Solutions

Spring 2004

1. Give the order of the following difference equations and state whether they are linear, nonlinear, autonomous or non-autonomous.

(a) \( m_{n+2} + 3m_{n} = x_{n} + 2 \)
(b) \( x_{n+1} + \cosh(x_{n}) = 2 \)

Solution,

Order,

(a) the order \((n+2) - (n-1) = 3\),
(b) the order \((n) - (n-1) = 1\).

Linear or nonlinear?

(a) \( m_{n+2} + 3m_{n} = x_{n} + 2 \) all terms present linearly. Thus the equation is linear.
(b) \( x_{n+1} + \cosh(x_{n}) \) non-linear. Thus the equation is nonlinear.

Autonomous or non-autonomous?

(a) \( m_{n+2} + 3m_{n} = x_{n} + 2 \) all terms present. Thus the equation is autonomous.
(b) \( x_{n+1} + \cosh(x_{n}) \) only \( x_{n} \) is present. Thus the equation is non-autonomous.

2. Consider the difference equation

\[ y_{k} = ky_{k-1}, \quad k = 1, 2, 3, \ldots \]

with initial condition \( y_{1} = 1 \).

(a) Calculate \( y_{1}, y_{2}, y_{3}, y_{4} \) and make a guess at the “closed-form” solution of \( y_{k}\).
(b) Verify that your formula satisfies the difference equation and the initial condition.

Solution

(a) \( y_{1} = 1, y_{2} = 2, y_{3} = 6, y_{4} = 24 \)

This sequence looks like \( y_{k} = k! \).
(b) If \( y_{k} = k! \) then \( y_{k+1} = (k+1)! \) Consider the RHS of the equation

\[ ky_{k-1} = k(k-1)! \]

Thus \( y_{k} = LHS \).

Furthermore \( y_{1} = 0! = 1 \) which is the initial condition.

3. Solve the following difference equations to obtain solutions in “closed form”.

(a) \( x_{n+1} = 2x_{n} + 1 = 0 \)
(b) \( x_{n+1} = 2x_{n} + 1 \)
(c) \( x_{n+1} = x_{n} + 1 = n \)

(Hint: Arithmetic-Geometric Series \( \sum_{k=1}^{n} (1)^{n} \cdot k = \frac{1}{4} (2n + 1) - \frac{1}{4} (1)^{n} \))

Solution,

The solution to the equation

\[ x_{n} = ax_{n-1} + b(n) \]

is

\[ x_{n} = a^{n}x_{1} + \sum_{k=1}^{n} b(k) a^{n-k} \]

(a) \( x_{n+1} = 2x_{n} + 1 \)

We have

\[ a = 2, \quad b(n) = 0, \]

Thus the solution is

\[ x_{n} = 2^{n}x_{1} \]

where \( x_{1} \) is the initial condition.

(b) \( x_{n+1} = 2x_{n} + 3 \)

We have

\[ a = 1, \quad b(n) = 3, \]

Thus the solution is

\[ x_{n} = x_{1} + 3n \]

where \( x_{1} \) is the initial condition.

(c) \( x_{n+1} = x_{n} + 1 \)

We have

\[ a = 1, \quad b(n) = n, \]

Thus the solution is

\[ x_{n} = (1)^{n}x_{1} + \sum_{k=1}^{n} (1)^{n} \cdot k \]

using the Hint,

\[ \sum_{k=1}^{n} (1)^{n} \cdot k = \frac{1}{4} (2n + 1) - \frac{1}{4} (1)^{n} \]

4. Consider the problem of modeling patient flow in a department of geriatric medicine. Each day the following activities occur:

- A number of new patients are admitted to the department for acute care.
- A fraction, \( a \), of the current patients are treated and discharged.
- A fraction, \( b \), of the current patients, unfortunately, die.
- A fraction, \( c \), of the current patients, \( c \), are transferred to another section.

(a) Write down a word equation that defines this problem.
(b) Write down, formally, the difference equation that describes the above scenario. Define all variables and explain your terms.

Solution

(a) \[
\left\{ \begin{array}{l}
\text{Change in patient numbers} = \{ \text{new patients} \} - \{ \text{patients discharged} \} - \{ \text{patients died} \} - \{ \text{patients transferred} \}
\end{array} \right.
\]

(b) Let \( P_d \) and \( P_{d+1} \) be the number of patients in the department on days \( d \) and \( d+1 \) respectively.

Let \( \alpha, \beta \) and \( \gamma \) be the fraction of patients that are discharged, die and transferred respectively.

Let \( N \) be the number of new patients arriving at the department.

Then

\[
P_{d+1} = P_d = N - \alpha P_d - \beta P_d - \gamma P_d
\]

\[
P_{d+1} = P_d = N + (1 - \alpha - \beta - \gamma) P_d
\]

5. Imagine this scenario, if you will. Economic rationalism has taken hold of your workplace and it's time to renegotiate your contract. Knowing a thing or two about maths, you make the following proposal. "Now, I've been far too greedy. But I've come to my senses, after reading Animal Farm, and propose a new pay scale. Starting tomorrow, I would like you to pay me two cents..." "It's a deal", "raised to the power of the number of days", "Sign here!" "...the commencement of my new..." "Next!", "contract."

Day one, you are paid 2\(c\). Day two, 4\(c\) (2 squared). Day three, 8\(c\) (2\(^3\)). Day four, 16\(c\) (2\(^4\)). Day five, 32\(c\). For week one, you take home 62\(c\).

(a) How much do you take home in week two?

(b) How much do you take home in week three?

(c) How much do you take home in week four?

Based on an article by Jeremy Clunen that appeared in Men's Style Summer 2003

Solution. The sequence 2, 4, 8, 16, 32, 64... is a Geometric Progression with \( k = 2 \) and \( a = 2 \). Thus the sum of the first \( n \) term is

\[
s_n = a \left( \frac{a^n - 1}{a - 1} \right) \]

\[
= 2 \left( \frac{2^n - 1}{1} \right)
\]

There are five working days in the first week. So the take home pay is

\[
s_5 = 2 \left( \frac{2^5 - 1}{1} \right)
\]

\[
= 2 \times 31
\]

\[
= 62c
\]

(a) The take home pay in week two is the amount of money earned in the first ten working days minus the money earned in the first week.

Pay in week 2 = \( s_{10} - s_5 \)

\[
= 2 \left( \frac{2^{10} - 1}{1} \right) - 62
\]

\[
= 1984c
\]

\[
= $1984
\]

(b) The take home pay in week three is the total amount of money earned in the first 15 working days minus the money earned in the first and second weeks.

Pay in week 3 = \( s_{15} - s_2 \)

\[
= 2 \left( \frac{2^{15} - 1}{1} \right) - 2 \left( \frac{2^{14} - 1}{1} \right)
\]

\[
= 63486c
\]

\[
= $634.86
\]

(c) The take home pay in week four is the total amount of money earned in the first 20 working days minus the money earned in the first three weeks.

Pay in week 4 = \( s_{20} - s_3 \)

\[
= 2 \left( \frac{2^{20} - 1}{1} \right) - 2 \left( \frac{2^{19} - 1}{1} \right)
\]

\[
= 2031616c
\]

\[
= $203161.6
\]