
\frac{dx}{dt} = \mu x - x^3.
\]
(\text{II11})

The set of steady-state solutions is given by
\[
x_1 = 0,
\]
\[
x_\pm = \pm \sqrt{\mu}, \quad \mu \geq 0.
\]

There is one steady-state solutions when \( \mu \leq 0 \) and three steady-state solutions when \( \mu > 0 \). The number of steady-state solutions to the differential equation changes at the point \((\mu, x) = (0, 0)\). Thus this point is a bifurcation point.

The particular type of bifurcation occurring in figure 12.3 (i.e., where on one side of a parameter value there are two fixed points and on the other side there are two fixed points) is known as a transcritical bifurcation. This bifurcation frequently occurs in mathematical epidemiology.

Question 12.6 Outline the reasons why a transcritical bifurcation is expected to occur in mathematical epidemiology.
Question 12.7

1. Show that the steady-state $x_1 = 0$ is stable for $\mu < 0$ and unstable for $\mu > 0$.

2. Show that the steady-state $x_{\pm} = \pm \sqrt{\mu}$ is stable for $\mu > 0$.

The steady-state diagram for equation (111) is shown in figure 24. Note that at the point $(\mu, x) = (0, 0)$, which we have already identified as a bifurcation point, two solution branches with distinct tangents intersect.
Figure 14: Steady-state diagram for the differential equation $\frac{dx}{dt} = \mu x - x^3$: a solid line indicates a stable solution whilst a dotted line indicates an unstable solution.

The particular type of bifurcation occurring in figure 12.4 (i.e., where on one side of a parameter value there is one fixed points and on the other side there are three fixed points) is known as a pitchfork bifurcation.
The study of steady-state solutions, their stability and their bifurcations is the process through which the properties of non-linear differential equations are elucidated in a systematic manner. These issues are left for another course.

2.6 Revision of key ideas

2.7 Concept map

Draw a concept map for this chapter relating the aims/key ideas of the chapter. If you are unfamiliar with the idea of a concept map see appendix A.