Note

Hadamard Matrices of Order $28m$, $36m$ and $44m$

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Communicated by Marshall Hall, Jr.

Received March 16, 1972

We show that if four suitable matrices of order $m$ exist then there are Hadamard matrices of order $28m$, $36m$, and $44m$. In particular we show that Hadamard matrices of orders $14(q + 1)$, $18(q + 1)$, and $22(q + 1)$ exist when $q$ is a prime power and $q = 1 \pmod{4}$.

Also we show that if $n$ is the order of a conference matrix there is an Hadamard matrix of order $4mn$.

As a consequence there are Hadamard matrices of the following orders less than 4000:

476, 532, 836, 1036, 1012, 1100, 1148, 1276, 1364, 1372, 1476, 1672, 1836, 2024, 2052, 2156, 2212, 2380, 2484, 2508, 2548, 2716, 3036, 3476, 3892.

All these orders seem to be new.

Suppose a square matrix $A = (a_{ij})$ of side $n$ has the property that the entry in position $(i, j)$ always equals the entry in position $(i + 1, j + 1)$, where these coordinates are reduced modulo $n$ if necessary. Then the matrix is completely determined by its first row; in fact if $T = T_n = (t_{ij})$ is the $n \times n$ matrix defined by

\[ t_{i,i+1} = 1, \quad i = 1, 2, \ldots, n - 1, \]
\[ t_{n,1} = 1, \]
\[ t_{i,j} = 0, \quad \text{otherwise}, \]

then $A$ can be written

\[ A = \sum_{j=1}^{n} a_{i}T^{j-1}. \]
We say \( A \) is a \textit{circulant} matrix, formed by \textit{circulating} the row

\[
(a_{11}, a_{12}, \ldots, a_{1n}).
\]

Similarly, if \( P \) is an \( n \times n \) array of \( m \times m \) submatrices \( P_{ij} \) where \( P_{i+j, j-1} = P_{ij} \) (subscripts reduced modulo \( n \)), that is

\[
P = \sum_{j=1}^{n} T^{j-1} \times P_{ij}
\]

(where \( \times \) denotes Kronecker product), we shall say \( P \) is formed by circulating

\[
(P_{11}, P_{12}, \ldots, P_{nn}).
\]

We denote by \( R \) a square back-diagonal matrix whose order shall be determined by context: if \( R = (r_{ij}) \) is of order \( n \) then

\[
r_{ij} = 1, \quad \text{when} \quad i + j = n + 1,
\]

\[
r_{ij} = 0, \quad \text{otherwise}.
\]

We consider a set of four \( n \times n \) arrays \( X, Y, Z, \) and \( W \) which are formed by circulating their first rows; the entries shall be \( m \times m \) matrices chosen from a set of four matrices \( \{A, B, C, D\} \).

\textbf{Lemma 1.} If \( A, B, C, \) and \( D \) commute in pairs then \( X, Y, Z, \) and \( W \) commute in pairs.

In particular, Lemma 1 is satisfied if \( A, B, C, \) and \( D \) are circulant.

\textbf{Lemma 2.} If \( S \) and \( P \) are chosen from \( \{X, Y, Z, W\} \) and if \( A, B, C, \) and \( D \) are circulant matrices then

\[
S R P^T = P R S^T.
\]

\textit{Proof.} It is known (see [6]) that equation (1) would hold if \( S \) and \( P \) were circulant. In particular

\[
E_i R F_j^T = F_j R E_i^T
\]

when \( E_i \) and \( F_j \) belong to \( \{A, B, C, D\} \), and

\[
T^{i} R T^{n-i} = T^{i} R T^{n-i}.
\]

If we write

\[
S = \sum_{i=0}^{n-1} T^{i} \times E_i, \quad P = \sum_{j=0}^{n-1} T^{j} \times F_j,
\]
then
\[
SRP^T = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (T^i \times E_i) R(T^{n-j} \times F_j^T)
\]
\[
= \sum \sum (T^i \times E_i)(R \times R)(T^{n-j} \times F_j^T)
\]
\[
= \sum \sum (T^iRT^{n-j} \times E_iRF_j^T)
\]
\[
= \sum \sum (T^iRT^{n-j} \times F_jRE_i^T)
\]
\[
= PRS^T.
\]

Suppose
\[
XX^T + YY^T + ZZ^T + WW^T = I_n \times n(AB^T + BB^T + CC^T + DD^T).
\]
(2)

Then it is easy to verify that the matrix
\[
H = \begin{bmatrix}
X & YR & ZR & WR \\
- YR & X & -W^T R & Z^T R \\
- ZR & W^T R & X & -Y^T R \\
- WR & -Z^T R & Y^T R & X
\end{bmatrix}
\]

which is a form of block-matrix introduced by Goethals and Seidel in [2])
satisfies
\[
HH^T = nI_n \times (AB^T + BB^T + CC^T + DD^T)
\]
(3)

provided that \(X, Y, Z,\) and \(W\) pairwise commute and pairwise satisfy (1).

**Lemma 3.** If \(A, B, C,\) and \(D\) are such that \(AB^T, AC^T, AD^T, BC^T,\)
\(BD^T,\) and \(CD^T\) are symmetric, then the first rows
\[
(C, A, -A, -B, -B, A, D) \quad \text{for } X,
\]
\[
(-D, -B, B, -A, -A, -B, C) \quad \text{for } Y,
\]
\[
(-A, C, -C, D, D, C, B) \quad \text{for } Z,
\]
\[
(B, -D, D, C, C, -D, A) \quad \text{for } W
\]
give matrices which satisfy (2) for the case \(n = 7,\) the first rows
\[
(C, B, -A, -A, A, C, A, B, -D) \quad \text{for } X,
\]
\[
(A, -C, -D, A, B, B, -B, -D, -B) \quad \text{for } Y,
\]
\[
(A, -C, D, B, A, D, C, C, -C) \quad \text{for } Z,
\]
\[
(-B, D, C, A, -B, C, -D, -D, D) \quad \text{for } W
\]
give matrices which satisfy (2) for the case \( n = 9 \), and the first rows

\[
\begin{align*}
(C, B, A, A, -A, B, -A, B, -B, A) & \quad \text{for } X, \\
(D, A, -B, -B, B, A, A, A, -A, -B) & \quad \text{for } Y, \\
(-A, D, C, C, -C, D, -C, D, -D, C) & \quad \text{for } Z, \\
(-B, C, -D, -D, D, C, D, C, -C, -D) & \quad \text{for } W
\end{align*}
\]

give matrices which satisfy (2) for the case \( n = 11 \).

The verification is straightforward.

If \( AA^T + BB^T + CC^T + DD^T = 4mI_m \) and if \( H \) has all its entries \( 1 \) or \(-1\), then equation (3) means that \( H \) is Hadamard. So, gathering together the foregoing results, we have the following theorem:

**Theorem 4.** If there exist square circulant \( (1, -1) \) matrices \( A, B, C, \) and \( D \) of order \( m \) which satisfy

\[
AA^T + BB^T + CC^T + DD^T = 4mI
\]

and are such that \( AB^T, AC^T, AD^T, BC^T, BD^T, \) and \( CD^T \) are symmetric, then there are Hadamard matrices of orders \( 28m, 36m, \) and \( 44m \).

Matrices \( A, B, C, \) and \( D \) satisfying the conditions of Theorem 4 were previously used to construct Hadamard matrices of orders \( 4m \) [11], \( 12m \) [1], and \( 20m \) (unpublished result of L. R. Welch, communicated to the author by L. D. Baumert). They are known to exist when \( m \) is a member of the set

\[
M = \{3, 5, 7, \ldots, 29, 37, 43\}
\]

[3], and when \( 2m - 1 \) is a prime power congruent to \( 1 \) modulo \( 4 \) [4, 10].

**Corollary 5.** There exist Hadamard matrices of orders \( 28m, 36m, \) and \( 44m \) whenever \( m \in M \).

**Corollary 6.** There exist Hadamard matrices of orders \( 14(q + 1) \), \( 18(q + 1) \), and \( 22(q + 1) \) whenever \( q \) is a prime power congruent to \( 1 \) modulo \( 4 \).

This gives Hadamard matrices of twenty-two orders less than 4000 for which no matrices were previously known, namely,

\[
476, 532, 836, 1012, 1036, 1030, 1100, 1148, 1276, 1364, 1372, 1476, 1672, 1836, 2024, 2052, 2156, 2212, 2380, 2484, 2508, 2548, 2716, 3036, 3476, 3892.
\]
A conference matrix \( N \) is a \((0, 1, -1)\) matrix with zero diagonal and every other element +1 or -1 which satisfies

\[
NN^T = (n - 1) I_n, \quad NJ = 0, \quad N^T = eN, \quad e = \pm 1,
\]

where \( J \) is the matrix with every element +1. These are discussed in [5, 7, 8, 9] where they are sometimes called \( n \)-type and skew-Hadamard matrices. Some of Turyn's constructions for complex Hadamard matrices are equivalent to conference matrices when \( n = 2 \) (mod 4).

Symmetric conference matrices are known to exist for orders \( p + 1 \) when \( p = 1 \) (mod 4) is a prime power and \((h - 1)^2 + 1 \) when \( h \) is the order of a skew-Hadamard matrix. The skew-Hadamard matrices (skew-symmetric conference matrices) are listed in [7, 8, 9] but in particular they exist for orders \( p + 1, p = 3 \) (mod 4), a prime power.

Then we have:

**Theorem 7.** Let

\[
A_1 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad A_2 = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{bmatrix},
\]

\[
A_3 = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}, \quad A_4 = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{bmatrix},
\]

and let \( N \) be the core of a conference matrix of order \( n \); if \( A, B, C, D \) are four \((1, -1)\) matrices which pairwise satisfy \( XY^T = YX^T \) and if

\[
AA^T + BB^T + CC^T + DD^T = 4mI_n
\]

then

\[
H = A_1 \times N \times A + A_1 \times I \times B + A_2 \times N \times -B + A_2 \times I \times A \\
+ A_3 \times N \times C + A_3 \times I \times D + A_4 \times N \times -D + A_4 \times I \times C
\]

is an Hadamard matrix of order \( 4mn \).

**Corollary 8.** Let \( p \) be any prime power and \( m \in M \); then there exists an Hadamard matrix of order \( 4m(p + 1) \).
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Printed by the St Catherine Press Ltd., Tempelhof 37, Bruges, Belgium.